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GENERALIZATIONS AND REFINEMENTS OF HERMITE-HADAMARD'S INEQUALITY

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ABSTRACT. In this article, with the help of the concept of the harmonic sequence of polynomials, the well known Hermite-Hadamard's inequality for convex functions is generalized to cases with bounded derivatives of nth order, including the so-called n-convex functions, from which Hermite-Hadamard's inequality is extended and refined.

1. Introduction. Let $f(x)$ be a convex function on the closed interval $[a, b]$, the well known Hermite-Hadamard's inequality [**6**] can be expressed as:

$$(1)\quad 0 \le \int_a^b f(t)\,dt - (b-a) f\left(\frac{a+b}{2}\right) \le (b-a)\frac{f(a)+f(b)}{2} - \int_a^b f(t)\,dt.$$

It is well known that Hermite-Hadamard's inequality is an important cornerstone in mathematical analysis and optimization. There is a growing literature considering its refinements and interpolations now.

A function $f(x)$ is said to be r-convex on $[a, b]$ with $r \ge 2$ if and only if $f^{(r)}(x)$ exists and $f^{(r)}(x) \ge 0$.

In terms of a trapezoidal formula and a midpoint formula for a real function $f(x)$ defined and integrable on $[a, b]$, using the first and second Euler-Maclaurin summation formulas, inequality (1) was generalized for $(2r)$-convex functions on $[a, b]$ with $r \ge 1$ in [**2**].

In this paper, for our own convenience, we adopt the following notation

$$(2)\qquad S_n = \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}$$

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for any n-times differentiable function f defined on the closed interval $[a, b]$.

In [**3, 4**], the following double integral inequalities were obtained.

Theorem A. *Let $f : [a, b] \to \mathbf{R}$ be a twice differentiable mapping and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t \in (a, b)$. Then we have*

$$\frac{\gamma(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(b-a)^2}{24}, \tag{3}$$

$$\frac{\gamma(b-a)^2}{12} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \le \frac{\Gamma(b-a)^2}{12}. \tag{4}$$

In [**11**]*, the above inequalities were refined as follows.*

Theorem B. *Let $f : [a, b] \to \mathbf{R}$ be a twice differentiable mapping and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t \in (a, b)$. Then we have*

$$\begin{aligned} \frac{3S_2 - 2\Gamma}{24}(b-a)^2 &\le \frac{1}{b-a}\int_a^b f(t)\,dt - f\left(\frac{a+b}{2}\right) \\ &\le \frac{3S_2 - 2\gamma}{24}(b-a)^2, \end{aligned} \tag{5}$$

$$\begin{aligned} \frac{3S_2 - \Gamma}{24}(b-a)^2 &\le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)\,dt \\ &\le \frac{3S_2 - \gamma}{24}(b-a)^2. \end{aligned} \tag{6}$$

If $f''(t) \le 0$, or $f''(t) \ge 0$, then we can set $\Gamma = 0$, or $\gamma = 0$, in Theorem A and Theorem B. Then Hermite-Hadamard's inequality (1) and those similar to the Hermite-Hadamard's inequality (1) can be obtained.

In this article, using the concept of the harmonic sequence of polynomials, the well known Hermite-Hadamard's inequality for convex functions is generalized to the cases with bounded derivatives of nth order, including the so-called n-convex functions, from which Hermite-Hadamard's inequality is extended and refined.

2. Some simple generalizations. In this section, we will generalize results above to the cases that the nth derivative of integrand is bounded for $n \in \mathbf{N}$.

Theorem 1. *Let $f(t)$ be n-times differentiable on the closed interval $[a,b]$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t \in [a,b]$ and $n \in \mathbf{N}$. Further, let $u \in [a,b]$ be a parameter. Then*

$$
\begin{aligned}
(7) \quad & (b-a)S_n \max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\} \\
&+ \left[\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!} - (b-a)\max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\}\right]\Gamma \\
&\le (-1)^n \int_a^b f(t)\,dt + \sum_{i=0}^{n-1} \frac{(u-a)^{n-i}-(u-b)^{n-i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u) \\
&\le (b-a)S_n \max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\} \\
&+ \left[\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!} - (b-a)\max\left\{\frac{(u-a)^n}{n!}, \frac{(b-u)^n}{n!}\right\}\right]\gamma.
\end{aligned}
$$

Proof. Define

$$
(8) \qquad p_n(t) = \begin{cases} (t-a)^n/n!, & t \in [a,u], \\ (t-b)^n/n!, & t \in (u,b]. \end{cases}
$$

By direct computation, we have

$$
(9) \qquad \int_a^b p_n(t)\,dt = \frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!}.
$$

Integrating by parts and using mathematical induction yields

$$
(10) \qquad \begin{aligned} \int_a^b p_n(t) f^{(n)}(t)\,dt = {} & \frac{(u-a)^n-(u-b)^n}{n!} f^{(n-1)}(u) \\ & - \int_a^b p_{n-1}(t) f^{(n-1)}(t)\,dt, \end{aligned}
$$

and then

$$\begin{aligned}\int_a^b p_n(t)f^{(n)}(t)\,dt+(-1)^{n+1}\int_a^b f(t)\,dt\\=\sum_{i=0}^{n-1}\frac{(u-a)^{n-i}-(u-b)^{n-i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u).\end{aligned}\tag{11}$$

Utilization of (9) and (11) yields

$$\begin{aligned}\int_a^b p_n(t)\big[f^{(n)}(t)-\gamma\big]\,dt\\=(-1)^n\int_a^b f(t)\,dt-\frac{(u-a)^{n+1}-(u-b)^{n+1}}{(n+1)!}\gamma\\+\sum_{i=0}^{n-1}\frac{(u-a)^{n-i}-(u-b)^{n-i}}{(n-i)!}(-1)^i f^{(n-i-1)}(u).\end{aligned}\tag{12}$$

Meanwhile,

$$\begin{aligned}&\int_a^b p_n(t)\big[f^{(n)}(t)-\gamma\big]\,dt\\&\le\int_a^b |p_n(t)|\,\big|f^{(n)}(t)-\gamma\big|\,dt\\&\le\max_{t\in[a,b]}|p_n(t)|\int_a^b\big(f^{(n)}(t)-\gamma\big)\,dt\\&\le\max\left\{\frac{(u-a)^n}{n!},\frac{(b-u)^n}{n!}\right\}\left[\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}-\gamma\right](b-a).\end{aligned}\tag{13}$$

The right inequality in (7) follows from combining (12) with (13).

The left inequality in (7) follows from similar arguments as above. □

Theorem 2. *Let $f(t)$ be n-times differentiable on the closed interval $[a,b]$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t \in [a,b]$ and $n \in \mathbf{N}$. Then*

$$\begin{aligned}
&\frac{1}{2^n}\,\frac{(b-a)^{n+1}}{n!}\left[S_n + \left(\frac{1+(-1)^n}{2(n+1)} - 1\right)\Gamma\right] \\
&\qquad \le (-1)^n \int_a^b f(t)\,dt \\
&\qquad\quad + \sum_{i=0}^{n-1} \frac{(b-a)^{n-i}}{(n-i)!}\,\frac{(-1)^{n+1}+(-1)^i}{2^{n-i}}\,f^{(n-i-1)}\left(\frac{a+b}{2}\right) \\
&\qquad \le \frac{1}{2^n}\,\frac{(b-a)^{n+1}}{n!}\left[S_n + \left(\frac{1+(-1)^n}{2(n+1)} - 1\right)\gamma\right].
\end{aligned} \tag{14}$$

Proof. This follows from taking $u = (a+b)/2$ in inequality (7). □

Remark 1. If taking $n = 2$ in (14), the double inequality (5) follows. □

Theorem 3. *Let $f(t)$ be n-times differentiable on the closed interval $[a,b]$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t \in [a,b]$ and $n \in \mathbf{N}$, and $u \in \mathbf{R}$. Then*

$$\begin{aligned}
&\left[(b-a)\max\left\{\frac{|a-u|^n}{n!}, \frac{|b-u|^n}{n!}\right\} + \frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\right]\gamma \\
&\quad - (b-a)S_n \max\left\{\frac{|a-u|^n}{n!}, \frac{|b-u|^n}{n!}\right\} \\
&\le (-1)^n \int_a^b f(t)\,dt \\
&\quad + \sum_{i=0}^{n-1} (-1)^i \frac{(b-u)^{n-i} f^{(n-i-1)}(b) - (a-u)^{n-i} f^{(n-i-1)}(a)}{(n-i)!} \\
&\le \left[(b-a)\max\left\{\frac{|a-u|^n}{n!}, \frac{|b-u|^n}{n!}\right\} + \frac{(b-u)^{n+1}-(a-u)^{n+1}}{(n+1)!}\right]\Gamma \\
&\quad - (b-a)S_n \max\left\{\frac{|a-u|^n}{n!}, \frac{|b-u|^n}{n!}\right\}.
\end{aligned}$$

Proof. Define

$$q_n(t) = \frac{(t-u)^n}{n!}, \quad u \in \mathbf{R}. \tag{16}$$

By direct computation, we have

$$\int_a^b q_n(t)\,dt = \frac{(b-u)^{n+1} - (a-u)^{n+1}}{(n+1)!}. \tag{17}$$

Integrating by parts and using mathematical induction yields

$$\begin{aligned}&\int_a^b q_n(t) f^{(n)}(t)\,dt + \int_a^b q_{n-1}(t) f^{(n-1)}(t)\,dt \\ &\qquad = \frac{(b-u)^n f^{(n-1)}(b) - (a-u)^n f^{(n-1)}(a)}{n!},\end{aligned} \tag{18}$$

and then

$$\begin{aligned}&\int_a^b q_n(t) f^{(n)}(t)\,dt + (-1)^{n+1}\int_a^b f(t)\,dt \\ &\qquad = \sum_{i=0}^{n-1}(-1)^i \frac{(b-u)^{n-i} f^{(n-i-1)}(b) - (a-u)^{n-i} f^{(n-i-1)}(a)}{(n-i)!}.\end{aligned} \tag{19}$$

Making use of (17) and (19) and direct calculation yields

$$\begin{aligned}&\int_a^b q_n(t)\big[\gamma - f^{(n)}(t)\big]\,dt \\ &= (-1)^{n+1}\int_a^b f(t)\,dt + \frac{(b-u)^{n+1} - (a-u)^{n+1}}{(n+1)!}\gamma \\ &\quad + \sum_{i=0}^{n-1}(-1)^{i+1} \frac{(b-u)^{n-i} f^{(n-i-1)}(b) - (a-u)^{n-i} f^{(n-i-1)}(a)}{(n-i)!}.\end{aligned} \tag{20}$$

It is easy to see that

$$\begin{aligned}&\int_a^b q_n(t)\big[\gamma - f^{(n)}(t)\big]\,dt \\ &\le \max_{t\in[a,b]} |q_n(t)| \int_a^b \big(f^{(n)}(t) - \gamma\big)\,dt \\ &\le \max\left\{\frac{|a-u^n|}{n!}, \frac{|b-u^n|}{n!}\right\}\left[\frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a} - \gamma\right](b-a).\end{aligned} \tag{21}$$

The left inequality in (15) follows from combining (20) with (21).

The right inequality in (15) follows from similar arguments as above. □

Theorem 4. *Let $f(t)$ be n-times differentiable on the closed interval $[a,b]$ such that $\gamma \le f^{(n)}(t) \le \Gamma$ for $t \in [a,b]$ and $n \in \mathbf{N}$. Then*

$$\begin{aligned} &\frac{1}{2^n}\,\frac{(b-a)^{n+1}}{n!}\left[\left(1+\frac{1+(-1)^n}{2(n+1)}\right)\gamma - S_n\right] \\ &\quad\le (-1)^n \int_a^b f(t)\,dt \\ &\qquad + \sum_{i=0}^{n-1} \frac{(b-a)^{n-i}}{(n-i)!}\,\frac{(-1)^{n+1} f^{(n-i-1)}(a) + (-1)^i f^{(n-i-1)}(b)}{2^{n-i}} \\ &\quad\le \frac{1}{2^n}\,\frac{(b-a)^{n+1}}{n!}\left[\left(1+\frac{1+(-1)^n}{2(n+1)}\right)\Gamma - S_n\right]. \end{aligned} \tag{22}$$

Proof. This follows from taking $u = (a+b)/2$ in (15). □

Corollary 1. *Let $f : [a,b] \to \mathbf{R}$ be a twice differentiable mapping on $[a,b]$ and suppose that $\gamma \le f''(t) \le \Gamma$ for $t \in (a,b)$. Then we have*

$$\frac{2\gamma - 3S_2}{12}\,(b-a)^2 \le \frac{1}{b-a}\int_a^b f(t)\,dt - \frac{f(a)+f(b)}{2} \le \frac{2\Gamma - 3S_2}{12}\,(b-a)^2. \tag{23}$$

Proof. If setting $n = 2$ in (22), then inequality (23) follows. □

3. More general generalizations. In this section, we will generalize Hermite-Hadamard's inequality to more general cases with the help of the concept of harmonic sequence of polynomials.

Definition 1. A sequence of polynomials $\{P_i(t,x)\}_{i=0}^{\infty}$ is called harmonic if it satisfies the following Appell condition

$$P_i'(t) \triangleq \frac{\partial P_i(t,x)}{\partial t} = P_{i-1}(t,x) \triangleq P_{i-1}(t) \tag{24}$$

and $P_0(t,x)=1$ for all defined (t,x) and $i \in \mathbf{N}$.

It is well-known that Bernoulli's polynomials $B_i(t)$ can be defined by the following expansion

$$\frac{xe^{tx}}{e^x-1} = \sum_{i=0}^{\infty} \frac{B_i(t)}{i!} x^i, \quad |x|<2\pi, \quad t \in \mathbf{R}, \tag{25}$$

and are uniquely determined by the following formulae

$$B_i'(t) = iB_{i-1}(t), \quad B_0(t)=1; \tag{26}$$

$$B_i(t+1) - B_i(t) = it^{i-1}. \tag{27}$$

Similarly, Euler's polynomials can be defined by

$$\frac{2e^{tx}}{e^x+1} = \sum_{i=0}^{\infty} \frac{E_i(t)}{i!} x^i, \quad |x|<\pi, \quad t \in \mathbf{R}, \tag{28}$$

and are uniquely determined by the following properties

$$E_i'(t) = iE_{i-1}(t), \quad E_0(t)=1; \tag{29}$$

$$E_i(t+1) + E_i(t) = 2t^i. \tag{30}$$

For further details about Bernoulli's polynomials and Euler's polynomials, please refer to [**1**, 23.1.5 and 23.1.6] or [**12**]. Moreover, some new generalizations of Bernoulli's numbers and polynomials can be found in [**7, 8, 9, 10**].

There are many examples of harmonic sequences of polynomials. For instance, for i being a nonnegative integer, $t,\tau,\theta \in \mathbf{R}$ and $\tau \neq \theta$,

$$P_{i,\lambda}(t) \triangleq P_{i,\lambda}(t;\tau;\theta) = \frac{[t-(\lambda\theta+(1-\lambda)\tau)]^i}{i!}, \tag{31}$$

$$P_{i,B}(t) \triangleq P_{i,B}(t;\tau;\theta) = \frac{(\tau-\theta)^i}{i!} B_i\left(\frac{t-\theta}{\tau-\theta}\right), \tag{32}$$

$$P_{i,E}(t) \triangleq P_{i,E}(t;\tau;\theta) = \frac{(\tau-\theta)^i}{i!} E_i\left(\frac{t-\theta}{\tau-\theta}\right). \tag{33}$$

As usual, let $B_i = B_i(0)$, $i \in \mathbf{N}$, denote Bernoulli's numbers. From properties (26) and (27), (29) and (30) of Bernoulli's and Euler's polynomials respectively, we can obtain easily that, for $i \geq 1$,

$$B_{i+1}(0) = B_{i+1}(1) = B_{i+1}, \quad B_1(0) = -B_1(1) = -\frac{1}{2}, \tag{34}$$

and, for $j \in \mathbf{N}$,

$$E_j(0) = -E_j(1) = -\frac{2}{j+1}\,(2^{j+1}-1)B_{j+1}. \tag{35}$$

It is also a well-known fact that $B_{2i+1} = 0$ for all $i \in \mathbf{N}$.

Theorem 5. *Let $\{P_i(t)\}_{i=0}^{\infty}$ be a harmonic sequence of polynomials, let $f(t)$ be n-times differentiable on the closed interval $[a,b]$ such that $\gamma \leq f^{(n)}(t) \leq \Gamma$ for $t \in [a,b]$ and $n \in \mathbf{N}$. Let α be a real constant. Then*

$$\begin{aligned} &\Big[\alpha + \max_{t\in[a,b]} |P_n(t)+\alpha|\Big] S_n \\ &\quad - \left(\max_{t\in[a,b]} |P_n(t)+\alpha| + \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} + \alpha\right)\Gamma \\ &\leq (-1)^{n+1}\left[\tfrac{1}{b-a}\int_a^b f(t)\,dt + \sum_{i=1}^{n}(-1)^i\,\tfrac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\right] \\ &\leq \Big[\alpha - \max_{t\in[a,b]} |P_n(t)+\alpha|\Big] S_n \\ &\quad + \left(\max_{t\in[a,b]} |P_n(t)+\alpha| - \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} - \alpha\right)\Gamma \end{aligned} \tag{36}$$

and

$$\begin{aligned} &\Big[\alpha - \max_{t\in[a,b]} |P_n(t)+\alpha|\Big] S_n \\ &\quad + \left(\max_{t\in[a,b]} |P_n(t)+\alpha| - \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} - \alpha\right)\gamma \\ &\leq (-1)^{n+1}\left[\tfrac{1}{b-a}\int_a^b f(t)\,dt + \sum_{i=1}^{n}(-1)^i\,\tfrac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\right] \\ &\leq \Big[\alpha + \max_{t\in[a,b]} |P_n(t)+\alpha|\Big] S_n \\ &\quad - \left(\max_{t\in[a,b]} |P_n(t)+\alpha| + \frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} + \alpha\right)\gamma. \end{aligned} \tag{37}$$

Proof. By successive integration by parts and mathematical induction we obtain

$$\begin{aligned}(-1)^n \int_a^b P_n(t) f^{(n)}(t)\,dt - \int_a^b f(t)\,dt \\ = \sum_{i=1}^{n} (-1)^i \big[P_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)\big].\end{aligned} \tag{38}$$

Using the definition of the harmonic sequence of polynomials yields

$$\int_a^b P_n(t)\,dt = P_{n+1}(b) - P_{n+1}(a). \tag{39}$$

Using (38) and (39) gives us

$$\begin{aligned}&\frac{1}{b-a}\int_a^b \big[P_n(t)+\alpha\big]\big[\Gamma - f^{(n)}(t)\big]\,dt \\ &= \frac{(-1)^{n+1}}{b-a}\int_a^b f(t)\,dt + \left(\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a} + \alpha\right)\Gamma \\ &+ \sum_{i=1}^{n}(-1)^{n+i+1}\frac{P_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)}{b-a} - \alpha S_n.\end{aligned} \tag{40}$$

Direct calculation shows

$$\begin{aligned}&\left|\frac{1}{b-a}\int_a^b \big[P_n(t)+\alpha\big]\big[\Gamma - f^{(n)}(t)\big]\,dt\right| \\ &\le \frac{1}{b-a}\max_{t\in[a,b]}|P_n(t)+\alpha| \int_a^b \big[\Gamma - f^{(n)}(t)\big]\,dt \\ &= \max_{t\in[a,b]}|P_n(t)+\alpha| \left[\Gamma - \frac{f^{(n-1)}(b) - f^{(n-1)}(a)}{b-a}\right].\end{aligned} \tag{41}$$

From combining (40) with (41), it follows that

$$
\begin{aligned}
&\big[\alpha+\max_{t\in[a,b]}|P_n(t)+\alpha|\,\big]S_n\\
&\quad-\left(\max_{t\in[a,b]}|P_n(t)+\alpha|+\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}+\alpha\right)\Gamma\\
&\le\frac{(-1)^{n+1}}{b-a}\int_a^b f(t)\,dt\\
&\quad+\sum_{i=1}^{n}(-1)^{n+i+1}\frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\\
&\le\big[\alpha-\max_{t\in[a,b]}|P_n(t)+\alpha|\,\big]S_n\\
&\quad+\left(\max_{t\in[a,b]}|P_n(t)+\alpha|-\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}-\alpha\right)\Gamma.
\end{aligned}
\tag{42}
$$

The inequality (36) follows.

Similarly, we can obtain the inequality (37). □

Remark 2. If taking $P_2(t)=(t-(a+b)/2)^2/2$, $\alpha=-(b-a)^2/8$, and $n=2$ in (36) and (37), then the inequality (6) follows easily.

Remark 3. If setting $P_n(t)=q_n(t)$ and $\alpha=0$ in (36) and (37), then we can deduce Theorem 3 from Theorem 5.

Theorem 6. *Let $\{E_i(t)\}_{i=0}^{\infty}$ be the Euler's polynomials and $\{B_i\}_{i=0}^{\infty}$ the Bernoulli's numbers. Let $f(t)$ be n-times differentiable on the closed interval $[a,b]$ such that $\gamma\le f^{(n)}(t)\le\Gamma$ for $t\in[a,b]$ and $n\in\mathbf{N}$. Then*

$$
\begin{aligned}
&\frac{(a-b)^n}{n!}\left[\left(\max_{t\in[0,1]}|E_n(t)|+\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\Gamma-\max_{t\in[0,1]}|E_n(t)|S_n\right]\\
&\le\frac{1}{b-a}\int_a^b f(t)\,dt\\
&\quad+2\sum_{i=1}^{[(n+1)/2]}\frac{(b-a)^{2(i-1)}}{(2i)!}\left[f^{(2(i-1))}(a)+f^{(2(i-1))}(b)\right](1-4^i)B_{2i}\\
&\le\frac{(a-b)^n}{n!}\left[\max_{t\in[0,1]}|E_n(t)|S_n-\left(\max_{t\in[0,1]}|E_n(t)|-\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\Gamma\right]
\end{aligned}
\tag{43}
$$

and

(44)
$$\frac{(a-b)^n}{n!}\left[\max_{t\in[0,1]}|E_n(t)|S_n-\left(\max_{t\in[0,1]}|E_n(t)|-\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\gamma\right]$$
$$\le\frac{1}{b-a}\int_a^b f(t)\,dt$$
$$+2\sum_{i=1}^{[(n+1)/2]}(1-4^i)\frac{(b-a)^{2(i-1)}}{(2i)!}\left[f^{(2(i-1))}(a)+f^{(2(i-1))}(b)\right]B_{2i}$$
$$\le\frac{(a-b)^n}{n!}\left[\left(\max_{t\in[0,1]}|E_n(t)|+\frac{4(2^{n+2}-1)}{(n+1)(n+2)}B_{n+2}\right)\gamma-\max_{t\in[0,1]}|E_n(t)|S_n\right],$$

where $[x]$ *denotes the Gauss function, whose value is the largest integer not more than* x.

Proof. Let

$$P_i(t)=P_{i,E}(t;b;a)=\frac{(b-a)^i}{i!}E_i\left(\frac{t-a}{b-a}\right). \tag{45}$$

Then, we have

$$\max_{t\in[a,b]}|P_n(t)|=\frac{(b-a)^n}{n!}\max_{t\in[0,1]}|E_n(t)|, \tag{46}$$

and

$$\frac{P_{n+1}(b)-P_{n+1}(a)}{b-a}=\frac{4(2^{n+2}-1)}{n+2}\,\frac{(b-a)^n}{(n+1)!}B_{n+2}. \tag{47}$$

Using formulae (35) and straightforward calculation yields

(48)
$$\sum_{i=1}^{n}(-1)^i\frac{P_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}$$
$$=\sum_{i=1}^{n}(-1)^i\frac{(b-a)^{i-1}}{i!}\left[E_i(1)f^{(i-1)}(b)-E_i(0)f^{(i-1)}(a)\right]$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}(-1)^i \, \frac{(b-a)^{i-1}}{i!} E_i(1) \left[f^{(i-1)}(a) + f^{(i-1)}(b)\right] \\
&= 2\sum_{i=1}^{n}(-1)^i \, \frac{(b-a)^{i-1}}{(i+1)!} \left[f^{(i-1)}(a) + f^{(i-1)}(b)\right] (2^{i+1}-1)B_{i+1} \\
&= 2 \sum_{i=1}^{[(n+1)/2]} (1-4^i) \frac{(b-a)^{2(i-1)}}{(2i)!} \left[f^{(2(i-1))}(a) + f^{(2(i-1))}(b)\right] B_{2i}.
\end{aligned}
$$

Substituting (45), (46), (47) and (48) into (36) and (37) and taking $\alpha = 0$ leads to (43) and (44). The proof is complete. □

Theorem 7. *Let* $\{P_i(t)\}_{i=0}^{\infty}$ *and* $\{Q_i(t)\}_{i=0}^{\infty}$ *be two harmonic sequences of polynomials,* α *and* β *two real constants,* $u \in [a,b]$. *Let* $f(t)$ *be* n*-times differentiable on the closed interval* $[a,b]$ *such that* $\gamma \le f^{(n)}(t) \le \Gamma$ *for* $t \in [a,b]$ *and* $n \in \mathbf{N}$. *Then*

$$
\begin{aligned}
&\left[\frac{Q_{n+1}(b) - P_{n+1}(a)}{b-a} + \frac{P_{n+1}(u) - Q_{n+1}(u)}{b-a} \right. \\
&\qquad \left. + \frac{(\alpha-\beta)u + (b\beta - a\alpha)}{b-a} + C(u)\right]\gamma - C(u)S_n \\
&\le \frac{(-1)^n}{b-a}\int_a^b f(t)\,dt \\
&\quad + \sum_{i=1}^{n}(-1)^{n+i}\frac{Q_i(b)f^{(i-1)}(b) - P_i(a)f^{(i-1)}(a)}{b-a} \\
&\quad + \sum_{i=1}^{n}(-1)^{n+i}\frac{P_i(u) - Q_i(u)}{b-a} f^{(i-1)}(u) \\
&\quad + \frac{\beta f^{(n-1)}(b) - \alpha f^{(n-1)}(a)}{b-a} + \frac{(\alpha-\beta)f^{(n-1)}(u)}{b-a} \\
&\le \left[\frac{Q_{n+1}(b) - P_{n+1}(a)}{b-a} + \frac{P_{n+1}(u) - Q_{n+1}(u)}{b-a} \right. \\
&\qquad \left. + \frac{(\alpha-\beta)u + (b\beta - a\alpha)}{b-a} - C(u)\right]\gamma + C(u)S_n
\end{aligned} \tag{49}
$$

and

$$\begin{aligned}
&\left[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\right.\\
&\qquad\left.+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}-C(u)\right]\Gamma+C(u)S_n\\
&\le\frac{(-1)^n}{b-a}\int_a^b f(t)\,dt\\
&\quad+\sum_{i=1}^{n}(-1)^{n+i}\frac{Q_i(b)f^{(i-1)}(b)-P_i(a)f^{(i-1)}(a)}{b-a}\\
&\quad+\sum_{i=1}^{n}(-1)^{n+i}\frac{P_i(u)-Q_i(u)}{b-a}f^{(i-1)}(u)\\
&\quad+\frac{\beta f^{(n-1)}(b)-\alpha f^{(n-1)}(a)}{b-a}+\frac{(\alpha-\beta)f^{(n-1)}(u)}{b-a}\\
&\le\left[\frac{Q_{n+1}(b)-P_{n+1}(a)}{b-a}+\frac{P_{n+1}(u)-Q_{n+1}(u)}{b-a}\right.\\
&\qquad\left.+\frac{(\alpha-\beta)u+(b\beta-a\alpha)}{b-a}+C(u)\right]\Gamma-C(u)S_n,
\end{aligned}\tag{50}$$

where

$$C(u)=\max\left\{\max_{t\in[a,u]}|P_n(t)+\alpha|,\ \max_{t\in(u,b]}|Q_n(t)+\beta|\right\}.\tag{51}$$

Proof. Define

$$\psi_n(t)=\begin{cases}P_n(t)+\alpha, & t\in[a,u],\\ Q_n(t)+\beta, & t\in(u,b].\end{cases}\tag{52}$$

It is easy to see that

$$\begin{aligned}
\int_a^b\psi_n(t)\,dt&=\int_a^u\psi_n(t)\,dt+\int_u^b\psi_n(t)\,dt\\
&=[Q_{n+1}(b)-P_{n+1}(a)]+[P_{n+1}(u)-Q_{n+1}(u)]\\
&\quad+(\alpha-\beta)u+(b\beta-a\alpha).
\end{aligned}\tag{53}$$

Direct computation produces

(54)

$$\begin{aligned}\int_a^b \psi_n(t) f^{(n)}(t)\,dt &= \int_a^u \psi_n(t) f^{(n)}(t)\,dt + \int_u^b \psi_n(t) f^{(n)}(t)\,dt \\ &= (-1)^n \int_a^b f(t)\,dt + (\alpha-\beta) f^{(n-1)}(u) \\ &\quad + \sum_{i=1}^{n} (-1)^{n+i} \left[Q_i(b) f^{(i-1)}(b) - P_i(a) f^{(i-1)}(a)\right] \\ &\quad + \sum_{i=1}^{n} (-1)^{n+i} \left[P_i(u) - Q_i(u)\right] f^{(i-1)}(u) \\ &\quad + \left[\beta f^{(n-1)}(b) - \alpha f^{(n-1)}(a)\right],\end{aligned}$$

and

$$\begin{aligned}&\left|\int_a^b \psi_n(t)\big[f^{(n)}(t)-\gamma\big]\,dt\right| \\ (55)\qquad &\leq \max_{t\in[a,b]} |\psi_n(t)| \int_a^b \big(f^{(n)}(t)-\gamma\big)\,dt \\ &\leq C(u)\left[f^{(n-1)}(b) - f^{(n-1)}(a) - \gamma(b-a)\right].\end{aligned}$$

Combining (53), (54), (55) and rearranging leads to (49).

The inequality (50) follows from the same arguments. The proof is complete. □

Remark 4. If taking $u = b$ in Theorem 7, then Theorem 5 is derived.

Remark 5. If taking $\alpha = \beta = 0$, $P_i(t) = ((t-a)^i/i!)$ and $Q_i(t) = (t-b)^i/i!$ in Theorem 7, then Theorem 1 follows.

Remark 6. If $f^{(n)}(t) \geq 0$, or $f^{(n)}(t) \leq 0$, for $t \in [a,b]$, then we can set $\gamma = 0$, or $\Gamma = 0$, and so some inequalities for the so-called n-convex, or n-concave, functions are obtained as consequences of theorems in this paper, which generalize or refine the well-known Hermite-Hadamard's inequality.

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