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GEOMETRY OF JUMP SYSTEMS

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ABSTRACT. A jump system is a set of lattice points satisfying a certain "two-step" axiom. We present a variety of results concerning the geometry of these objects, including a characterization of two-dimensional jump systems, necessary (though not sufficient) properties of higher-dimensional jump systems, and a characterization of constant-sum jump systems.

1. Introduction. A jump system is a set of lattice points that satisfy a simple "two-step" axiom. They were introduced by Bouchet and Cunningham [**1**] in order to simultaneously generalize delta-matroids (hence matroids) and degree sequences of subgraphs.

Fix a finite set S. We consider elements of $\mathbf{Z}^S$ together with the 1-norm $|x| = \sum_{i\in S} |x_i|$ and the corresponding distance $d(x,y) = |x-y|$.

For elements $x, y \in \mathbf{Z}^S$, we say $z \in \mathbf{Z}^S$ is a *step from x toward* (*in the direction of*) y if $|z-x| = 1$ and $|z-y| < |x-y|$. Note that if z is a step from x toward y, then $z = x \pm e_i$ for some standard unit vector e_i. For notational convenience, we will use $x \xrightarrow{y} z$ to denote a step from x to z in the direction of y.

Given a collection of points $J \subseteq \mathbf{Z}^S$, we say that J is a *jump system* if it satisfies Axiom 1.1.

Axiom 1.1 (2-step axiom). *If $x, y \in J$ and $x \xrightarrow{y} z$ with $z \notin J$, then there exists $z' \in J$ with $z \xrightarrow{y} z'$.*

The following well-known operations all preserve Axiom 1.1, see [**1, 3, 4, 5**]. They allow us to simplify many of the later proofs concerning various properties of jump systems.

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Let J be a jump system. Let $a \in \mathbf{Z}^S$. We call $\{x + a : x \in J\}$ the *translation* of J by a. Let $N \subseteq S$. We call $\{x' : x \in J, x'_j = x_j \textit{ for } j \notin N, x'_j = -x_j \textit{ for } j \in N\}$ the *reflection* of J in N. We call $\{x' \in \mathbf{Z}^{(S \setminus N) \dot\cup \{0\}} : x \in J, x'_j = x_j \text{ for } j \notin N, x'_0 = \sum_{j \in N} x_j\}$ the *reduction* of J by N. Let J_1 and J_2 be jump systems on $\mathbf{Z}^S$. We call $\{x + y : x \in J_1, y \in J_2\}$ the *sum* of J_1 and J_2.

Example 1.1. Let $J_1 = \{(2,2),(2,3)\}$, $J_2 = \{(0,0),(1,0),(3,0)\}$. The translation of J_1 by $(2,4)$ is $\{(4,6),(4,7)\}$. The reflection of J_1 in $\{2\}$ is $\{(2,-2),(2,-3)\}$. The reduction of J_2 by $\{1,2\}$ is $\{(0),(1),(3)\}$. The sum of J_1 and J_2 is $\{(2,2),(2,3),(3,2),(3,3),(5,2),(5,3)\}$.

Let $-\infty \leq a_i \leq b_i \leq \infty$ for all $i \in S$. The set of points $\{x \in \mathbf{Z}^S : x_i \in [a_i, b_i]\}$ is called a *box*. The following useful theorem characterizes jump systems.

Theorem 1.1 (Lovász). *Let J be a jump system, and let $B^1 \subseteq B^2 \subseteq \cdots \subseteq B^r$ be boxes in $\mathbf{Z}^S$. Then there is some point $x \in J$ such that x is simultaneously of minimal distance to $B^1, B^2, \ldots, B^r$.*

For v an element of $\mathbf{R}^S$, define $\bar{v} \in \{-1,0,1\}^S$ by

$$\bar{v}_i = \left\{ \begin{array}{rl} 1 & v_i > 0 \\ 0 & v_i = 0 \\ -1 & v_i < 0 \end{array} \right\}.$$

We will first consider the polytope determined by a jump system, and in particular the faces of this polytope. Then, we will proceed to properties specific to two-dimensional jump systems, including a characterization. We will then consider properties for higher-dimensional jump systems. We will conclude with analysis of the constant-sum jump systems, which are equivalent to faces of jump systems.

2. Faces and polytopes. In this section we lay the groundwork for the remainder. We present generalizations, corollaries, and elementary proofs of some known results; we fix notation, and we give several illustrative examples.

Let $V = \{v : v \in \{-1, 0, 1\}^S, v \neq 0)\}$, and $V' = \{v : v \in \mathbf{R}^S, v \neq 0\}$. If $v \in V'$ (and, in particular, if $v \in V$), we set $\omega_v = \sup_{x \in J} v^T x$. We call $\mathfrak{f}_v = \{x : x \in J, v^T x = \omega_v\}$ a *face* of J. For finite jump systems, each such $\mathfrak{f}_v$ is nonempty. Lovász has shown in [**4**] that $\mathfrak{f}_v$ is, in turn, a jump system.

Example 2.1. Let $J = \{(2,3),(3,2),(3,3),(5,2),(5,3)\}$. Then $\omega_{(1,0)} = 5$ and $\mathfrak{f}_{(1,0)} = \{(5,2),(5,3)\}$. $\omega_{(-1,-1)} = -5$ and $\mathfrak{f}_{(-1,-1)} = \{(2,3),(3,2)\}$.

The following useful result is a simple corollary of Theorem 1.1.

Theorem 2.1. *Let $v_1, v_2, \ldots, v_r \in V'$ have disjoint support. Then $\mathfrak{f}_{v_1} \cap \mathfrak{f}_{v_1+v_2} \cap \cdots \cap \mathfrak{f}_{v_1+v_2+\cdots+v_r} \neq \varnothing$.*

Proof. Let $M = \max_{z \in J, i \in S} |z_i|$. For $1 \leq j \leq r$, set $w^j = v_1 + v_2 + \cdots + v_j$. Let B^j be defined as

$$B_i^j = \left\{ \begin{array}{cl} (-\infty, -M] & \text{if } w_i^j = -1, \\ (-\infty, \infty) & \text{if } w_i^j = 0, \\ [M, \infty) & \text{if } w_i^j = 1. \end{array} \right\}.$$

Clearly $B^r \subseteq \cdots \subseteq B^2 \subseteq B^1$. $\square$

Every point x of the jump system satisfies $v^T x \leq \omega_v$ for every $v \in V'$. The set of all such points in $\mathbf{Z}^S$ (not necessarily in J) we call the *polytope associated with* J, denoted P_J or P.

We call $\{x : x \in P, v^T x = \omega_v\}$ a *face* of P. We call those points *surface* points, while the other points of P are called *interior* points. We call points in $P \setminus J$ *gaps*.

The following two results show that V' and V induce equivalent geometry. This was first shown in [**1**] using bisubmodular polyhedra; we present elementary proofs. The first result, concerning faces, is actually a bit stronger, and generalizes a result in [**4**].

Theorem 2.2. *Let $v' \in V'$. Then $\mathfrak{f}_{v'} \subseteq \mathfrak{f}_{\bar{v}'}$.*

Proof. By reindexing and reflecting if necessary, we may assume without loss of generality that $v'_1 \geq v'_2 \geq \cdots \geq v'_{|S|} \geq 0$. Let m be between 1 and $|S|$ such that $v'_m > 0$, but $v'_{m+1} = 0$. Set $v = (1, \ldots, 1, 0, \ldots, 0)$ where the first m coordinates are 1, and the remaining $|S| - m$ coordinates are 0.

Suppose $x \in \mathfrak{f}_{v'} \setminus \mathfrak{f}_v$. Let $y \in \mathfrak{f}_v$ be such that $d(x, y)$ is minimal. Since $v^T y > v^T x$, there must exist $i \leq m$ for which $y_i > x_i$. Now, since ${v'}^T x \geq {v'}^T y$, there must exist $j \leq m$ for which $x_j > y_j$.

Consider the step $y \xrightarrow{x} y + e_j$. Because $v^T(y + e_j) > v^T y = \omega_v$, we know that $y + e_j \notin J$. Therefore Axiom 1.1 states that there exists a second step $y + e_j \xrightarrow{x} y + e_j + s \in J$. Because s is a step and $v^T y + 1 + v^T s = v^T(y + e_j + s) \leq \omega_v = v^T y$, we must have $s = -e_k$ for some $k \leq m$. But then $y + e_j + s \in \mathfrak{f}_v$ and $d(y + e_j + s, x) < d(y, x)$, which violates the minimal choice of y. $\square$

If P is the polytope induced by V', consider $\overline{P}$, the polytope analogously induced by V. It is obvious that $P \subseteq \overline{P}$. The following result shows that, in fact, $\overline{P} = P$.

Theorem 2.3. *Let x be a surface point of P. Then x is a surface point of $\overline{P}$.*

Proof. The result holds for x in the jump system by Theorem 2.2. Suppose it did not hold for all surface points in the polytope. Let $x \in P$ be such a surface point. Let $v' \in V'$ be such that ${v'}^T x = \omega_{v'}$. By reindexing and reflecting if necessary, we may assume without loss of generality that $v'_1 \geq v'_2 \geq \cdots \geq v'_{|S|} \geq 0$. Finally, by translation, we can assume that $x = (0, 0, \ldots, 0)$, and therefore that $\omega_{v'} = 0$. However, by our assumption on x, we must have $\omega_v > 0 = v^T x$ for each $v \in V$.

We can now write $v' = \lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k$, where $\lambda_i > 0$ and each u_i is a 0-1 vector of the form $u_i = e_1 + e_2 + \cdots + e_{t_i}$, for some $1 \leq t_1 < t_2 < \cdots < t_k$. Observe that each $u_i \in V$, and hence that each $\omega_{u_i} > 0$.

Let $M = \max_{z \in J, i \in S} z_i$. Consider the boxes $B^i = [M, \infty) \times \cdots \times [M, \infty) \times \mathbf{Z}^{|S|-t_i}$. Because $B^1 \subseteq B^2 \subseteq \cdots \subseteq B^k$, we can apply Theorem 1.1, which gives us some y contained in $\mathfrak{f}_{u_1} \cap \mathfrak{f}_{u_2} \cap \cdots \cap \mathfrak{f}_{u_k}$. Now, observe that ${v'}^T y = (\lambda_1 u_1 + \lambda_2 u_2 + \cdots + \lambda_k u_k)^T y = \lambda_1 \omega_{u_1} + \lambda_2 \omega_{u_2} + \cdots + \lambda_k \omega_{u_k} > 0 = \omega_{v'}$. This is impossible, and therefore no such x could have existed. □

The following result shows that two "similar" points in a face force a variety of other points to be in the face as well.

Theorem 2.4. *Let J be a jump system, and let $v \in V$. Suppose a and b are in $\mathfrak{f}_v$. Let $T = \{i \mid v_i(a_i - b_i) \neq 0\}$. Suppose that T contains only two coordinates, α and β. Then $a' = (a_1, a_2, \ldots, a_{\alpha-1}, b_\alpha, a_{\alpha+1}, \ldots, a_{\beta-1}, b_\beta, a_{\beta+1}, \ldots, a_{|S|})$ is in $\mathfrak{f}_v$, as well as every lattice point between a and a'.*

Proof. By reindexing, reflection, and translation, we may assume without loss of generality that for some $m > 0$, $1 = v_1 = \cdots = v_m$, $0 = v_{m+1} = \cdots = v_{|S|}$, that $T = \{1, 2\}$, that $a = (0, 0, \ldots, 0)$, and that $b = (b_1, -b_1, 0, \ldots, 0, b_{m+1}, \ldots, b_n)$ where $b_1 > 0$.

It is enough to prove that $(1, -1, 0, \ldots, 0) \in J$ because then we can recursively set $a = (1, -1, 0, \ldots, 0)$ and prove that $(2, -2, 0, \ldots, 0) \in \mathfrak{f}_v$, and so on. Consider the step $a \xrightarrow{b} (1, 0, \ldots, 0)$. We see that $(1, 0, \ldots, 0) \notin J$ because $v^T(1, 0, \ldots, 0) > v^T a$. By Axiom 1.1, we can take a second step that will get us back into J. The only possible second step is $(1, 0, \ldots, 0) \xrightarrow{b} (1, -1, 0, \ldots, 0)$, because no step is possible in the direction of b between the third and mth coordinates, and a step in any of the last $|S| - m$ coordinates will take us out of $\mathfrak{f}_v$. Therefore, $(1, -1, 0, \ldots, 0) \in J$. □

Example 2.2. Let $v = (1, 1, 1, 0)$. Suppose that both $(3, 5, 6, 7)$ and $(0, 5, 9, 10)$ are in $\mathfrak{f}_v$. Then $(0, 5, 9, 7), (1, 5, 8, 7)$, and $(2, 5, 7, 7)$ are also in $\mathfrak{f}_v$. Furthermore, so are $(3, 5, 6, 10), (2, 5, 7, 10)$, and $(1, 5, 8, 10)$.

This last corollary will be used in our characterization of two-dimensional jump systems. A more general, unpublished, result of

Sebő appears in Geelen's unpublished lecture notes [**2**]. This elementary result, however, is all we need in the sequel.

Corollary 2.1. *If $J \subseteq \mathbf{Z}^2$ is a jump system and $a, b \in \mathfrak{f}_{(\pm 1, \pm 1)}$, then all the points between a and b are also in the jump system.*

3. Two-dimensional properties and characterization. One-dimensional jump systems are easily characterized: there can be no two adjacent gaps. Two-dimensional jump systems are more difficult. In this section we provide such a characterization, together with a theorem about gaps. In [**2**] it is shown that any gap must lie on a line segment between some two points of the jump system. We provide an elementary proof of this result for two-dimensional jump systems.

We start by giving the still weaker result in the special case where the gap is a surface gap. This is needed in our characterization of two-dimensional jump systems.

Lemma 3.1. *Let $J \subseteq \mathbf{Z}^2$ be a jump system and a be a surface gap. Then there exist points $x, y \in J$ (both on the same face of P as a) such that a lies on the line segment connecting x and y.*

Proof. By translation we will assume without loss of generality that $a = (0, 0)$. By hypothesis, let $v \in V$ be such that $v^T a = \omega_v = 0$. Let $x \in \mathfrak{f}_v$. There are two candidates $w \in V$ such that $|w - v| = 1$, and a simple calculation shows that for at least one of them, $x \notin \mathfrak{f}_w$. Now, by Theorem 2.1, there must be some $y \in \mathfrak{f}_v \cap \mathfrak{f}_w$. Observe that x and y are on the same line (the face of P), and a is between them. $\square$

We are now ready to present a characterization of two-dimensional jump systems.

Theorem 3.1. *Let $J \subseteq \mathbf{Z}^2$. Then J is a jump system if and only if*

1. *Each face of J is a jump system, and*

2a. *For any two adjacent gaps x', x'' in P, all points on the line containing them are not in J.*

(*or, equivalently*)

2b. *The following configuration is forbidden:* $x \in J$, x' *and* x'' *gaps, with* $x + 2\alpha = x' + \alpha = x''$ *for some* $|\alpha| = 1$.

Proof. ($\Rightarrow$). Assume that J is a jump system. The first condition is obviously true.

We will show the second condition by way of contradiction. By reindexing, reflection, and translation, we may assume that $x = (0,0) \in J$, $x' = (1,0) \in P \setminus J$, $x'' = (2,0) \in P \setminus J$. This implies that there are no points in J of the form $(k,0)$ where $k > 0$.

Observe that x' cannot be on a face of P. If it were, the face would have to be $(0,\pm 1)$ to allow x, x'' to be in P, and then we get an immediate contradiction from Lemma 3.1.

We therefore assume that x' is an interior point. Since our jump system is finite, there must be some surface gap $y = (m,0)$ with $m > 1$. If y is on the $(1,\pm 1)$ face, then we immediately get a contradiction from Lemma 3.1 and Corollary 2.1. Otherwise, y is on the $(1,0)$ face. We must have $(m,1) \in J$ by Lemma 3.1 and the one-dimensional classification. However, $(m,1), x$, and y violate Axiom 1.1 since the step $(m,1) \xrightarrow{x} y$ will not have a second valid step toward x.

($\Leftarrow$). Assume (1) and (2). We need to show Axiom 1.1 holds for any $x, y \in J$. By reflection and translation, we can assume that $x = (0,0)$ and $y = (p,q)$ where $p, q \geq 0$. Furthermore, we can intersect J with a box to assume that all points in J have nonnegative coordinates. That is, we assume that $x = \mathfrak{f}_{(-1,-1)} \cap \mathfrak{f}_{(-1,0)} \cap \mathfrak{f}_{(0,-1)}$.

We will first show Axiom 1.1 if $pq = 0$. Suppose that $q = 0$. $(1,0)$ and $(2,0)$ are both in P, and hence by (2) cannot both be gaps. Axiom 1.1 follows. The case $p = 0$ follows by symmetry.

We may assume, by reindexing if necessary, that the given first step is $x \xrightarrow{y} (1,0)$. Suppose that $(1,0) \notin J$. We will show that either $(2,0)$ or $(1,1)$ is in J, proving Axiom 1.1.

Apply Theorem 2.1 to get $z \in \mathfrak{f}_{(0,-1)} \cap \mathfrak{f}_{(1,-1)}$. If $z \neq x$, then $z = (m,0)$ and therefore $(1,0), (2,0)$ are both in P. Hence $(2,0)$ cannot be a gap by (2). On the other hand, if $z = x$, then $\omega_{(1,-1)} = 0$, and therefore $(1,1) \in \mathfrak{f}_{(1,-1)}$. By Lemma 3.1 and Corollary 2.1 we must have $(1,1) \in J$. $\square$

The following strengthens Lemma 3.1 to all gaps in two-dimensional jump systems.

Theorem 3.2. *Let $J \subseteq \mathbf{Z}^2$ be a jump system and a a gap. Then there exist points $x, y \in J$ such that a lies on the line segment connecting x and y.*

Proof. Without loss of generality, we assume that $a = (0,0)$. We further assume that a is an interior gap, or else Lemma 3.1 would apply.

Consider $(1,0)$ and $(-1,0)$. If both are in J, the theorem follows. If either is a gap, however, then all $(k,0)$ must be gaps for $k \in \mathbf{Z}$. Similarly we can assume that all $(0,k)$ are gaps. By Theorem 2.1, let $c^1 \in \mathfrak{f}_{(-1,-1)} \cap \mathfrak{f}_{(-1,0)}, c^2 \in \mathfrak{f}_{(-1,-1)} \cap \mathfrak{f}_{(0,-1)}$. Since a is an interior point, we must have $c_1^1 < 0$, $c_2^2 < 0$. By Corollary 2.1, we know that all points between c^1 and c^2 are in J. Therefore, we can choose some $c \in J$ with $c_1 < 0$, $c_2 < 0$.

We will now show that $(1,1)$ is in J. Suppose otherwise; let $b = (p,q) \in J$ have $p > 0$, $q > 0$ with $p+q$ minimal. By reindexing if necessary, assume that $p > 1$. Consider $b \xrightarrow{c} (p-1,q)$. This new point must not be in J by the minimality of b. Hence, by Axiom 1.1 we must have a second step $(p-1,q) \xrightarrow{c} d$, with $d \in J$. But then either d violates the minimality of b, or else has a coordinate equal to zero, which is also forbidden. This contradiction shows that $(1,1)$ is in J.

By a symmetric argument, we must have $(-1,-1)$ in J. And now the theorem follows. $\square$

4. Geometry of higher-dimensional jump systems. In this section we include several additional geometric results. The configuration of Theorem 3.1, while no longer forbidden, imposes a variety of restrictions on J, particularly for three-dimensional jump systems. We also include another forbidden configuration (that, unfortunately, does not characterize higher-dimensional jump systems). But first, we have the following result, that each hyperquadrant relative to a gap must contain some point of J.

Theorem 4.1. *Let x be a gap. Let $v \in \{-1,1\}^{|S|}$. Then $\{y \in J : v_i(y_i - x_i) \geq 0\}$ is nonempty.*

Proof. By translation, we can assume without loss of generality that x is the origin. By reflection, we may assume without loss of generality that $v = (1, 1, \dots, 1)$. For convenience, for each $T \subseteq \{1, 2, \dots, |S|\}$, we define the set $N_T = \{y \in J : \sum_{i \in T} y_i \geq \sum_{i \in T} x_i\}$. The theorem follows if we can show that $N_1 \cap N_2 \cap \cdots \cap N_{|S|}$ is nonempty.

We will show this in $|S|$ steps. Each step will allow for any permutation $i_1, i_2, \dots i_{|S|}$ of $1, 2, \dots, |S|$. The first step is to show that $N_{i_1} \cap N_{i_1 i_2} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}}$ is nonempty. This follows immediately from Theorem 2.1, as $f_{e_{i_1}} \cap f_{e_{i_1}+e_{i_2}} \cap \cdots \cap f_{e_{i_1}+e_{i_2}+\cdots+e_{i_{|S|}}} \subseteq N_{i_1} \cap N_{i_1 i_2} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}}$. We say that this step admits one coordinate, as there is one term N_{i_1} with just one coordinate.

We now assume that we have completed step k, for $1 \leq k \leq |S| - 1$. That step admits k coordinates: $N_{i_1} \cap N_{i_2} \cap \cdots \cap N_{i_k} \cap N_{i_1 i_2 \cdots i_k i_{k+1}} \cap N_{i_1 i_2 \cdots i_k i_{k+1} i_{k+2}} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}} \neq \varnothing$. It suffices to show that we can admit $k+1$ coordinates: $N_{i_1} \cap N_{i_2} \cap \cdots \cap N_{i_k} \cap N_{i_{k+1}} \cap N_{i_1 i_2 \cdots i_k i_{k+1} i_{k+2}} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}} \neq \varnothing$.

Choose $z \in N_{i_1} \cap N_{i_2} \cap \cdots \cap N_{i_k} \cap N_{i_1 i_2 \cdots i_k i_{k+1}} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}}$ with $z_{i_{k+1}}$ maximal. If $z_{i_{k+1}} \geq 0$ this $(k+1)^{\text{th}}$ step is complete, so assume otherwise. Because $z \in N_{i_1 i_2 \cdots i_k i_{k+1}}$, we have $z_{i_1} + z_{i_2} + \cdots + z_{i_{k+1}} \geq 0$. But $z_{i_{k+1}} < 0$, so for some other coordinate (say i_1), $z_{i_1} > 0$. Now, choose $y \in N_{i_2} \cap N_{i_3} \cap \cdots \cap N_{i_{k+1}} \cap N_{i_1 i_2 \cdots i_k i_{k+1}} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}}$. Because $z_{i_{k+1}} < 0 \leq y_{i_{k+1}}$, we have $z \xrightarrow{y} z + e_{i_{k+1}}$. By the maximal choice of z, we must have $z + e_{i_{k+1}} \notin J$. So, by Axiom 1.1, we must have $z + e_{i_{k+1}} \xrightarrow{y} z + e_{i_{k+1}} + \alpha$. But, again by the maximal choice of z, we must have $z + e_{i_{k+1}} + \alpha \notin N_{i_1} \cap N_{i_2} \cap \cdots \cap N_{i_k} \cap N_{i_1 i_2 \cdots i_k i_{k+1}} \cap \cdots \cap N_{i_1 i_2 \cdots i_{|S|}}$. Since this last step was in the direction of y, we must have $z + e_{i_{k+1}} + \alpha \notin N_{i_1}$. That is, $\alpha = -e_{i_1}$ and $z_{i_1} = 0$. But this is a contradiction since $z_{i_1} > 0$. $\square$

If the dimension of J is greater than two, the configuration of Theorem 3.1 is no longer forbidden. However, it does impose some conditions on J, as the following two results demonstrate. The first shows that the configuration prohibits a variety of points from being in J.

Theorem 4.2. *Let $x \in J$, x' and x'' gaps, and $x+2\alpha = x'+\alpha = x''$ for some $|\alpha| = 1$. For any $y \in J$, decompose y as $y = x + k_y\alpha + \hat{y}$, for $\hat{y} \cdot \alpha = 0$. Then, we must have $|\hat{y}| \geq k_y$.*

Proof. Suppose otherwise. Choose some $y = x+k_y\alpha+\hat{y}$ with $|\hat{y}| < k_y$ and k_y minimal. $\hat{y}$ cannot be 0, since then $y \xrightarrow{x} y - \alpha$ would violate Axiom 1.1. Therefore, there must be some step y' from y toward x in some coordinate not corresponding to α. That is, $y \xrightarrow{x} y'$. But we have $|\hat{y}'| < |\hat{y}| < k_y = k_{y'}$. By the minimal choice of y, we must have $y' \notin J$. Now, by applying Axiom 1.1, we get $y'' \in J$, with $y' \xrightarrow{x} y''$. By the minimal choice of y, we must have $|\hat{y}''| \geq k_{y''}$. However, this is a contradiction, since either $k_{y''} = k_y - 1 > |\hat{y}| - 1 = |\hat{y}''|$ or $k_{y''} = k_y > |\hat{y}| \geq |\hat{y}''|$. $\square$

For three-dimensional jump systems, this configuration actually forces quite a bit more.

Theorem 4.3. *Let $J \subseteq \mathbf{Z}^3$. Let $x \in J$, x' and x'' gaps, and $x+2e_3 = x'+e_3 = x''$. Then the eight points in $\{x'' \pm e_1 \pm e_2; x' \pm e_1; x' \pm e_2\}$ are all in J.*

Proof. By translation, we may assume without loss of generality that x is the origin. By reflection, the theorem will follow if we can show that the three points $x'' + e_1 + e_2, x' + e_1, x' + e_2$ are all in J. By Theorem 4.1, there must be some $a \in J$ with $a_1 \geq 0, a_2 \geq 0, a_3 \geq 2$. Now, consider $x \xrightarrow{a} x'$. By Axiom 1.1, there must be a second step $x' \xrightarrow{a} b$. We must have $b \in J$, with $b = x' + e_1$ or $b = x' + e_2$. By reindexing if necessary, we may assume without loss of generality that $b = x' + e_1$.

By Theorem 4.1, there must be some $c \in J$ with $c_1 \leq 0,\ c_2 \geq 0,\ c_3 \geq 2$. Consider $b \xrightarrow{c} x'' + e_1$. This is not in J by Theorem 4.2. Thus, by Axiom 1.1, there must be a second step $x'' + e_1 \xrightarrow{a} d$. We must have $d \in J$, with $d = x'' + e_1 + e_3, d = x''$, or $d = x'' + e_1 + e_2$. The first is impossible by Theorem 4.2, and the second is impossible by the hypotheses. Hence we must have $d = x'' + e_1 + e_2 \in J$.

Finally, consider $d \xrightarrow{x} x'' + e_2$. By Theorem 4.2, this is not in J. Thus, by Axiom 1.1, there must be a second step $x'' + e_2 \xrightarrow{x} e$. We must therefore have $e \in J$, with $e = x' + e_2$. $\square$

Our final result of this section concerns rifts. For $v \in V$ and $b \in \mathbf{Z}$, the set of points $\{x : v^T x = b\}$ is called a *rift* $R(v, b)$ whenever none of those points is in J. We say that J *admits* $R(v, b)$. The result states that if J admits two adjacent rifts, it must be entirely on one side or the other of the rifts.

Theorem 4.4. *Suppose J admits both $R(v, b)$ and $R(v, b+1)$. Then, either $\omega_v < b$, or $\omega_{-v} < -(b + 1)$.*

Proof. Let $x, y \in J$ be such that $v^T x < b$, $v^T y > b + 1$ is chosen so that $|x - y|$ is minimal. Consider any step $x \xrightarrow{y} z$. If $v^T z = b$, then z is in the rift $R(v, b)$ and hence not in J. If $v^T z < b$, then by the minimal choice of x we again have that $z \notin J$. Hence, by Axiom 1.1, there is some step $z \xrightarrow{y} w$ with $w \in J$. Since $v^T z \leq b$, $v^T w \leq b + 1$. But we must therefore have $v^T w < b$, because of the two rifts. And now, $w, y \in J$ violate the minimality of x, y. $\square$

5. Constant-sum jump systems. We now turn our attention to the special case where, for some $v \in V$, we have $J = f_v$. The result of this section is a characterization of these constant-sum jump systems in terms of an operation we call strong reduction.

Let J be a collection of points, let $T \subseteq S$ with $|T| \geq 2$, and let $\alpha \in \{-1, 0, 1\}^T$. Then the *strong reduction* $J\,[\alpha \cdot T]$ is defined by $J\,[\alpha \cdot T] = \{x' \in \mathbf{Z}^{(S \setminus T) \dot{\cup} 0} : x \in J,\ x'_0 = \sum_{i \in T} \alpha_i x_i,\ x'_j = x_j$ (for $j \notin T)\}$. Observe that the operation is equivalent to reflection followed by projection followed by reduction.

Example 5.1. Let $J = \{(1,1,1),(2,1,1),(1,2,1),(2,2,1),(1,1,2),(2,2,2)\}$. This is a jump system. $J[x_1 + 0x_2] = \{(1,1),(2,1),(1,2),(2,2)\}$, $J[x_1 + x_2] = \{(2,1),(3,1),(4,1),(2,2),(4,2)\}$, $J[x_1 - x_2] = \{(-1,1),(0,1),(1,1),(0,2)\}$, $J[0x_1 - x_2 - x_3] = \{(-4),(-3),(-2)\}$.

Because all of its constituent operations preserve Axiom 1.1, strong reduction does as well. Our final result is a partial converse to this fact, restricted to constant-sum jump systems. Unfortunately, it cannot be generalized to arbitrary jump systems, as $J = \{(0,0),(2,0),(2,2)\}$ is a collection of points that does not satisfy Axiom 1.1, but every strong reduction of which does.

Theorem 5.1. *Let J be a collection of points, with $v \in V$ such that $v^T x$ is constant for all $x \in J$. Then J is a jump system if and only if every strong reduction of J is a jump system.*

Proof. Suppose that, in violation of Axiom 1.1, there are points $a, b \in J$ and a step $a \xrightarrow{b} s$ with $s \notin J$, and no second step from s to b in J. By translation, we may assume that a is the origin (and hence the origin is in any strong reduction of J), and hence that $\omega_v = 0$. By reindexing, we may assume that $s = \pm e_1 = \bar{b}_1$. By reindexing the remaining coordinates, we may assume that all coordinates after the mth are zero, for some $1 \le m \le |S|$. Finally, by reflection, we may assume that either $v = (1,1,\ldots,1,0,\ldots,0)$, or $v = (0,1,\ldots,1,0,\ldots,0)$. These two cases will be treated separately. In both, consider $J' = J\,[x_1 + 0x_2]$, a jump system by hypothesis. Set $e = (\bar{b}_1, 0, \ldots, 0)$. This is a step from the origin toward $(b_1, b_3, \ldots, b_n) \in J'$. In both cases, we will show that $e \notin J'$, and hence by Axiom 1.1 we must have $e \xrightarrow{(b_1,b_3,\ldots,b_n)} f$, with $f = e + \bar{b}_j \in J'$ for some $1 \le j \le |S|$. We will then show that this leads to a contradiction, for any possible j.

Case 1. Since $v^T b = 0$, there must be some coordinate k between 2 and m with $\bar{b}_k = -\bar{b}_1$. Without loss of generality, we may reindex and assume that $k = 2$. Now, if $e \in J'$, then $(\bar{b}_1, \alpha, 0, \ldots, 0) \in J$ for some α, hence by the constant-sum property we have $\alpha = -\bar{b}_1$. But then this is a step in J from s toward b, which by assumption we cannot have. Hence $e \notin J'$, and so we get $f = e + \bar{b}_j \in J'$.

First, we will consider the cases where $j = 1$ or $j > m$. By the constant-sum property, this implies that $c = (2\bar{b}_1, 2\bar{b}_2, 0, \ldots, 0) \in J$ (respectively, that $c = (\bar{b}_1, \bar{b}_2, 0, \ldots, \bar{b}_j, \ldots, 0) \in J$). Consider $J'' = J[x_1 - x_2]$, a jump system by hypothesis. If $e \in J''$, then $(\alpha, \beta, 0, \ldots, 0) \in J$ for some $\alpha - \beta = \bar{b}_1$. By the constant-sum property, we also know that $\alpha + \beta = 0$. These two equations cannot be satisfied by integers, and therefore $e \notin J''$. If $2e \in J''$, then $(\alpha, \beta, 0, \ldots, 0) \in J$ for some $\alpha - \beta = 2\bar{b}_1$. By the constant-sum property, we also know that $\alpha + \beta = 0$. There is one solution to these equations—that $(\bar{b}_1, -\bar{b}_1, 0, \ldots, 0) \in J$; however, this is a step from s toward b, which by assumption is disallowed. Hence $2e \notin J''$. Finally, if $j > m$ and $f \in J''$, then $(\alpha, \beta, \ldots, 0, \bar{b}_j, 0, \ldots, 0) \in J$, for some $\alpha - \beta = \bar{b}_1$. By the constant-sum property, we also have that $\alpha + \beta = 0$. These two equations cannot be satisfied by integers, and therefore $f \notin J''$. Now, because $c \in J$, we have $c' = (4\bar{b}_1, 0, \ldots, 0) \in J''$ (respectively, $c' = (2\bar{b}_1, 0, \ldots, \bar{b}_j, \ldots, 0) \in J''$). So, we have a step $0 \xrightarrow{c'} e$, with $e \notin J''$. By Axiom 1.1, we must have a second step from e toward c' in J''. However, the only possibilities are $2e$ and f, and we have shown that neither can be in J''.

Now, we will consider the cases where $3 \leq j \leq m$. Without loss of generality, we will reindex and assume that $j = 3$. By the constant-sum property, and our assumption, we must have that $c = (\bar{b}_1, 2\bar{b}_2, \bar{b}_1, 0, \ldots, 0) \in J$. Consider $J''' = J[x_3 - x_2]$, a jump system by hypothesis. Set $e' = (0, \bar{b}_1, 0, \ldots, 0)$, $e'' = (\bar{b}_1, \bar{b}_1, 0, \ldots, 0)$. If $e' \in J'''$, then $(\bar{b}_1, \alpha, \beta, 0, \ldots, 0) \in J$, for some $\beta - \alpha = 0$. By the constant-sum property, we also know that $\bar{b}_1 + \alpha + \beta = 0$. These two equations cannot be satisfied by integers, and therefore $e' \notin J'''$. If $e'' \in J$, then $(\bar{b}_1, \alpha, \beta, 0, \ldots, 0) \in J$, for some $\beta - \alpha = \bar{b}_1$. By the constant-sum property, we also know that $\bar{b}_1 + \alpha + \beta = 0$. There is one solution to these equations–that $(\bar{b}_1, -\bar{b}_1, 0, \ldots, 0) \in J$; however, this is a step from s toward b, which by assumption is disallowed. Hence $e'' \notin J'''$. Now, because $c \in J$, we have $c' = (3\bar{b}_1, \bar{b}_1, 0, \ldots, 0) \in J'''$. So, we have a step $0 \xrightarrow{c'} e'$, with $e' \notin J'''$. By Axiom 1.1, we must have some second step in J''' from e' toward c. But the only possible second step is e'', which we have shown cannot be in J'''.

Case 2. If $e \in J'$, then by the constant sum property, $(\bar{b}_1, 0, \ldots, 0) = s \in J$, which violates the hypothesis. So $e \notin J'$, and we get $f = e + \bar{b}_j \in J'$. If $j = 1$ or $j > m$, then $(2\bar{b}_1, 0, \ldots, 0) \in J$ (respectively, $(\bar{b}_1, 0, \ldots, \bar{b}_j, \ldots, 0) \in J$), which is a step from s toward b, violating the hypothesis. By reindexing, we now assume without loss of generality that $j = 3 \leq m$. Since $f \in J'$, we must have $c = (\bar{b}_1, -\bar{b}_3, \bar{b}_3, 0, \ldots, 0) \in J$.

We now consider $J'' = J[x_3 - x_2]$, a jump system by hypothesis. Since $c \in J$, we must have $c' = (2\bar{b}_3, \bar{b}_1, 0, \ldots, 0) \in J''$. Consider $(0, \bar{b}_1, 0 \ldots, 0)$. This is not in J'' since $s \notin J$. However, it is a step from the origin toward c'. Hence, by Axiom 1.1, we must have a second step toward c'. This can only be $(\bar{b}_3, \bar{b}_1, 0, \ldots, 0) \in J''$. But then $(\bar{b}_1, \alpha, \beta, 0, \ldots, 0) \in J$, for some $\beta - \alpha = \bar{b}_3$. But, by the constant-sum property, we have $\beta + \alpha = 0$. These two equations cannot be satisfied by integers, and hence J'' cannot be a jump system, in violation of the hypothesis. $\square$

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