ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 36, Number 5, 2006

GENERAL NONDIAGONAL CUBIC HERMITE-PADÉ APPROXIMATION TO THE EXPONENTIAL FUNCTION

CHENG-DE ZHENG AND REN-HONG WANG

ABSTRACT. General nondiagonal cubic Hermite-Padé approximation to the exponential function with coefficient polynomials of degree at most n, m, s, l, respectively is considered. Explicit formulas and differential equations are obtained for the coefficient polynomials. An exact asymptotic expression is obtained for the error function and it is also shown that these generalized Padé-type approximations can be used to asymptotically minimize the expressions on the unit disk.

1. Introduction. We consider approximations of e^{-x} generated by finding polynomials $P_n,\ Q_m,\ R_s$ and S_l so that

$$\begin{aligned} \mathbf{E}_{nmsl}(x) &:= P_n(x)e^{-3x} + Q_m(x)e^{-2x} + R_s(x)e^{-x} + S_l(x) \\ &= O(x^{n+m+s+l+3}), \end{aligned} \tag{1.1}$$

with P_n, Q_m, R_s, S_l being algebraic polynomials of degree at most n, m, s, l, respectively, and P_n has leading coefficient 1. The approximation of e^{-x} is given by

$$\delta_{nmsl}(x) := \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} + \sqrt[3]{-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} - \frac{Q_m}{3P_n},$$

2000 AMS *Mathematics Subject Classification.* Primary 41A21.
Key words and phrases. Padé-type approximant, cubic Hermite-Padé approximation, asymptotic formula, differential equation.
Supported by the Natl. Natural Sci. Foundation of China (No. 69973010 and 10271022) and the Natural Sci. Foundation of Guangdong Province, China (No. 021755).
Received by the editors on Jan. 28, 2003, and in revised form on July 7, 2003.

or

$$\delta_{nmsl}(x) := \omega_1 \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} + \omega_2 \sqrt[3]{-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} - \frac{Q_m}{3P_n},$$

or

$$\delta_{nmsl}(x) := \omega_2 \sqrt[3]{-\frac{b}{2} + \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} + \omega_1 \sqrt[3]{-\frac{b}{2} - \sqrt{\left(\frac{b}{2}\right)^2 + \left(\frac{a}{3}\right)^3}} - \frac{Q_m}{3P_n},$$

which is real and closest to 0, where

$$a = \frac{3P_nR_s - Q_m^2}{3P_n^2}, \qquad b = \frac{2Q_m^3 - 9P_nQ_mR_s + 27P_n^2S_l}{27P_n^3};$$

$$\omega_1 = \frac{-1+\sqrt{3}\,i}{2}, \qquad \omega_2 = \frac{-1-\sqrt{3}\,i}{2}.$$

Obviously, $\delta_{nmsl}(x)$ is the natural cubic generalization of the Padé approximant $-\widehat{Q}_m/\widehat{P}_n$ satisfying

$$\widehat{P}_n(x)e^{-x} + \widehat{Q}_m(x) = O(x^{n+m+1}) \tag{1.2}$$

and the nondiagonal quadratic Hermite-Padé approximant [**3**]

$$\Big(-\widetilde{Q}_m + \sqrt{\widetilde{Q}_m^2 - 4\widetilde{P}_n\widetilde{R}_s}\,\Big)\Big/\big(2\widetilde{P}_n\big)$$

satisfying

$$\widetilde{P}_n(x)e^{-2x} + \widetilde{Q}_m(x)e^{-x} + \widetilde{R}_s(x) = O(x^{n+m+s+2}). \tag{1.3}$$

Our primary aim is to derive the exact asymptotic formula for $\{\mathbf{E}_{nmsl}\}$, the explicit formulae of $\{P_n\}$, $\{Q_m\}$, $\{R_s\}$, $\{S_l\}$, $\{\mathbf{E}_{nmsl}\}$

and to treat some minimization problems concerning related approximations on the unit disk in $\mathbf{C}$.

Hermite considered the expression of the form

$$T_r(x)e^{\alpha_r x} + T_{r-1}(x)e^{\alpha_{r-1}x} + \cdots + T_1(x)e^{\alpha_1 x} = O(x^q), \tag{1.4}$$

where $T_1, T_2, \dots, T_r$ are chosen polynomials so that q is as large as possible [**4**]. Included, of course, the expression of type (1.4) are both the Padé approximations (1.2), the nondiagonal quadratic Hermite-Padé approximations (1.3) and the general nondiagonal cubic Hermite-Padé approximations (1.1). Since we do not treat approximations of the generality of (1.4), we are able to provide an exact asymptotic formula of $\{\mathbf{E}_{nmsl}\}$ and the explicit formulae for $\{P_n\}$, $\{Q_m\}$, $\{R_s\}$, $\{S_l\}$, $\{\mathbf{E}_{nmsl}\}$ rather than the estimates included in [**4**].

The general problem of Hermite-Padé approximation is the following:

Given functions $f_1, f_2, \dots, f_r$, and integers $\tilde{d}_0, \tilde{d}_1, \dots, \tilde{d}_r$, find polynomials $\tilde{p}_0, \tilde{p}_1, \dots, \tilde{p}_r$ $(\deg(\tilde{p}_i) \le \tilde{d}_i)$ so that

$$\tilde{p}_0(x) + \tilde{p}_1(x)\, f_1(x) + \cdots + \tilde{p}_r(x)\, f_r(x) = O(x^q), \tag{1.5}$$

where q is as large as possible. Particularly some interesting cases arise where the f_i are related, for example, f_i is the ith power (or ith derivative) of a fixed f. The case of $i = 1$ is a Padé case and the case of $i = 2$ is a quadratic function approximation.

Exact results concerning best rational approximation to the exponential function, particularly the Meinardus conjecture, have attracted much attention [**2**]. Theorem 4 can be viewed as a cubic version of this conjecture on the disk. A linear version, due to Trefethen appeared in [**6**]; a quadratic version on the disk given by Driver can be found in [**3**]. Virtually no completely worked out examples for higher-dimensional approximations exist, and exp is a natural candidate for such a complete analysis. This analysis constitutes the thrust of this paper.

2. Explicit formulae. We derive the cubic approximations from the Padé approximation as follows.

It is well known that if we set

$$\widehat{P}_n(x) := \frac{1}{m!}\int_0^\infty t^m(t+x)^n e^{-t}\,dt = n!\sum_{k=0}^{n}\binom{n+m-k}{m}\frac{x^k}{k!},$$
$$\widehat{Q}_m(x) := -\frac{1}{m!}\int_0^\infty t^n(t-x)^m e^{-t}dt = -n!\sum_{k=0}^{m}\binom{n+m-k}{n}\frac{(-x)^k}{k!};$$

then $-\widehat{Q}_m(x)/\widehat{P}_n(x)$ is the (n,m) Padé approximation of e^{-x}. Also, $\widehat{P}_n$ and $\widehat{Q}_m$ satisfy (1.2) (see, for example, [**1**]) and

$$\begin{aligned}\widehat{P}_n(x)e^{-x} + \widehat{Q}_m(x) &= (-1)^{m+1}\,\frac{x^{n+m+1}}{m!}\int_0^1 (1-u)^m u^n e^{-ux}\,du\\ &= \frac{(-1)^{m+1}n!}{(n+m+1)!}\,x^{n+m+1}e^{-n/(n+m)x}(1+o(1)),\end{aligned}$$

as $n+m\to\infty$, uniformly for x in any compact set of the plane.

If we further let

$$\widetilde{P}_n(x) := \frac{e^{2x}2^{s+1}}{s!}\int_x^\infty (t-x)^s\widehat{P}_n(t)e^{-2t}\,dt = n!\sum_{j=0}^{n}\frac{p_j x^j}{j!}, \tag{2.1}$$

where

$$p_j := \sum_{k=0}^{n-j} 2^{-k}\binom{n+m-k-j}{m}\binom{s+k}{s}; \tag{2.2}$$

and

$$\widetilde{Q}_m(x) := \frac{e^{x}2^{s+1}}{s!}\int_x^\infty (t-x)^s\widehat{Q}_m(t)e^{-t}\,dt = -2^{s+1}n!\sum_{j=0}^{m}\frac{q_j x^j}{j!}, \tag{2.3}$$

where

$$q_j := \sum_{k=0}^{m-j}(-1)^{k+j}\binom{n+m-k-j}{n}\binom{s+k}{s}; \tag{2.4}$$

and

$$\begin{aligned}\widetilde{R}_s(x) &:= (-1)^{s+1}\,\frac{2^{s+1}}{s!}\int_0^x (x-t)^s\left\{\widehat{P}_n(t)e^{-t}+\widehat{Q}_m(t)\right\}e^{-t}\,dt \\ &= 2^{s-n}n!\,(-1)^m\sum_{j=0}^{s}\frac{r_jx^j}{j!},\end{aligned}\tag{2.5}$$

where

$$r_j := \sum_{k=0}^{s-j}(-1)^j\binom{s+m-k-j}{m}\binom{n+k}{n}2^{-k};\tag{2.6}$$

then $\left(-\widetilde{Q}_m+\sqrt{\widetilde{Q}_m^2-4\widetilde{P}_n\widetilde{R}_s}\right)/\left(2\widetilde{P}_n\right)$ is the nondiagonal quadratic Hermite-Padé approximant [**3**] to e^{-x}. Also, $\widetilde{P}_n$, $\widetilde{Q}_m$ and $\widetilde{R}_s$ satisfy (1.3) and

$$\begin{aligned}&\widetilde{P}_n(x)e^{-2x}+\widetilde{Q}_m(x)e^{-x}+\widetilde{R}_s(x) \\ &\qquad = (-1)^{m+s}\,\frac{2^{s+1}x^{n+m+s+2}}{m!s!} \\ &\qquad\quad \cdot\int_0^1\int_0^1(1-u)^m\,u^ne^{-uvx}(1-v)^s\,v^{n+m+1}e^{-vx}\,du\,dv.\end{aligned}\tag{2.7}$$

Now let

(2.8)

$$\begin{aligned}P_n(x) &:= \frac{e^{3x}3^{l+1}}{l!}\int_x^\infty (t-x)^l\,\widetilde{P}_n(t)e^{-3t}\,dt \\ &= \frac{e^{3x}3^{l+1}}{l!}\int_x^\infty (t-x)^l\,e^{-3t}n!\sum_{j=0}^{n}\frac{p_jt^j}{j!}\,dt \\ &= 3^{l+1}\,\frac{n!}{l!}\int_0^\infty u^l\,e^{-3u}\sum_{j=0}^{n}\frac{p_j}{j!}\,(u+x)^j\,du \quad (u=t-x) \\ &= 3^{l+1}\,\frac{n!}{l!}\sum_{j=0}^{n}\frac{p_j}{j!}\sum_{i=0}^{j}\binom{j}{i}x^i\int_0^\infty u^{l+j-i}e^{-3u}\,du \\ &= 3^{l+1}\,\frac{n!}{l!}\sum_{j=0}^{n}\sum_{i=0}^{j}\frac{p_jx^i}{i!(j-i)!}\,\frac{(l+j-i)!}{3^{l+j+1-i}}\end{aligned}$$

$$= n! \sum_{i=0}^{n} \sum_{j=i}^{n} \frac{p_j}{3^{j-i}} \binom{l+j-i}{l} \frac{x^i}{i!} = n! \sum_{i=0}^{n} \frac{a_i x^i}{i!}, \tag{2.9}$$

with

$$a_i := \sum_{j=i}^{n} 3^{i-j}\, p_j \binom{l+j-i}{l}. \tag{2.10}$$

Note that P_n is a polynomial of degree n with highest coefficient 1.
Let

$$\begin{aligned} Q_m(x) : &= \frac{e^{2x} 3^{l+1}}{l!} \int_x^{\infty} (t-x)^l\, \widetilde{Q}_m(t) e^{-2t}\, dt \\ &= -\frac{e^{2x} 3^{l+1}}{l!} \int_x^{\infty} (t-x)^l\, e^{-2t} n!\, 2^{s+1} \sum_{j=0}^{m} \frac{q_j t^j}{j!}\, dt \end{aligned} \tag{2.11}$$

from (2.4). Substituting $u = t - x$ in (2.11) and using the Binomial theorem and a Laplace transform, we obtain

$$Q_m(x) = -3^{l+1} n!\, 2^{s-l} \sum_{i=0}^{m} \frac{b_i x^i}{i!}, \tag{2.12}$$

where

$$b_i := \sum_{j=i}^{m} q_j\, 2^{i-j} \binom{l+j-i}{l}. \tag{2.13}$$

Let

$$\begin{aligned} R_s(x) : &= \frac{e^x 3^{l+1}}{l!} \int_x^{\infty} (t-x)^l\, \widetilde{R}_s(t) e^{-t} dt, \\ &= (-1)^m\, 2^{s-n}\, \frac{n!}{l!}\, e^x\, 3^{l+1} \int_x^{\infty} (t-x)^l\, e^{-t} \sum_{j=0}^{s} \frac{r_j t^j}{j!}\, dt. \end{aligned} \tag{2.14}$$

Similar to above demonstration, we have

$$R_s(x) = (-1)^m n!\, 2^{s-n}\, 3^{l+1} \sum_{i=0}^{s} \frac{c_i x^i}{i!}, \tag{2.15}$$

where

$$c_i := \sum_{j=i}^{s} r_j \binom{l+j-i}{l}. \tag{2.16}$$

Define S_l by

(2.17)

$$\begin{aligned}
S_l(x) &:= -\frac{3^{l+1}}{l!}\int_0^\infty (t-x)^l e^{-t}\left\{\widetilde{P}_n(t)e^{-2t} + \widetilde{Q}_m(t)e^{-t} + \widetilde{R}_s(t)\right\} dt \\
&= -\frac{3^{l+1}}{l!}\int_0^\infty \sum_{i=0}^{l}\binom{l}{i} t^{l-i}(-x)^i e^{-t}\left\{e^{-2t} n! \sum_{j=0}^{n}\frac{p_j}{j!}\, t^j - e^{-t}\, n!\, 2^{s+1}\right. \\
&\qquad \left. \cdot \sum_{j=0}^{m}\frac{q_j}{j!}\, t^j + (-1)^m\, n!\, 2^{s-n}\sum_{j=0}^{s}\frac{r_j}{j!}\, t^j\right\} dt \\
&= -3^{l+1}\,\frac{n!}{l!}\sum_{i=0}^{l}\binom{l}{i}(-x)^i\left\{\sum_{j=0}^{n}\frac{p_j}{j!}\int_0^\infty t^{l+j-i}e^{-3t}\,dt - 2^{s+1}\right. \\
&\qquad \left. \cdot\sum_{j=0}^{m}\frac{q_j}{j!}\int_0^\infty t^{l+j-i}e^{-2t}\,dt + (-1)^m\, 2^{s-n}\sum_{j=0}^{s}\frac{r_j}{j!}\int_0^\infty t^{l+j-i}e^{-t}\,dt\right\} \\
&= -3^{l+1}\, n!\sum_{i=0}^{l}\frac{(-x)^i}{i!}\left\{\sum_{j=0}^{n}\binom{l+j-i}{j}\frac{p_j}{3^{l+j+1-i}} - 2^{s+1}\right. \\
&\qquad \left. \cdot\sum_{j=0}^{m}\binom{l+j-i}{j}\frac{q_j}{2^{l+j+1-i}} + (-1)^m\, 2^{s-n}\sum_{j=0}^{s}\binom{l+j-i}{j} r_j\right\}
\end{aligned}$$

(2.18)

$$:= -3^{l+1}\, n! \sum_{i=0}^{l}\frac{d_i x^i}{i!},$$

where

(2.19)

$$d_i := (-1)^i \left\{ \sum_{j=0}^{n} \binom{l+j-i}{j} \frac{p_j}{3^{l+j+1-i}} - 2^{s+1} \sum_{j=0}^{m} \binom{l+j-i}{j} \frac{q_j}{2^{l+j+1-i}} + (-1)^m \, 2^{s-n} \sum_{j=0}^{s} \binom{l+j-i}{j} r_j \right\}.$$

Finally, let

(2.20)

$$\begin{aligned}\mathbf{E}_{nmsl}(x) :&= (-1)^{m+s+l} \, \frac{3^{l+1}2^{s+1}}{m!s!\,l!} \int_0^x (x-t)^l e^{-t} \, t^{n+m+s+2} \\ &\quad \cdot \int_0^1 \int_0^1 (1-u)^m \, u^n e^{-uvt} (1-v)^s \, v^{n+m+1} e^{-vt} \, du \, dv \, dt \\ &= (-1)^{m+s+l} \, \frac{3^{l+1}2^{s+1}}{m!s!\,l!} \, x^{n+m+s+l+3} \\ &\quad \cdot \int_0^1 \int_0^1 \int_0^1 (1-u)^m \, u^n e^{-uvwx} (1-v)^s \, v^{n+m+1} e^{-vwx} \\ &\quad \cdot (1-w)^l \, w^{n+m+s+2} e^{-wx} \, du \, dv \, dw\end{aligned}$$

(2.21) $$(t = wx).$$

Now we may establish the basic theorem.

Theorem 1.

$$\begin{aligned}\mathbf{E}_{nmsl}(x) &:= P_n(x)e^{-3x} + Q_m(x)e^{-2x} + R_s(x)e^{-x} + S_l(x) \\ &= O(x^{n+m+s+l+3}),\end{aligned}$$

where $\mathbf{E}_{nmsl}$, P_n, Q_m, R_s *and* S_l *are given by* (2.20), (2.8), (2.11), (2.14) *and* (2.17), *respectively.*

Proof. By (2.20) and (2.7)

$$\begin{aligned}\mathbf{E}_{nmsl}(x) &= (-1)^{l+1}\frac{3^{l+1}}{l!}\int_0^x (x-t)^l e^{-t}\big\{\widetilde{P}_n(t)e^{-2t}+\widetilde{Q}_m(t)e^{-t}+\widetilde{R}_s(t)\big\}dt\\ &= \frac{3^{l+1}}{l!}\int_x^\infty (t-x)^l\,\widetilde{P}_n(t)e^{-3t}\,dt\\ &\quad+\frac{3^{l+1}}{l!}\int_x^\infty (t-x)^l\,\widetilde{Q}_m(t)e^{-2t}\,dt\\ &\quad+\frac{3^{l+1}}{l!}\int_x^\infty (t-x)^l\,\widetilde{R}_s(t)e^{-t}\,dt-\frac{3^{l+1}}{l!}\\ &\quad\cdot\int_0^\infty (t-x)^l e^{-t}\big\{\widetilde{P}_n(t)e^{-2t}+\widetilde{Q}_m(t)e^{-t}+\widetilde{R}_s(t)\big\}\,dt\\ &= P_n(x)e^{-3x}+Q_m(x)e^{-2x}+R_s(x)e^{-x}+S_l(x).\end{aligned}$$

With (2.21), the theorem has been proved. □

3. Asymptotics. We now turn to asymptotic estimate for $\{\mathbf{E}_{nmsl}\}$ defined in (2.20) as $n+m\to\infty$.

In order to give the asymptotic, we need the following lemma.

Lemma 1 [**6**].

$$\begin{aligned}\int_0^1 (1-t)^{\alpha p}\,t^{\beta p}\,e^{-\gamma t}\,dt &\sim\; e^{-(\beta\gamma)/(\alpha+\beta)}\int_0^1 (1-t)^{\alpha p}\,t^{\beta p}\,dt\\ &=\; e^{-(\beta\gamma)/(\alpha+\beta)}\,\frac{(\alpha p)!\,(\beta p)!}{((\alpha+\beta)p+1)!},\end{aligned}$$

where αp, βp all are positive integers.

From the above lemma, we have the following relation as $n+m\to\infty$

$$(3.1)\qquad \int_0^1 (1-u)^m u^n e^{-ux}\,du=\frac{n!\,m!}{(n+m+1)!}\,e^{-(n/n+m)x}(1+o(1)).$$

Now we can give the asymptotic estimates for $\{\mathbf{E}_{nmsl}\}$.

Theorem 2.

$$\mathbf{E}_{nmsl}(x) = \frac{(-1)^{m+s+l} n!\, 2^{s+1} 3^{l+1}}{(n+m+s+l+3)!}\, x^{n+m+s+l+3} \cdot e^{-(2m+3n+s+1)/(n+m+s+l+1)x}\,(1+o(1)) \tag{3.2}$$

as $n+m\to\infty$. *The asymptotic is uniform on bounded subsets of* $\mathbf{C}$.

Proof. Applying (3.1) successively to the triple integral on the right-hand side of (2.20), we have as $n+m\to\infty$,

(3.3)

$$\begin{aligned}
&\int_0^1\int_0^1\int_0^1 (1-u)^m\, u^n e^{-uvwx} (1-v)^s v^{n+m+1} \\
&\qquad\qquad \cdot e^{-vwx} (1-w)^l\, w^{n+m+s+2} e^{-wx}\, du\, dv\, dw \\
&= \int_0^1\int_0^1 (1-v)^s\, v^{n+m+1} e^{-vwx} (1-w)^l\, w^{n+m+s+2} \\
&\qquad\qquad \cdot e^{-wx}\, \frac{n!\, m!}{(n+m+1)!}\, e^{-n/(n+m)vwx}\,(1+o(1))\, dv\, dw \\
&= \int_0^1 (1-w)^l\, w^{n+m+s+2} e^{-wx}\, \frac{n!\, m!}{(n+m+1)!}\, \frac{(n+m+1)!\, s!}{(n+m+s+2)!} \\
&\qquad\qquad \cdot e^{-(n+m+1)/(n+m+s+1)\cdot(1+n)/(n+m)wx}\,(1+o(1))\, dw \\
&= \frac{n!\, m!\, s!}{(n+m+s+2)!} \int_0^1 (1-w)^l\, w^{n+m+s+2} e^{-wx} \\
&\qquad\qquad \cdot e^{-(m+2n)/(n+m+s+1)wx}\,(1+o(1))\, dw \\
&= \frac{n!\, m!\, s!}{(n+m+s+2)!}\, \frac{l!\,(n+m+s+2)!}{(n+m+s+l+3)!} \\
&\qquad \cdot e^{-(n+m+s+2)/(n+m+s+l+2)\cdot(2m+3n+s+1)/(n+m+s+1)x}\,(1+o(1)) \\
&= \frac{n!\, m!\, s!\, l!}{(n+m+s+l+3)!}\, e^{-(2m+3n+s+1)/(n+m+s+l+2)x}\,(1+o(1)) \\
&\qquad\qquad (n+m\to\infty).
\end{aligned}$$

The asymptotic (3.2) follows immediately from (2.20) and (3.3). It is easy to check directly from (2.20) that

$$\mathbf{E}_{nmsl}(x) \Big/ \left(\frac{n!\, 2^{s+1} 3^{l+1} x^{n+m+s+l+3} e^{-x(2m+3n+s+1)/(n+m+s+l+1)}}{(n+m+s+l+3)!} \right)$$

is uniformly bounded on compact subsets of $\mathbf{C}$ as $n+m \to \infty$. The uniformity of the asymptotic now follows from Vitali's theorem [**1**].

Remark. For ray sequences in the Hermite-Padé table, that is, multi-indices $(n+1,\, m+1,\, s+1,\, l+1)$ that satisfy

$$\text{(3.4)} \qquad \lim_{n\to\infty} \frac{m}{n} = \alpha, \quad \lim_{n\to\infty} \frac{s}{n} = \beta, \quad \lim_{n\to\infty} \frac{l}{n} = \gamma, \quad \alpha,\, \beta,\, \gamma \geq 0,$$

the asymptotic (3.2) can be written

$$\text{(3.5)} \qquad \begin{aligned} E_{nmsl}(x) = \frac{(-1)^{m+s+l}\, n!\, 2^{s+1} 3^{l+1}}{(n+m+s+l+3)!}\, x^{n+m+s+l+3} \\ \cdot\, e^{-(2\alpha+\beta+3)/(\alpha+\beta+\gamma+1)x}\, (1+o(1)) \end{aligned}$$

as $n \to \infty$. Note that when $\alpha + 2 = \gamma$, the exponential term on the right-hand side of (3.5) loses its dependence on β and reduces to e^{-x}.

4. An exact minimization. We wish to uniformly minimize over $D := \{z \in \mathbf{C} : |z| \leq 1\}$,

$$\text{(4.1)} \qquad W_{nmsl}(z) := h_n(z)e^{-3z} + t_m(z)e^{-2z} + u_s(z)e^{-z} + v_l(z),$$

where h_n, t_m, u_s, v_l being algebraic polynomials of degree at most n, m, s, l respectively; and h_n has highest coefficient 1. In the case when the positive integers n, m, s, l satisfy (3.4), we can prove the following results.

Firstly, we have

Theorem 3. *Suppose* (3.4) *holds. Then, for* $|z| = 1$*, as* $n \to \infty$*,*

$$\text{(4.2)} \qquad \left| \mathbf{E}_{nmsl}\left(z + \frac{(2\alpha+\beta+3)/(\alpha+\beta+\gamma+1)}{n+m+s+l+3} \right) \right| \sim \frac{n!\, 2^{s+1}\, 3^{l+1}}{(n+m+s+l+3)!}.$$

Proof. This follows from Theorem 2 and the observation that

$$\left(z + \frac{k}{n+m+s+l+3}\right)^{n+m+s+l+3} \sim z^{n+m+s+l+3} e^{k/z},$$
$$e^{-k(z+k)/(n+m+s+l+3)} \sim e^{-kz},$$

and the fact that, for $|z| = 1$, $k \in \mathbf{R}$, $|e^{k(1/z-z)}| = 1$.

Let

$$\begin{aligned} z^* &= z + \frac{(2\alpha+\beta+3)/(\alpha+\beta+\gamma+1)}{n+m+s+l+3}, \\ P_n^*(z) &= P_n(z^*), \quad Q_m^*(z) = Q_m(z^*), \\ R_s^*(z) &= R_s(z^*), \quad S_l^*(z) = S_l(z^*). \end{aligned} \tag{4.3}$$

Let $||\cdot||_D$ denote the supremum norm on D.

Now we need the following lemma.

Lemma 2 [**5**]. *Let real numbers* $\lambda_1 < \lambda_2 < \cdots < \lambda_\nu$ *and positive integers* $m_1, m_2, \dots, m_\nu$ *be such that* $m_1 + m_2 + \cdots + m_\nu = r$. *Let* $f_1(z), f_2(z), \dots, f_r(z)$ *denote* $e^{\lambda_1 z}, ze^{\lambda_1 z}, \dots, z^{m_1-1}e^{\lambda_1 z}$; $e^{\lambda_2 z}, ze^{\lambda_2 z}, \dots, z^{m_2-1}e^{\lambda_2 z}$; $\dots, \dots$; $e^{\lambda_\nu z}, ze^{\lambda_\nu z}, \dots, z^{m_\nu-1}e^{\lambda_\nu z}$, *respectively. If* $|c_1|+|c_2|+\cdots+|c_{m_1}| > 0$, $|c_{r-m_\nu+1}|+\cdots+|c_{r-1}|+|c_r| > 0$, N *denotes the zero number of function* $c_1f_1(z)+c_2f_2(z)+\cdots+c_rf_r(z)$ *in the region* $\Omega = \{z : \xi \le \operatorname{Im} z \le \eta\}$. *Then*

$$\frac{(\lambda_\nu-\lambda_1)(\eta-\xi)}{2\pi} - r + 1 \le N \le \frac{(\lambda_\nu-\lambda_1)(\eta-\xi)}{2\pi} + r - 1.$$

By this lemma, we can prove the main result of this section.

Theorem 4. (a)

$$\left|\left|P_n^*(z)e^{-3z} + Q_m^*(z)e^{-2z} + R_s^*(z)e^{-z} + S_l^*(z)\right|\right|_D \sim \frac{n!\,2^{s+1}\,3^{l+1}}{(n+m+s+l+3)!}.$$

(b) *Let*

$$W^*_{nmsl} := \min_{h,\,t,\,u,\,v} \left|\left|h_n(z)e^{-3z} + t_m(z)e^{-2z} + u_s(z)e^{-z} + v_l(z)\right|\right|_D,$$

where h, t, u, v *being algebraic polynomials of degree at most* n, m, s, l, *respectively;* $h = z^n + \cdots$, *then*

$$W^*_{nmsl} \sim \frac{n!\,2^{s+1}\,3^{l+1}}{(n+m+s+l+3)!}.$$

Proof. Part (a) is just a restatement of Theorem 3. Observe that P^*_m has leading coefficient 1.

To prove part (b) we use the fact that a nonzero expression of the form

$$\phi_1(z)e^{-3z} + \phi_2(z)e^{-2z} + \phi_3(z)e^{-z} + \phi_4(z), \tag{4.3}$$

where $\phi_1,\ \phi_2,\ \phi_3,\ \phi_4$ are polynomials, the sum of whose degrees is k, can have at most $k+3$ zeros in D. This is a winding number argument and is proved in Lemma 2 with $\nu = 4$, $\xi = -1$, $\eta = 1$, $\lambda_1 = -3$, $\lambda_2 = -2$, $\lambda_3 = -1$, $\lambda_4 = 0$ and $r = k+4$. Thus

$$W^*_{nmsl} \geq \min_{|z|=1} \left|P^*_n(z)e^{-3z} + Q^*_m(z)e^{-2z} + R^*_s(z)e^{-z} + S^*_l(z)\right|.$$

If this were not the case, we could find $h \in \pi_n$, $t \in \pi_m$, $u \in \pi_s$, $v \in \pi_l$ with h having leading coefficient 1 so that, for $|z| = 1$,

$$\left|he^{-3z} + te^{-2z} + ue^{-z} + v\right| < \left|P^*_n e^{-3z} + Q^*_m e^{-2z} + R^*_s e^{-z} + S^*_l\right|.$$

By Rouché's theorem this would imply that

$$(h - P^*_n)e^{-3z} + (t - Q^*_m)e^{-2z} + (u - R^*_s)e^{-z} + (v - S^*_l) \tag{4.4}$$

has at least $n + m + s + l + 3$ zeros in D. However, since $h - P^*_n$ has degree at most $n-1$, the sum of the degrees of the coefficient polynomials in (4.4) is at most $n + m + s + l - 1$, and we have contradicted the above result from Lemma 2. □

We note that if the positive integers $n,\ m,\ s$ and l satisfy (3.4), then we have

$$\left|\left|P_n e^{-3z} + Q_m e^{-2z} + R_s e^{-z} + S_l\right|\right|_D \sim \frac{n!\,2^{s+1}\,3^{l+1}}{(n+m+s+l+3)!}\,e^{(2\alpha+\beta+3)/(\alpha+\beta+\gamma+1)}$$

and so, up to a small constant, the cubic Hermite-Padé approximant is optimal in the sense of Theorem 4. The trick of shifting the center of the approximation to make the error curve have an asymptotically constant norm on D is due to Braess [**2**] who used it to get the right constant in Meinardus's conjecture.

5. Differential equations. The coefficient polynomials are linked by the following fourth-order differential equations.

Theorem 5.

(a) $-6nP_{n-1}(x) = P_n''''(x) - 6P_n'''(x) + 11P_n''(x) - 6P_n'(x)$.

(b) $-6nQ_{m-1}(x) = Q_m''''(x) - 2Q_m'''(x) - Q_m''(x) + 2Q_m'(x)$.

(c) $-6nR_{s-1}(x) = R_s''''(x) + 2R_s'''(x) - R_s''(x) - 2R_s'(x)$.

(d) $-6nS_{l-1}(x) = S_l''''(x) + 6S_l'''(x) + 11S_l''(x) + 6S_l'(x)$.

Proof. We suppress the variable x in the coefficient polynomials and start with the relation

$$P_n e^{-3x} + Q_m e^{-2x} + R_s e^{-x} + S_l = O(x^{n+m+s+l+3}). \tag{5.1}$$

Then

$$(P_n' - 3P_n)e^{-3x} + (Q_m' - 2Q_m)e^{-2x} + (R_s' - R_s)e^{-x} + S_l' = O(x^{n+m+s+l+2}). \tag{5.2}$$

Adding triple (5.1) to (5.2) gives

$$P_n' e^{-3x} + (Q_m' + Q_m)e^{-2x} + (R_s' + 2R_s)e^{-x} + S_l' + 3S_l = O(x^{n+m+s+l+2}) \tag{5.3}$$

and differentiating again gives

(5.4)

$$(P_n''-3P_n')e^{-3x}+(Q_m''-Q_m'-2Q_m)e^{-2x}+(R_s''+R_s'-2R_s)e^{-x}+S_l''+3S_l' = O(x^{n+m+s+l+1}).$$

Now adding twice (5.3) to (5.4) gives

(5.5)

$$(P_n''-P_n')e^{-3x}+(Q_m''+Q_m')e^{-2x}+(R_s''+3R_s'+2R_s)e^{-x}+S_l''+5S_l'+6S_l = O(x^{n+m+s+l+1}).$$

and differentiating yields

$$\begin{aligned}(P_n'''-4P_n''+3P_n')e^{-3x}+(Q_m'''-Q_m''-2Q_m')e^{-2x}\\ +(R_s'''+2R_s''-R_s'-2R_s)e^{-x}+S_l'''+5S_l''+6S_l'\\ = O(x^{n+m+s+l}).\end{aligned} \tag{5.6}$$

Finally, adding (5.5) to (5.6) and differentiating yields

(5.7)

$$\begin{aligned}(P_n''''-6P_n'''+11P_n''-6P_n')e^{-3x}+(Q_m''''-2Q_m'''-Q_m''+2Q_m')e^{-2x}\\ +(R_s''''+2R_s'''-R_s''-2R_s')e^{-x}+S_l''''+6S_l'''+11S_l''+6S_l'\\ = O(x^{n+m+s+l-1}).\end{aligned}$$

Since the degrees of the coefficient polynomials in (5.7) are at most $n-1$, $m-1$, $s-1$ and $l-1$, respectively, we see that up to a constant multiple of $-6n$, (5.7) must equal $P_{n-1}e^{-3x}+Q_{m-1}e^{-2x}+R_{s-1}e^{-x}+S_{l-1}$. (Here we have appealed to uniqueness, as in the proof of Theorem 1.)

6. Numerical examples. The following table gives the values of coefficient polynomials and the error at $x = 1$.

n	m	s	l	$\delta_m(1) - (1/e)$
1	1	1	2	1.39975×10^{-5}
1	1	5	1	-6.45200×10^{-9}
2	2	2	4	9.89194×10^{-11}
3	3	2	3	1.49192×10^{-11}
3	3	6	3	2.94625×10^{-17}
3	3	4	5	-5.68769×10^{-17}
4	4	4	4	-3.41816×10^{-18}
4	4	4	9	-1.70369×10^{-25}
5	5	5	1	-2.30800×10^{-19}
6	6	5	6	5.73937×10^{-28}
6	6	6	7	8.39593×10^{-31}
7	7	7	6	1.80156×10^{-34}
8	8	8	4	9.92927×10^{-41}
8	8	8	8	-4.53470×10^{-42}

REFERENCES

1. P.B. Borwein, *Quadratic Hermite-Padé approximation to the exponential function*, Constr. Approx. **2** (1986), 291–302.

2. D. Braess, *On the conjecture of Meinardus on rational approximation of* e^x, II, J. Approx. Theory **40** (1984), 375–379.

3. K. Driver, *Nondiagonal quadratic Hermite-Padé approximation to the exponential function*, J. Comp. Appl. Math. **65** (1995), 125–134.

4. Ch. Hermite, *Sur la généralisation des fractions continues algébriques*, Ann. Mat. Pura Appl. (2A) **21** (1883), 289–308.

5. G. Pólya and G. Szegö, *Problems and theorems in analysis*, I, Springer-Verlag, Berlin, 1972.

6. L.N. Trefethen, *The asymptotic accuracy of rational best approximation to* e^z *on a disk*, J. Approx. Theory **40** (1984), 380–383.

Department of Mathematics, Dalian Jiaotong University, Dalian, 116028, P.R. China
E-mail address: chdzheng.student@sina.com

Institute of Mathematics Science, Dalian University of Technology, Dalian, 116024, P.R. China