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GENERALIZED FREE PRODUCTS OF RESIDUALLY P-FINITE GROUPS

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ABSTRACT. In this note, we characterize the residual p-finiteness of generalized free products and tree products of certain residually p-finite groups with non-trivial center amalgamating infinite cyclic subgroups and the tree products of certain one-relator groups. We then apply our results to tree products of finitely generated torsion-free nilpotent groups and free groups.

1. Introduction. Let p be a prime. A group G is said to be residually p-finite if for each non-trivial element x of G, there exists a normal subgroup N of index a power of p in G such that $x \notin N$. It is well known that free groups and finitely generated torsion-free nilpotent groups are residually p-finite for all primes p (Iwasawa [**5**], Gruenberg [**3**]). In [**4**], Higman proved that a generalized free product of two finite p-groups amalgamating a cyclic subgroup, is residually p-finite. Kim and McCarron [**6**] then generalized Higman's result by proving that the generalized free product of residually p-groups amalgamating a finite cyclic subgroup, is residually p-finite. In the same paper [**6**], they also proved a sufficient condition for a free product of finitely many residually p-finite groups amalgamating a single infinite cyclic subgroup, to be residually p-finite. From this, they showed that a generalized free product of finitely many free groups or finitely generated torsion-free nilpotent groups amalgamating a maximal cyclic subgroup is residually p-finite for all primes p. In [**11**], Wong and Tang extended Kim and McCarron's result to finite tree product of residually p-finite groups, amalgamating infinite cyclic subgroups. Thus, the finite tree products of finitely many free groups or finitely generated torsion-free nilpotent groups amalgamating maximal cyclic subgroups are residually p-finite for all primes p.

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More recently, Kim and Tang [**9**] characterized the residual p-finiteness of generalized free products and tree products of finitely generated torsion-free nilpotent groups and free groups amalgamating maximal cyclic subgroups while in [**7**], Kim and McCarron characterized the residual p-finiteness of certain one-relator groups. In this note we continue the work of Kim, McCarron and Tang. In the first part we characterize the residual p-finiteness of generalized free products and tree products of certain residually p-finite groups with non-trivial center amalgamating infinite cyclic subgroups. In the second part we characterize the residual p-finiteness of tree products of certain one-relator groups. Then by applying our results to tree products of finitely generated torsion-free nilpotent groups and free groups amalgamating cyclic subgroups, we obtain partial extensions of the results in [**7, 9**].

The notation used here is standard. In addition, the following will be used for any group G: p shall denote a prime. $N \lhd_p G$ means N is a normal subgroup of index a power of p in G. $Z(G)$ denotes the center of G. $C_G(x)$ denotes the centralizer of the element x in G.

2. Preliminaries. In this section, we give some definitions and lemmas which will be used to prove our results.

Definition 2.1 [**6**, Definition 2.2]. Let H be a subgroup of a group G. Then H is said to be p-closed in G if for $x \notin H$, there exists $N \lhd_p G$ such that $x \notin HN$.

Lemma 2.2 [**9**, Lemma 2.4]. *Let G be a residually p-finite group and $\langle h \rangle$ is p-closed in G. Then $\langle h^n \rangle$ is p-closed in G where $|n| = p^\alpha$, $\alpha \geq 0$.*

Definition 2.3. A subgroup H of a group G is called a retract of G if G has a normal subgroup L such that $G = LH$ and $L \cap H = 1$.

Lemma 2.4 [**8**, Lemma 3.2]. *Let G be a residually p-finite group and $\langle h \rangle$ be a retract of G. Then $\langle h \rangle$ is p-closed in G.*

Lemma 2.5. *Let G be a residually p-finite group and $x \in G$. If $C_G(x) = \langle x \rangle$, then $\langle x \rangle$ is p-closed in G.*

Proof. Let $g \in G \setminus \langle x \rangle$. Since $C_G(x) = \langle x \rangle$, then $[g, x] \neq 1$. By residual p-finiteness of G, there exists $N \lhd_p G$ such that $[g, x] \notin N$. Clearly $g \notin \langle x \rangle N$ and we are done. □

Lemma 2.6. *Let G be a free group or a finitely generated torsion-free nilpotent group, and let $\langle h \rangle$ be a maximal cyclic subgroup or a retract of G or $C_G(h) = \langle h \rangle$. Then $\langle h^n \rangle$ is p-closed in G where $|n| = p^\alpha$, $\alpha \geq 0$.*

Proof. Let p be any prime. It is well known that free groups and finitely generated torsion-free nilpotent groups are residually p-finite [**3, 5**]. Furthermore, maximal cyclic subgroups of a free group are p-closed (see Theorem 3.9 of Kim and McCarron [**6**]). Also maximal cyclic subgroups of a finitely generated (non-cyclic) torsion-free nilpotent group are isolated and hence are p-closed (see Theorem 2.5 of Baumslag [**2**]). By Lemma 2.4, retracts of a residually p-finite group are p-closed. By Lemma 2.5, the cyclic subgroup $\langle h \rangle$ is p-closed since $C_G(h) = \langle h \rangle$. The result now follows from Lemma 2.2. □

Lemma 2.7 [**11**, Lemma 2]. *Let $G = A \underset{a=b}{*} B$. Suppose that A, B are residually p-finite groups and $\langle a \rangle$ is p-closed in A and $\langle b \rangle$ is p-closed in B. If K is a subgroup of A and K is p-closed in A, then K is p-closed in G.*

Definition 2.8. Let A_i, $1 \leq i \leq n$, be groups where n is finite. Then $G = \langle A_1, \ldots, A_n; a_{ij} = a_{ji},\ i \neq j,\ 1 \leq i, j \leq n \rangle$ where $a_{ij} \in A_{ij}$, shall denote a tree product of the groups A_i, with the cyclic subgroups $\langle a_{ij} \rangle$ of A_i and $\langle a_{ji} \rangle$ of A_j amalgamated.

Theorem 2.9 [**11**, Theorem 3]. *Let $G = \langle A_1, \ldots, A_n; a_{ij} = a_{ji},\ i \neq j,\ 1 \leq i, j \leq n \rangle$ where $a_{ij} \in A_{ij}$ and n is finite, be a tree product of the groups A_i, amalgamating the cyclic subgroups $\langle a_{ij} \rangle$ of A_i and $\langle a_{ji} \rangle$ of A_j. Suppose each A_i is residually p-finite and $\langle a_{ij} \rangle$ is p-closed in A_i. Then G is residually p-finite.*

3. Generalized free products and tree products of residually p-finite groups with non-trivial center. In this section, we characterize the residual p-finiteness of generalized free products and tree products of certain residually p-finite groups with non-trivial center amalgamating infinite cyclic subgroups.

Lemma 3.1. *Let* $G = A \underset{a=b^m}{*} \langle b \rangle$, *and suppose that there exists* $c \in A \backslash \langle a \rangle$ *such that* $[c, a^t] = 1$ *for some* $t \in \mathbf{Z}$, $t \neq 0$. *If* G *is residually* p*-finite, then* $|m| = p^\beta$, $\beta \geq 0$.

Proof. Suppose that $m = qm_1$ where $q \neq p$ is a prime. By our assumption, there exists $c \in A \backslash \langle a \rangle$ such that $[c, a^t] = 1$ for some $t \in \mathbf{Z}$, $t \neq 0$. $\square$

Case 1. q does not divide t. Since $c \in A \backslash \langle a \rangle$ and $b^{m_1 t} \notin \langle b^m \rangle$, we have $x = [c, b^{m_1 t}] \neq 1$. Since G is residually p-finite, there exists $N \lhd_p G$ such that $x \notin N$. Let $\overline{G} = G/N$. Since $\bar{b}$ has order a power of p in $\overline{G}$ and $p \neq q$, we have $\bar{b} = \bar{b}^{qs}$ for some s. Therefore, $\bar{x} = [\bar{c}, \bar{b}^{m_1 t}] = [\bar{c}, \bar{b}^{m_1 qst}] = [\bar{c}, \bar{a}^{st}] = \bar{1}$ since $[\bar{c}, \bar{a}^t] = \bar{1}$, a contradiction. This implies that $m = 1$ or m has no prime factor other than p, that is, $|m| = p^\beta$, $\beta \geq 0$.

Case 2. q divides t. Let $t = q^r m_2$ where $(q, m_2) = 1$ and $r \neq 0$. Since $c \in A \backslash \langle a \rangle$ and $b^{m_1 m_2} \notin \langle b^m \rangle$, we have $x = [c, b^{m_1 m_2}] \neq 1$. By residual p-finiteness of G, there exists $N \lhd_p G$ such that $x \notin N$. Since $\bar{b}$ has order a power of p in $\overline{G} = G/N$ and $p \neq q$, we have $\bar{b} = \bar{b}^{qs}$ for some s. Therefore, $\bar{x} = [\bar{c}, \bar{b}^{m_1 m_2}] = [\bar{c}, \bar{b}^{m_1 qsm_2}] = \cdots = [\bar{c}, \bar{b}^{m_1 qq^r m_2 s^{r+1}}] = [\bar{c}, \bar{b}^{mts^{r+1}}] = [\bar{c}, \bar{a}^{ts^{r+1}}] = \bar{1}$, a contradiction. The result now follows as in Case 1.

Lemma 3.2 [**7**, Theorem 1.1]. *Let* $G = \langle a \rangle \underset{a^n=b^m}{*} \langle b \rangle$. *Then* G *is residually* p*-finite if and only if either* $|n| = 1$ *or* $|m| = 1$ *or* $|n| = p^\alpha$ *and* $|m| = p^\beta$, α, $\beta > 0$.

Theorem 3.3. *Let* $G = A \underset{a^n=b^m}{*} B$, *and suppose that there exist* $c \in A \backslash \langle a \rangle, d \in B \backslash \langle b \rangle$ *such that* $[c, a^t] = 1$ *and* $[d, b^s] = 1$ *for some* $s, t \in \mathbf{Z}$. *If* G *is residually* p*-finite, then* $|n| = p^\alpha$ *and* $|m| = p^\beta$, α, $\beta \geq 0$.

Proof. Since G is residually p-finite, then the subgroup $\langle a\rangle \underset{a^n=b^m}{*} \langle b\rangle$ is residually p-finite and hence, by Lemma 3.2, $|n| = 1$ or $|m| = 1$ or $|n| = p^\alpha$ and $|m| = p^\beta$, α, $\beta > 0$. If $|n| = 1$, then the subgroup $A \underset{a=b^m}{*} \langle b\rangle$ is residually p-finite and hence, by Lemma 3.1, $|m| = p^\beta$, $\beta \geq 0$. Similarly, if $|m| = 1$, then $|n| = p^\alpha$, $\alpha \geq 0$. The result now follows. □

Lemma 3.4. *Let A be a group with $Z(A) \neq 1$ and $a \in A$. If $A \neq \langle a\rangle$, then there exists $c \in A\backslash\langle a\rangle$ such that $[c, a^t] = 1$ for some $t \in \mathbf{Z}$.*

Proof. Suppose that there exists some t such that $a^t \in Z(A)$. Since $A \neq \langle a\rangle$, we can choose $c \in A\backslash\langle a\rangle$. Clearly $[c, a^t] = 1$, and we are done. So suppose $Z(A) \cap \langle a\rangle = 1$. Since $Z(A) \neq 1$, we can choose $1 \neq c \in Z(A)$. Then $[c, a] = 1$ and $c \notin \langle a\rangle$. □

Lemma 3.5. *Let $G = A \underset{a=b^m}{*} \langle b\rangle$ where $Z(A) \neq 1$. If G is residually p-finite, then either $A = \langle a\rangle$ or $|m| = p^\beta$, $\beta \geq 0$.*

Proof. Suppose that $A \neq \langle a\rangle$. Then by Lemma 3.4, there exists $c \in A\backslash\langle a\rangle$ such that $[c, a^t] = 1$ for some $t \in \mathbf{Z}$. Hence $|m| = p^\beta$, $\beta \geq 0$ by Lemma 3.1. □

We now state and prove the main results of this section, that is, Theorem 3.6 and Theorem 3.8 which are partial extensions of Theorem 4.4 and Theorem 5.4 of [**9**], respectively.

Theorem 3.6. *Let $G = A \underset{a^n=b^m}{*} B$ where $Z(A) \neq 1$ and $Z(B) \neq 1$. If G is residually p-finite, then $|m| = 1$ and $G= A$ or $|n| = 1$ and $G= B$ or $|n| = p^\alpha$ and $|m| = p^\beta$, α, $\beta \geq 0$.*

Proof. Since G is residually p-finite, then the subgroup $\langle a\rangle \underset{a^n=b^m}{*} \langle b\rangle$ is residually p-finite and hence, by Lemma 3.2, $|n| = 1$ or $|m| = 1$ or $|n| = p^\alpha$ and $|m| = p^\beta$, α, $\beta > 0$. If $|n| = 1$, then the subgroup $A \underset{a=b^m}{*} \langle b\rangle$ is residually p-finite and hence by Lemma 3.5, either $A = \langle a\rangle$ which implies $G = B$ or $|m| = p^\beta$, $\beta \geq 0$. Similarly, if $|m| = 1$, then either $B = \langle b\rangle$ which implies $G = A$ or $|n| = p^\alpha$, $\alpha \geq 0$. The result now follows. □

Next we extend Theorem 3.6 to finite tree products of residually p-finite groups with non-trivial center amalgamating infinite cyclic subgroups. First we have the following definition.

Definition 3.7 [**9**, Definition 5.3]. Let G be the tree product of a tree T and H the tree product of a subtree S of T. Then H is called a subtree product of G. If $G = H$, then G is said to be contractible to H.

Let I be a finite set.

Theorem 3.8. *Let* $G = \langle G_1, \dots , G_n; u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}},\ i \neq j,\ i, j \in I\rangle$ *where* $u_{ij} \in G_{ij}$ *and* I *is finite, be a tree product of the groups* G_i *with* $Z(G_i) \neq 1,\ i \in I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}}\rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}}\rangle$ *of* G_j.

(a) *If* G *is residually* p*-finite, then* G *is contractible to a subtree product of* $G_i,\ i \in J \subset I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}}\rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}}\rangle$ *of* G_j *where* $|n_{ij}| = p^{\alpha_{ij}}$ *and* $|n_{ji}| = p^{\beta_{ji}}$, $\alpha_{ij}, \beta_{ji} \geq 0$.

(b) *Suppose each* G_i *is a residually* p*-finite group and* $\langle u_{ij}^{n_{ij}}\rangle$ *is* p*-closed in* G_i. *If* G *is contractible to a subtree product of* $G_i,\ i \in J \subset I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}}\rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}}\rangle$ *of* G_j, *then* G *is residually* p*-finite.*

Proof. (a) Suppose G_i is connected to G_j and $|n_{ij}| \neq p^{\alpha_{ij}}$, $\alpha_{ij} \geq 0$ where $i, j \in I$. Since $T_1 = G_i \ {}_{u_{ij}^{n_{ij}} \overset{*}{=} u_{ji}^{n_{ji}}} G_j$ is residually p-finite, then by Theorem 3.6, $|n_{ji}| = 1$ and $T_1 = G_i$. Now, if G_j is connected to G_k, then $T_2 = G_i \ {}_{u_{ij}^{n_{ij}} \overset{*}{=} u_{ji}^{n_{ji}}} G_j \ {}_{u_{jk}^{n_{jk}} \overset{*}{=} u_{kj}^{n_{kj}}} G_k = G_i \ {}_{u_{ij}^{n_{ij} n_{jk} r} \overset{*}{=} u_{kj}^{n_{kj}}} G_k$ for some $r \in \mathbf{Z}$. Since $|n_{ij}| \neq p^{\alpha_{ij}}$, $\alpha_{ij} \geq 0$, then $|n_{ij} n_{jk} r| \neq p^{\alpha_{ij}}$, $\alpha_{ij} \geq 0$. Therefore by Theorem 3.6, $|n_{kj}| = 1$ and $T_2 = G_i$. Continuing in this way, we see that the tree product of the subtree connected to G_i by the subgroup $\langle u_{ij}\rangle = G_i$ if $|n_{ij}| \neq p^{\alpha_{ij}},\ \alpha_{ij} \geq 0$. This implies that $G =$ a subtree product of $G_i,\ i \in J$, amalgamating $\langle u_{ij}^{n_{ij}}\rangle$ of G_i and $\langle u_{ji}^{n_{ji}}\rangle$ of G_j where $|n_{ij}| = p^{\alpha_{ij}}$ and $|n_{ji}| = p^{\beta_{ji}}$, $\alpha_{ij}, \beta_{ji} \geq 0$.

(b) Follows from Theorem 2.9. □

Note that Theorem 3.8 can only be applied to residually p-finite groups with non-trivial center but this class of groups is not small. It includes the finitely generated torsion-free nilpotent groups which are residually p-finite for all primes p as well as the two-generator one-relator groups $\langle x, y; x^{p^\alpha} = y^{p^\beta} \rangle$ and $\langle x, y; [x, y^{p^\beta}] = 1 \rangle$ which are residually p-finite for that particular prime p [**7, 10**]. Thus from Theorem 3.8 and Lemma 2.6, we have the following extensions of Theorem 5.4 and Corollary 5.5 of [**9**].

Corollary 3.9 (see [**9**]). *Let* $G = \langle G_1, \ldots, G_n; u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}, \ i \neq j, \ i, j \in I \rangle$ *where* $u_{ij} \in G_{ij}$ *and* I *is finite, be a tree product of the groups* G_i, $i \in I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}} \rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}} \rangle$ *of* G_j. *Suppose the* G_i, $i \in I$, *are finitely generated torsion-free nilpotent groups and* $\langle u_{ij} \rangle$ *is a maximal cyclic subgroup or a retract of* G_i. *Then* G *is residually* p*-finite if and only if* G *is contractible to a subtree product of* G_i, $i \in J \subset I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}} \rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}} \rangle$ *of* G_j *where* $|n_{ij}| = p^{\alpha_{ij}}$ *and* $|n_{ji}| = p^{\beta_{ji}}$, $\alpha_{ij}, \beta_{ji} \geq 0$.

Corollary 3.10 (see [**9**]). *Let* $G = \langle G_1, \ldots, G_n; u_{ij}^{n_{ij}} = u_{ji}^{n_{ji}}, \ i \neq j, \ i, j \in I \rangle$ *where* $u_{ij} \in G_{ij}$ *and* I *is finite, be a tree product of the groups* G_i, $i \in I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}} \rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}} \rangle$ *of* G_j. *Suppose the* G_i, $i \in I$, *are finitely generated torsion-free nilpotent groups and* $\langle u_{ij} \rangle$ *is a maximal cyclic subgroup or a retract of* G_i *or* $C_{G_i}(u_{ij}) = \langle u_{ij} \rangle$. *Then* G *is residually* p*-finite for all primes* p *if and only if* G *is contractible to a subtree product of* G_i, $i \in J \subset I$, *amalgamating the cyclic subgroups* $\langle u_{ij}^{n_{ij}} \rangle$ *of* G_i *and* $\langle u_{ji}^{n_{ji}} \rangle$ *of* G_j *where* $|n_{ij}| = 1 = |n_{ji}|$.

4. Tree products of certain one-relator groups. In this section we characterize the residual p-finiteness of tree products of certain one-relator groups. We start with the following theorem of Kim and McCarron in [**7**].

Theorem 4.1 [**7**, Theorem 3.2]. *The group* $G = \langle x, y; [x^s, y^t] = 1 \rangle$ *is residually* p*-finite if and only if* $|s| = p^\alpha$ *and* $|t| = p^\beta$, $\alpha, \beta \geq 0$.

We shall extend Theorem 4.1 to certain tree products. First we have the following lemma.

Lemma 4.2. *Let $G = \langle x, y;\, [x^s, y^t] = 1\rangle$. If $|s| = p^\alpha$ and $|t| = p^\beta$, α, $\beta \geq 0$, then $\langle x\rangle$ and $\langle y\rangle$ are p-closed in G.*

Proof. Since G is residually p-finite and $\langle x\rangle$ and $\langle y\rangle$ are retracts of G, then $\langle x\rangle$ and $\langle y\rangle$ are p-closed in G by Lemma 2.4. □

Before proceeding further, we give here an example. Let $G = \langle a_1, a_2, a_3, a_4, a_5; [a_1^{s_{12}}, a_2^{s_{21}}] = 1, [a_1^{s_{13}}, a_3^{s_{31}}] = 1, [a_2^{s_{24}}, a_4^{s_{42}}] = 1, [a_2^{s_{25}}, a_5^{s_{52}}] = 1\rangle$. We note that no sequences of relations of the form $[a_i^{s_{ij_1}}, a_{j_1}^{s_{j_1 i}}] = 1, [a_{j_1}^{s_{j_1 j_2}}, a_{j_2}^{s_{j_2 j_1}}] = 1, \ldots,\ [a_{j_r}^{s_{j_r i}}, a_i^{s_{ij_r}}] = 1$ for $r \geq 2$, occurs in G. Then by using this fact, we can show that G can be considered as a tree product and extend the result of Theorem 4.1 to this group.

Let $G_1 = \langle a_1\rangle$, $G_{21} = \langle a_2, x_{12};\, [a_2^{s_{21}}, x_{12}^{s_{12}}] = 1\rangle$, $G_{31} = \langle a_3, x_{13}; [a_3^{s_{31}}, x_{13}^{s_{13}}] = 1\rangle, G_{42} = \langle a_4, x_{24};\, [a_4^{s_{42}}, x_{24}^{s_{24}}] = 1\rangle, G_{52} = \langle a_5, x_{25};\, [a_5^{s_{52}}, x_{25}^{s_{25}}] = 1\rangle$. Next we form the generalized free products $H_{21} = \langle G_1, G_{21};\, a_1 = x_{12}\rangle$, $H_{31} = \langle G_1, G_{31};\, a_1 = x_{13}\rangle$, $H_{42} = \langle G_{21}, G_{42};\, a_2 = x_{24}\rangle$ and $H_{52} = \langle G_{21}, G_{52};\, a_2 = x_{25}\rangle$. Now let T be the group with presentation obtained by taking the union of the presentations of the generalized free products H_{21}, H_{31}, H_{42} and H_{52}. We associated with T a linear graph, where the vertices are the groups $G_1, G_{21}, G_{31}, G_{42}$ and G_{52} with an edge joining each following pair of the vertices $\{G_1, G_{21}\}$, $\{G_1, G_{31}\}$, $\{G_{21}, G_{42}\}$ and $\{G_{21}, G_{52}\}$. Clearly there are no loops in this graph, and hence this graph is a tree. Therefore, T is called a tree product of the groups G_{ij} amalgamating the cyclic subgroups $\langle a_i\rangle$ of G_{ij} and $\langle x_{ik}\rangle$ of G_{ki}. Since T is isomorphic to G by Tietze transformations, by abuse of notation we say that G is a tree product.

Now we extend Theorem 4.1 to the following class of tree products. Let $G = \langle a_1, \ldots, a_n;\, [a_i^{s_{ij}}, a_j^{s_{ji}}] = 1,\ i \neq j,\ 1 \leq i, j \leq n\rangle$ be such that no sequences of relations of the form $[a_i^{s_{ij_1}}, a_{j_1}^{s_{j_1 i}}] = 1, [a_{j_1}^{s_{j_1 j_2}}, a_{j_2}^{s_{j_2 j_1}}] = 1, \ldots,\ [a_{j_r}^{s_{j_r i}}, a_i^{s_{ij_r}}] = 1$ for $r \geq 2$, occurs in G. We shall show that G can be considered as a finite tree product.

Suppose $[a_1^{s_{1p_1}}, a_{p_1}^{s_{p_11}}] = 1, [a_1^{s_{1p_2}}, a_{p_2}^{s_{p_21}}] = 1, \dots [a_1^{s_{1p_r}}, a_{p_r}^{s_{p_r1}}] = 1$ are all the relations in G which involve a_1. Then $p_i \neq p_j$ if $i \neq j$. Now for each $1 < i \leq r$, define the group $G_{p_i1} = \langle a_{p_i}, x_{1p_i}; [a_{p_i}^{s_{p_i1}}, x_{1p_i}^{s_{1p_i}}] = 1$ and then form the generalized free product $H_{p_i1} = \langle G_1, G_{p_i1}; a_1 = x_{1p_i}\rangle$.

Suppose inductively, for each relation of the form $[a_i^{s_{ij}}, a_j^{s_{ji}}] = 1,\ j \neq i,\ 1 \leq i, j \leq n\rangle$ which involve a_j in G, we have defined $G_{ij} = \langle a_i, x_{ji}; [a_i^{s_{ij}}, x_{ji}^{s_{ji}}] = 1,\ i \neq j,\ 1 \leq i, j \leq n\rangle$ and form the generalized free product $H_{ij} = \langle G_{jh}, G_{ij}; a_j = x_{ji}\rangle$. Now suppose that the relation $[a_i^{s_{ik}}, a_k^{s_{ki}}] = 1$, where $i \neq k$, $1 \leq i$, $k \leq n$, holds in G. Then we define $G_{ki} = \langle a_k, x_{ik}; [a_k^{s_{ki}}, x_{ik}^{s_{ik}}] = 1,\ i \neq k,\ 1 \leq i, k \leq n\rangle$ and form the generalized free product $H_{ki} = \langle G_{ij}, G_{ki}; a_i = x_{ik}\rangle$. We proceed in this manner until all the relations $[a_r^{s_{rt}}, a_t^{s_{tr}}] = 1$, where $r \neq t$, $1 \leq r$, $t \leq n$, of G have been considered.

Let T be the group with presentation obtained by presenting each of the generalized free product $\langle G_{ij}, G_{ki}; a_i = x_{ik}\rangle$ of the groups G_{ij} and G_{ki} with the cyclic subgroups $\langle a_i\rangle$ of G_{ij} and $\langle x_{ik}\rangle$ of G_{ki} amalgamated under $a_i = x_{ik}$ and then taking the union of these presentations. With T we associated a linear graph, where the vertices are the groups G_{ij} and each of whose edges joins two vertices G_{ij} and G_{ki} if $a_i = x_{ik}$. When this graph is a tree, T is called a tree product of the groups G_{ij} amalgamating the cyclic subgroups $\langle a_i\rangle$ of G_{ij} and $\langle x_{ik}\rangle$ of G_{ki}.

Since no sequences of relations of the form $[a_i^{s_{ij_1}}, a_{j_1}^{s_{j_1i}}] = 1, [a_{j_1}^{s_{j_1j_2}}, a_{j_2}^{s_{j_2j_1}}] = 1, \dots, [a_{j_r}^{s_{j_ri}}, a_i^{s_{ij_r}}] = 1$ for $r \geq 2$, occurs in G, then the linear graph associated with T contains no loop. Hence the linear graph is a tree and so T is a tree product. Furthermore, T is isomorphic to G by Tietze transformations. By abuse of notation we say that G is a tree product if T is a tree product. Now we can extend Theorem 4.1.

Theorem 4.3. *Let* $G = \langle a_1, \dots, a_n; [a_i^{s_{ij}}, a_j^{s_{ji}}] = 1,\ i \neq j,\ 1 \leq i, j \leq n\rangle$ *and suppose that* G *is a tree product. Then* G *is residually* p*-finite if and only if* $|s_{ij}| = p^{\alpha_{ij}}$ *and* $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.

Proof. As in the discussion above, we can assume that G is a tree product of the groups $G_{ij} = \langle a_i, x_{ji}; [a_i^{s_{ij}}, x_{ji}^{s_{ji}}] = 1\rangle$ amalgamating the cyclic subgroups $\langle a_i\rangle$ of G_{ij} and $\langle x_{ik}\rangle$ of G_{ki}.

Suppose G is residually p-finite. Then the subgroup G_{ij} is residually p-finite. By Theorem 4.1, $|s_{ij}| = p^{\alpha_{ij}}$ and $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.

Conversely, suppose $|s_{ij}| = p^{\alpha_{ij}}$ and $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$. By Theorem 4.1, G_{ij} is residually p-finite. Moreover, by Lemma 4.2, $\langle a_i \rangle$ and $\langle x_{ji} \rangle$ are p-closed in G_{ij}. Hence G is residually p-finite by Theorem 2.9. $\square$

To further extend Theorem 4.3, we have the following two lemmas.

Lemma 4.4. *Let A, B be residually p-finite groups, and let $\langle a \rangle, \langle b \rangle$ be infinite cyclic subgroups of A, B respectively. Suppose that $\langle a \rangle$ is p-closed in A and $\langle b \rangle$ is p-closed in B. If $|s| = p^{\alpha}$ and $|t| = p^{\beta}$, α, $\beta \geq 0$, then the group $G = \langle A, B;\, [a^s, b^t] = 1 \rangle$ is residually p-finite and $\langle a \rangle, \langle b \rangle$ are p-closed in G.*

Proof. Let $G_1 = \langle x, y;\, [x^s, y^t] = 1 \rangle$. Then G can be written as $G = A \underset{a=x}{*} G_1 \underset{y=b}{*} B$. By Theorem 4.1, G_1 is residually p-finite and, by Lemma 4.2, $\langle x \rangle$ and $\langle y \rangle$ are p-closed in G_1. Hence, G is residually p-finite by Theorem 2.9 and $\langle a \rangle, \langle b \rangle$ are p-closed in G by Lemma 2.7. $\square$

Next we extend Theorem 4.3 to the following class of tree products. Let A_i, $1 \leq i \leq n$, be residually p-finite groups and $\langle a_{ij} \rangle$ an infinite cyclic subgroup of A_i where a_{ij} is not a proper power of another element. Let $G = \langle A_1, \dots, A_n;\, [a_{ij}^{s_{ij}}, a_{ji}^{s_{ji}}] = 1,\ i \neq j,\ 1 \leq i, j \leq n \rangle$. Further, suppose that no sequences of relations of the form $[a_i^{s_{ij_1}}, a_{j_1}^{s_{j_1 i}}] = 1$, $[a_{j_1}^{s_{j_1 j_2}}, a_{j_2}^{s_{j_2 j_1}}] = 1, \dots, [a_{j_r}^{s_{j_r i}}, a_i^{s_{ij_r}}] = 1$ for $r \geq 2$, occurs in G. Then, as in the discussion before Theorem 4.3, we show that G can be considered as a finite tree product as follows. Let $G_1 = A_1$ and $G_{ij} = \langle A_i, x_{ji};\, [a_{ij}^{s_{ij}}, x_{ji}^{s_{ji}}] = 1,\ a_{ij} \in A_i \rangle$ for $i \neq j$, $1 \leq i,\ j \leq n$. Then G can be considered as a tree product of the groups $G_{ij} = \langle A_i, x_{ji};\, [a_{ij}^{s_{ij}}, x_{ji}^{s_{ji}}] = 1,\ a_{ij} \in A_i \rangle$ amalgamating the cyclic subgroups $\langle a_i \rangle$ of G_{ij} and $\langle x_{ik} \rangle$ of G_{ki}. Now we can prove the following theorem.

Theorem 4.5. *For each $1 \leq i \leq n$, let A_i be residually p-finite such that $\langle a_{ij} \rangle$ is an infinite cyclic subgroup of A_i where a_{ij} is not a proper power of another element. Further, suppose that, for each A_i, $\langle a_{ij} \rangle$ is p-closed in A_i. Let $G = \langle A_1, \dots, A_n;\ [a_{ij}^{s_{ij}}, a_{ji}^{s_{ji}}] = 1,\ i \neq j,\ 1 \leq i, j \leq n \rangle$, and suppose that G is a tree product. Then G is residually p-finite if and only if $|s_{ij}| = p^{\alpha_{ij}}$ and $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.*

Proof. If all the A_i are cyclic, the result follows from Theorem 4.3. So we may assume that there exists at least one i such that A_i is non-cyclic. As in the discussion in Theorem 4.3 and above, we can assume that G is a tree product of the groups $G_{ij} = \langle A_i, x_{ji};\ [a_{ij}^{s_{ij}}, x_{ji}^{s_{ji}}] = 1,\ a_{ij} \in A_i \rangle$ amalgamating the cyclic subgroups $\langle a_i \rangle$ of G_{ij} and $\langle x_{ik} \rangle$ of G_{ki}.

Suppose that G is residually p-finite. Then the subgroup $\langle a_{ij}, x_{ji};\ [a_{ij}^{s_{ij}}, x_{ji}^{s_{ji}}] = 1 \rangle$ of G_{ij} is residually p-finite and hence, by Theorem 4.1, $|s_{ij}| = p^{\alpha_{ij}}$ and $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.

Conversely, suppose that $|s_{ij}| = p^{\alpha_{ij}}$ and $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$. By Theorem 4.1, Lemmas 4.2 and 4.4, G_{ij} is residually p-finite and $\langle a_{ij} \rangle$, $\langle x_{ji} \rangle$ are p-closed in G_{ij}. Hence G is residually p-finite by Theorem 2.9. □

We apply Theorem 4.5 to free groups and finitely generated torsion-free nilpotent groups.

Corollary 4.6. *Let A_i, $1 \leq i \leq n$, be free groups or finitely generated torsion-free nilpotent groups, and let $\langle a_{ij} \rangle$ be a maximal cyclic subgroup or retract of A_i or $C_{A_i}(a_{ij}) = \langle a_{ij} \rangle$. Let $G = \langle A_1, \dots, A_n;\ [a_{ij}^{s_{ij}}, a_{ji}^{s_{ji}}] = 1,\ i \neq j,\ 1 \leq i, j \leq n \rangle$, and suppose that G is a tree product. Then G is residually p-finite if and only if $|s_{ij}| = p^{\alpha_{ij}}$ and $|s_{ji}| = p^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.*

Proof. Clearly a_{ij} is a non-proper power. Since A_i is residually p-finite and for each A_i which is non-cyclic, $\langle a_{ij} \rangle$ is p-closed in A_i, then the result follows from Theorem 4.5. □

In [**7**], Kim and McCarron also proved the following theorem:

Theorem 4.7 [**7**, Theorem 3.4]. *The group* $G = \langle x, y; x^{-s}y^tx^s = y^{-t}\rangle$ *is residually* p*-finite if and only if* $p = 2$ *and* $|s| = 2^\alpha$ *and* $|t| = 2^\beta$, α, $\beta \geq 0$.

As above we shall extend Theorem 4.7 with the help of the next lemma.

Lemma 4.8. *Let* $G = \langle x, y; x^{-s}y^tx^s = y^{-t}\rangle$. *If* $|s| = 2^\alpha$ *and* $|t| = 2^\beta$, α, $\beta \geq 0$, *then* $\langle x\rangle$ *and* $\langle y\rangle$ *are* 2*-closed in* G.

Proof. Since G is residually 2-finite and $\langle x\rangle$ is a retract of G, $\langle x\rangle$ is 2-closed in G by Lemma 2.4. To show $\langle y\rangle$ is 2-closed in G, we let $u \in G\backslash\langle y\rangle$. Clearly there is a homomorphism θ from G to $\langle x\rangle$ defined by $x\theta = x$ and $y\theta = 1$. Since $u\theta \neq 1$ and $\langle x\rangle$ is residually 2-finite, our result follows. □

Theorem 4.9. *Let* A_i, $1 \leq i \leq n$, *be residually* p*-finite groups, and* $\langle a_{ij}\rangle$ *is an infinite cyclic subgroup of* A_i *such that* a_{ij} *is not a proper power of another element. Further, suppose that, for each* A_i *which is non-cyclic,* $\langle a_{ij}\rangle$ *is* p*-closed in* A_i. *Let* $G = \langle A_1, \ldots, A_n; a_{ij}^{-s_{ij}}a_{ji}^{s_{ji}}a_{ij}^{s_{ij}} = a_{ji}^{-s_{ji}}, i \neq j, 1 \leq i, j \leq n\rangle$, *and suppose that* G *is a tree product. Then* G *is residually* p*-finite if and only if* $p = 2$ *and* $|s_{ij}| = 2^{\alpha_{ij}}$ *and* $|s_{ji}| = 2^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.

Corollary 4.10. *Let* A_i, $1 \leq i \leq n$, *be free groups or finitely generated torsion-free nilpotent groups, and let* $\langle a_{ij}\rangle$ *be a maximal cyclic subgroup or retract of* A_i *or* $C_{A_i}(a_{ij}) = \langle a_{ij}\rangle$. *Let* $G = \langle A_i; a_{ij}^{-s_{ij}}a_{ji}^{s_{ji}}a_{ij}^{s_{ij}} = a_{ji}^{-s_{ji}}, 1 \leq i \leq n\rangle$, *and suppose that* G *is a tree product. Then* G *is residually* p*-finite if and only if* $p = 2$ *and* $|s_{ij}| = 2^{\alpha_{ij}}$ *and* $|s_{ji}| = 2^{\beta_{ji}}$, α_{ij}, $\beta_{ji} \geq 0$.

Since elements of finite orders in residually p-finite groups are of orders p^α we restate Baumslag's result [**1**, Lemma 2]:

Theorem 4.11. *The group* $G = \langle x, y; (x^ly^m)^t = 1\rangle$, $t > 1$, *is residually* p*-finite if and only if* $t = p^\alpha$, $\alpha > 0$.

Lemma 4.12. *Let $G = \langle x, y;\, (x^l y^m)^t = 1\rangle$, $t > 1$. If $t = p^\alpha$, $\alpha > 0$, then $\langle x\rangle$ and $\langle y\rangle$ are p-closed in G.*

Proof. By Theorem 4.11, G is residually p-finite. We shall now consider the following cases.

Case 1. $|l| \neq 1 \neq |m|$. Note that $G = \langle x\rangle \underset{x^l=c}{*} G_0$ where $G_0 = \langle c, y;\, (cy^m)^t = 1\rangle$. Then $C_G(x) = \langle x\rangle$, and hence, $\langle x\rangle$ is p-closed in G by Lemma 2.5. Similarly, we can show that $\langle y\rangle$ is p-closed in G.

Case 2. $|l| = 1 = |m|$. Without loss of generality, we may assume that $G = \langle x, y;\, (xy)^t = 1\rangle$. Let $z = xy$. Note that $G = \langle x, z;\, z^t = 1\rangle = \langle x\rangle * \langle z;\, z^t = 1\rangle$. Clearly, $C_G(x) = \langle x\rangle$ and hence $\langle x\rangle$ is p-closed in G by Lemma 2.5. Similarly, $\langle y\rangle$ is p-closed in G.

Case 3. $|l| = 1,\ |m| \neq 1$. Without loss of generality, we may assume that $G = \langle x, y;\, (xy^m)^t = 1\rangle$. Let $z = xy^m$. Then $G = \langle y, z;\, z^t = 1\rangle = \langle y\rangle * \langle z;\, z^t = 1\rangle$. Now, $C_G(x) = C_G(zy^{-m}) = \langle zy^{-m}\rangle = \langle x\rangle$, $C_G(y) = \langle y\rangle$ and we are done by Lemma 2.5.

Case 4. $|l| \neq 1,\ |m| = 1$. This case is similar to Case 3.

As above we can extend Theorem 4.11 to the following theorem:

Theorem 4.13. *Let A_i, $1 \le i \le n$, be residually p-finite groups, and let $\langle a_{ij}\rangle$ be an infinite cyclic subgroup of A_i where a_{ij} is not a proper power of another element. Further, suppose that for each A_i which is non-cyclic, $\langle a_{ij}\rangle$ is p-closed in A_i. Let $G = \langle A_1, \dots, A_n;\, (a_{ij}^{l_{ij}} a_{ji}^{m_{ji}})^{t_{ij}} = 1,\ i \neq j,\ 1 \le i, j \le n\rangle$, $t_{ij} > 1$, and suppose that G is a tree product. Then G is residually p-finite if and only if $|t_{ij}| = p^{\gamma_{ij}}$, $\gamma_{ij} > 0$.*

Corollary 4.14. *Let A_i, $1 \le i \le n$, be free groups or finitely generated torsion-free nilpotent groups, and let $\langle a_{ij}\rangle$ be a maximal cyclic subgroup or retract of A_i or $C_{A_i}(a_{ij}) = \langle a_{ij}\rangle$. Let $G = \langle A_1, \dots, A_n;$ $(a_{ij}^{l_{ij}} a_{ji}^{m_{ji}})^{t_{ij}} = 1,\ i \neq j,\ 1 \le i, j \le n\rangle$, $t_{ij} > 1$, and suppose that G is a tree product. Then G is residually p-finite if and only if $|t_{ij}| = p^{\gamma_{ij}}$, $\gamma_{ij} > 0$.*

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