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LOCALLY EUCLIDEAN METRICS ON S^2 IN WHICH SOME OPEN BALLS ARE NOT CONNECTED

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ABSTRACT. Let $S_r^2 \subset \mathbf{R}^3$ be the 2-sphere with center O and radius r. For all $0 < s \leq 1$, we define a locally Euclidean metric d^s on S_r^2 which is equivalent to the Euclidean metric. These metrics are invariant under Euclidean isometries, and if $0 < s < 1$ then some open balls in (S_r^2, d^s) are not connected.

1. Introduction. Let $S_r^2 \subset \mathbf{R}^3$ be the 2-sphere with center $O = (0,0,0)$ and radius $r > 0$. We write d_E to denote the Euclidean metric on S_r^2. A metric d on the set S_r^2 is called *locally Euclidean* if, for all $P \in S_r^2$, there exists $t > 0$ such that

$$d(Q,R) = d_E(Q,R) \quad \text{for all} \quad Q,R \in B_t(P) = \{S \in S_r^2 \mid d(P,S) < t\}.$$

As usual, two metrics d_1 and d_2 on the set S_r^2 are called *equivalent* if the identity mapping of (S_r^2, d_1) onto (S_r^2, d_2) is a homeomorphism. Notice that the following trivial metric d_T is locally Euclidean but not equivalent to d_E.

$$d_T(P,Q) = \begin{cases} 0 & \text{if } P = Q \\ 1 & \text{if } P \neq Q. \end{cases}$$

In this paper we define a locally Euclidean metric d^s, which is equivalent to d_E and invariant under Euclidean isometries. Notice that the Euclidean metric d_E is trivially locally Euclidean. In fact, the metric d^1 will turn out to be the Euclidean metric d_E. Every open ball in (S_r^2, d_E) is connected. However, if $0 < s < 1$, then some open balls in (S_r^2, d^s) are not connected.

Suppose that $0 < s \leq 1$. Let $-P$ denote the antipodal point of $P \in S_r^2$. Let

$$\alpha = \sin^{-1}\left(\frac{\sqrt{2-s^2}-s}{2}\right), \quad \text{where} \quad 0 \leq \alpha < \pi/4.$$

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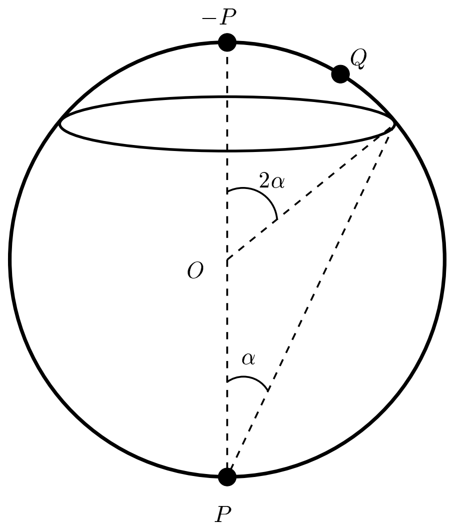

FIGURE 1. S_r^2.

Notice that $\sin\alpha$ is a decreasing function of s and hence so is α. We are going to use this function α to define the new metric d^s on S_r^2. For all $P, Q \in S_r^2$, let (see Figure 1)

$$d^s(P,Q) = \begin{cases} d_E(P,Q) & \text{if } \angle POQ \leq \pi - 2\alpha \\ 2rs + d_E(-P,Q) & \text{if } \angle POQ > \pi - 2\alpha, \end{cases}$$

where α is defined as above. Notice that if $s = 1$ then $d^1 = d_E$ and $d^1(P,-P) = 2r$ for all $P \in S_r^2$.

In the next section we will prove

Theorem 1.1. *For all $0 < s \leq 1$, d^s is a metric on S_r^2.*

Notice that if $d^s(P,Q) < 2rs$ then $d^s(P,Q) = d_E(P,Q)$ for all $P, Q \in S_r^2$. We write $B_t^s(P)$ to denote the open ball in (S_r^2, d^s) with center P and radius t.

Suppose that $P \in S_r^2$ and $Q, R \in B_{rs}^s(P)$. Since $d^s(Q,R) \leq d^s(Q,P) + d^s(P,R) < 2rs$, we have $d^s(Q,R) = d_E(Q,R)$. Therefore d^s is locally Euclidean for all $0 < s \leq 1$.

The following theorem which is proven in the next section implies that d^s is equivalent to d_E for all $0 < s \leq 1$.

Theorem 1.2. *$sd_E(P,Q) \leq d^s(P,Q) \leq d_E(P,Q)$ for all $P,Q \in S_r^2$.*

Not all locally Euclidean metrics on S_r^2, which are equivalent to d_E, are invariant under Euclidean isometries. However, we can show

Proposition 1.3. *d^s is invariant under any Euclidean isometry, for all $0 < s \leq 1$.*

Proof. Suppose that $\phi : S_r^2 \to S_r^2$ is an Euclidean isometry and $P,Q \in S_r^2$. Notice that $\angle\phi(P)O\phi(Q) = \angle POQ$. If $\angle POQ \leq \pi - 2\alpha$, then

$$d^s(\phi(P),\phi(Q)) = d_E(\phi(P),\phi(Q)) = d_E(P,Q) = d^s(P,Q).$$

Suppose that $\angle POQ > \pi - 2\alpha$. Since $2r = d_E(P,-P) = d_E(\phi(P),\phi(-P))$, we have $\phi(-P) = -\phi(P)$. Therefore

$$\begin{aligned} d^s(\phi(P),\phi(Q)) &= 2rs + d_E(-\phi(P),\phi(Q)) \\ &= 2rs + d_E(\phi(-P),\phi(Q)) = 2rs + d_E(-P,Q) = d^s(P,Q). \end{aligned} \qquad \square$$

Notice that the trivial metric d_T is invariant under any Euclidean isometry but not equivalent to d_E.

Suppose that $0 < s < 1$. Notice that $\sqrt{2r^2 - r^2s^2} > rs$. By the following theorem, some open balls in (S_r^2, d^s) are not connected.

Proposition 1.4. *Let $0 < s < 1$ and $2rs < t < \sqrt{2r^2 - r^2s^2} + rs$. Let $P \in S_r^2$ be arbitrary. Then the open ball $B_t^s(P)$ is not connected.*

Proof. Let $P \in S_r^2$, $U = B_t^1(P)$ and $V = B_{t-2rs}^1(-P)$. We will show that U and V are nonempty disjoint open sets in (S_r^2, d^s) whose union is $B_t^s(P)$. Notice that $P \in U$, $-P \in V$, hence U and V are nonempty by Theorem 1.2. Since d^s is equivalent to $d_E = d^1$, U and V are open sets in (S_r^2, d^s).

If $Q \in U \cap V$, then $4r^2 = d_E(P, -P)^2 = d_E(P, Q)^2 + d_E(Q, -P)^2 < t^2 + (t - 2rs)^2 < 4r^2$. This is a contradiction. Therefore $U \cap V = \varnothing$.

Suppose that $Q \in B^s_t(P)$. If $d^s(P, Q) = d_E(P, Q)$, then $d_E(P, Q) < t$. If $d^s(P, Q) \neq d_E(P, Q)$, then $d_E(-P, Q) = d^s(P, Q) - 2rs < t - 2rs$. Therefore $B^s_t(P) \subset U \cup V$.

If $Q \in U$ then, by Theorem 1.2, we have $Q \in B^s_t(P)$. Suppose that $Q \in V$. Since $d_E(-P, Q) < t - 2rs < \sqrt{2r^2 - r^2s^2} - rs$, by Lemma 2.1 in the next section, we have $\angle(-P)OQ < 2\alpha$. Therefore, $\angle POQ > \pi - 2\alpha$ and $d^s(P, Q) = 2rs + d_E(-P, Q) < t$. Hence, $Q \in B^s_t(P)$. Thus, $U \cup V \subset B^s_t(P)$. □

This paper is motivated by the Poincaré conjecture. In his work on the Poincaré conjecture, the author was interested in discontinuous functions from (S^2_1, d_E) to the closed interval $[0, a]$. Any countable-to-one function from (S^2_1, d_E) to $[0, a]$ is discontinuous. Let B^3 be the closed unit ball in $\mathbf{R}^3$ and d_E the Euclidean metric on B^3. Define locally Euclidean metrics on the set B^3 as on S^2_r. Using the metric d^s on S^2_r, the author [**1**] constructed a family of pseudo metrics on B^3. Some of these pseudo metrics are locally Euclidean metrics which are equivalent to d_E, and in which some open balls are not connected. As an application of this construction, the author obtained a result on countable-to-one functions from (S^2_1, d_E) to $[0, a]$, see [**1**] for details.

2. Proof of theorems. In this section we prove Theorem 1.1 and Theorem 1.2. Recall that $0 \leq \alpha < \pi/4$. If $\angle POQ > \pi - 2\alpha$, then $\angle(-P)OQ = \pi - \angle POQ < 2\alpha < \pi - 2\alpha$ and hence

$$d_E(P, Q) > d_E(-P, Q). \tag{1}$$

Since $d_E(P, Q)^2 = 2r^2 - 2r^2 \cos \angle POQ$, $d_E(P, Q)$ is an increasing function of $\angle POQ$ on the interval $0 \leq \angle POQ \leq \pi$. We will make use of the following lemma.

Lemma 2.1. *If* $\angle POQ = \pi - 2\alpha$, *then* $d_E(P, Q) = \sqrt{2r^2 - r^2s^2} + rs$. *If* $\angle POQ = 2\alpha$ *then* $d_E(P, Q) = \sqrt{2r^2 - r^2s^2} - rs$.

Proof. Suppose that $\angle POQ = \pi - 2\alpha$. Since $0 \le \alpha < \pi/4$ and

$$\cos^2\alpha = 1 - \sin^2\alpha = 1 - \frac{2 - 2s\sqrt{2-s^2}}{4} = \left(\frac{\sqrt{2-s^2}+s}{2}\right)^2,$$

we have

$$\cos\alpha = \frac{\sqrt{2-s^2}+s}{2}. \tag{2}$$

Therefore $d_E(P,Q) = 2r\cos\alpha = r\left(\sqrt{2-s^2}+s\right) = \sqrt{2r^2 - r^2s^2} + rs$. Note that

$$d_E(P,Q) - d_E(-P,Q) = 2r(\cos\alpha - \sin\alpha) = 2rs. \tag{3}$$

Suppose that $\angle POQ = 2\alpha$. Since $\angle(-P)OQ = \pi - 2\alpha$, from equation (3), we have

$$\begin{aligned} d_E(P,Q) &= d_E(-P,Q) - 2rs = \sqrt{2r^2 - r^2s^2} + rs - 2rs \\ &= \sqrt{2r^2 - r^2s^2} - rs. \qquad \square \end{aligned}$$

We will also make use of the following lemma.

Lemma 2.2. *If* $P, Q, R, S \in S_r^2$ *and* $\angle POQ + \angle ROS \ge 2\alpha$, *then*

$$d_E(P,Q) + d_E(R,S) \ge \sqrt{2r^2 - r^2s^2} - rs.$$

Proof. Notice that we may assume $\angle ROS \le \angle POQ$. Due to Lemma 2.1, we may assume that $0 < \angle ROS \le \angle POQ < 2\alpha$. Since $0 \le \alpha < \pi/4$, we have $0 < \angle ROS \le \angle POQ < \pi/2$. Consider the great circle on S_r^2 through the two points P and Q. On this great circle, there exist two points S_0 and S_1 such that $\angle QOS_0 = \angle QOS_1 = \angle ROS$, $\angle POQ + \angle QOS_0 = \angle POS_0$ and $\angle POQ - \angle QOS_1 = \angle POS_1$, see Figure 2.

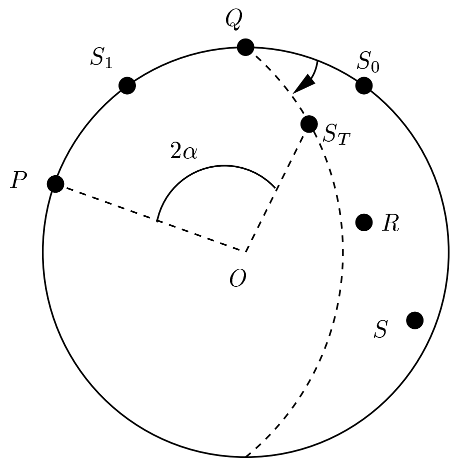

FIGURE 2. $d_E(Q,S_0) = d_E(Q,S_1) = d_E(Q,S_T) = d_E(R,S)$.

Fixing Q, rotate the arc QS_0 clockwise toward the arc QS_1 in the time interval $[0,1]$, see Figure 2. Let QS_t be the rotating arc at time t. Notice that $\angle POS_t$ is a continuous function on $[0,1]$,

$$\angle POS_0 = \angle POQ + \angle QOS_0 = \angle POQ + \angle ROS \geq 2\alpha$$

and

$$\angle POS_1 < 2\alpha.$$

Therefore, by the intermediate value theorem, there exists $S_T \in S_r^2$ such that $\angle POS_T = 2\alpha$ for some $T \in [0,1]$. From Lemma 2.1, we have $d_E(P,S_T) = \sqrt{2r^2 - r^2s^2} - rs$. Since $d_E(Q,S_T) = d_E(R,S)$, we have

$$\begin{aligned} d_E(P,Q) + d_E(R,S) &= d_E(P,Q) + d_E(Q,S_T) \geq d_E(P,S_T) \\ &= \sqrt{2r^2 - r^2s^2} - rs. \qquad \square \end{aligned}$$

We will need the following theorem, see [**2**, Chapter VII] for a proof.

Theorem 2.3. *For $P,Q \in S_r^2$, let $\rho(P,Q) = \angle POQ$. Then ρ is a metric on S_r^2.*

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. It is clear that d^s is nonnegative. Since $\angle POP = 0$, we have $d^s(P,P) = d_E(P,P) = 0$ for all $P \in S_r^2$. If $d^s(P,Q) = 0$, then $d_E(P,Q) = d^s(P,Q) = 0$ and hence $P = Q$.

Suppose that $P, Q \in S_r^2$. If $\angle POQ > \pi - 2\alpha$, then

$$\begin{aligned} d^s(Q,P) &= 2rs + d_E(-Q,P) \\ &= 2rs + d_E(Q,-P) \\ &= 2rs + d_E(-P,Q) \\ &= d^s(P,Q). \end{aligned}$$

If $\angle POQ \leq \pi - 2\alpha$, then $d^s(Q,P) = d_E(Q,P) = d_E(P,Q) = d^s(P,Q)$.

Suppose that $P, Q, R \in S_r^2$. If $\angle POQ$, $\angle QOR$, $\angle ROP \leq \pi - 2\alpha$, then the triangle inequality of d^s is trivial from that of d_E.

Suppose that only one angle, e.g. $\angle POQ$, is greater than $\pi - 2\alpha$. Then $d^s(Q,R) = d_E(Q,R)$ and $d^s(R,P) = d_E(R,P)$. Since $\angle(-P)OQ = \pi - \angle POQ < 2\alpha$, from Lemma 2.1, we have

$$\begin{aligned} d^s(Q,R) + d^s(R,P) &= d_E(Q,R) + d_E(R,P) \\ &\geq d_E(P,Q) \\ &> \sqrt{2r^2 - r^2 s^2} + rs \\ &> d_E(-P,Q) + 2rs \\ &= d^s(P,Q). \end{aligned}$$

By Theorem 2.3, $\angle(-P)OQ + \angle QOR + \angle ROP \geq \angle(-P)OP = \pi$. Since $\angle QOR \leq \pi - 2\alpha$, we have $\angle(-P)OQ + \angle ROP \geq 2\alpha$. Hence, from Lemma 2.2, we have $d_E(-P,Q) + d_E(R,P) \geq \sqrt{2r^2 - r^2 s^2} - rs$. Therefore,

$$\begin{aligned} d^s(P,Q) + d^s(R,P) &= 2rs + d_E(-P,Q) + d_E(R,P) \\ &\geq 2rs + \sqrt{2r^2 - r^2 s^2} - rs \\ &\geq d_E(Q,R) \\ &= d^s(Q,R). \end{aligned}$$

Similarly we can show that $d^s(P,Q) + d^s(Q,R) \geq d^s(R,P)$.

If two angles, e.g., $\angle POQ$ and $\angle QOR$, are greater than $\pi - 2\alpha$, then

$$\begin{aligned} d^s(P,Q) &= 2rs + d_E(-P,Q) = 2rs + d_E(P,-Q) \\ d^s(Q,R) &= 2rs + d_E(-Q,R) \\ d^s(R,P) &= d_E(R,P). \end{aligned}$$

Therefore the triangle inequality of d^s is trivial from that of d_E.

If all of the three angles are greater than $\pi - 2\alpha$, then from equation (1), we have

$$\begin{aligned} d^s(P,Q) + d^s(Q,R) &= 4rs + d_E(-P,Q) + d_E(-Q,R) \\ &= 4rs + d_E(P,-Q) + d_E(-Q,R) \\ &\geq 4rs + d_E(P,R) \\ &> 4rs + d_E(-P,R) \\ &> d^s(P,R) \\ &= d^s(R,P). \end{aligned}$$

Similarly we can show that $d^s(Q,R) + d^s(R,P) \geq d^s(P,Q)$ and $d^s(R,P) + d^s(P,Q) \geq d^s(Q,R)$.

We now prove Theorem 1.2.

Proof of Theorem 1.2. Let $P, Q \in S_r^2$. We may assume that $\angle POQ > \pi - 2\alpha$ and $s \neq 1$. Let $\angle POQ = \pi - 2\beta$. Notice that $0 \leq \beta < \alpha < \pi/4$ and $\cos x - \sin x$ is decreasing on $0 \leq x < \pi/4$. Since $d_E(P,Q) = 2r\cos\beta$ and $d_E(-P,Q) = 2r\sin\beta$, from equation (2), we have

$$\begin{aligned} d_E(P,Q) - d^s(P,Q) &= d_E(P,Q) - 2rs - d_E(-P,Q) \\ &= 2r\cos\beta - 2rs - 2r\sin\beta \\ &= 2r(\cos\beta - \sin\beta - s) \\ &\geq 2r(\cos\alpha - \sin\alpha - s) \\ &= 0 \\ d^s(P,Q) - sd_E(P,Q) &= 2rs + d_E(-P,Q) - sd_E(P,Q) \\ &= 2rs + 2r\sin\beta - 2rs\cos\beta \\ &= 2rs(1-\cos\beta) + 2r\sin\beta \\ &\geq 0. \qquad \square \end{aligned}$$

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