ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 37, Number 1, 2007

CLASSICAL OPERATORS ON MIXED-NORMED SPACES WITH PRODUCT WEIGHTS

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ABSTRACT. We prove norm inequalities for a variant of the Hardy-Littlewood maximal function on weighted mixed-norm spaces. These results are applied to singular integral operators, including the double Hilbert transform.

1. Introduction. Let f be a locally integrable function on $\mathbf{R}^n$. We define the *Hardy-Littlewood maximal function* Mf of f by

$$Mf(x) = \sup \frac{1}{|Q|} \int_Q |f(y)| \, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbf{R}^n$ containing x. In 1930, Hardy and Littlewood proved that this operator is bounded on L^p for $1 < p \leq \infty$. This result has been generalized in many directions. Fefferman and Stein [**4**] proved a vector-valued version:

(1.1)
$$\left(\int_{\mathbf{R}^n} \left(\sum_{j=1}^{\infty} |Mf_j(x)|^q \right)^{p/q} dx \right)^{1/p} \leq C \left(\int_{\mathbf{R}^n} \left(\sum_{j=1}^{\infty} |f_j(x)|^q \right)^{p/q} dx \right)^{1/p}$$

for $1 < p,\ q < \infty$. A key element of their proof is a weighted-norm inequality:

$$\left(\int_{\mathbf{R}^n} |Mf(x)|^p \, w(x) \, dx \right)^{1/p} \leq C \left(\int_{\mathbf{R}^n} |f(x)|^p \, Mw(x) \, dx \right)^{1/p}$$

which holds for any $p > 1$. If there is a constant $C > 0$ so that $Mw(x) \leq Cw(x)$, which is known as the A_1 condition, then we have

$$\left(\int_{\mathbf{R}^n} |Mf(x)|^p \, w \, dx \right)^{1/p} \leq C \left(\int_{\mathbf{R}^n} |f(x)|^p \, w \, dx \right)^{1/p} = \|f\|_{p,w}.$$

2000 AMS *Mathematics Subject Classification.* Primary 42B25, 42E30, Secondary 42B20.
Key words and phrases. Mixed-norm space, weight, maximal function.
Received by the editors on June 26, 2003, and in revised form on Oct. 10, 2004.

Muckenhoupt [**11**] characterized the weights for which the Hardy-Littlewood maximal function is bounded on L^p_w, $1 < p < \infty$, by introducing the A_p condition:

$$\left(\int_Q w(x)\ dx\right)\left(\int_Q w(x)^{1-p'}\ dx\right)^{p-1} \leq C\,|Q|^p\,.$$

The smallest such C is called the A_p norm of w, denoted by $\|w\|_{A_p}$. See, for example, Chapter IV in [**8**] and Chapter V in [**17**]. These results were unified by Andersen and John [**1**] who proved

$$\left(\int_{\mathbf{R}^n}\left(\sum_{j=1}^{\infty}|Mf_j(x)|^q\right)^{p/q} w(x)\ dx\right)^{1/p} \leq C\left(\int_{\mathbf{R}^n}\left(\sum_{j=1}^{\infty}|f_j(x)|^q\right)^{p/q} w(x)\ dx\right)^{1/p} \tag{1.2}$$

for $1 < p$, $q < \infty$ and $w \in A_p$.

The purpose of this paper is to study such operators on weighted mixed-norm spaces. Mixed-norm spaces were developed by Benedek and Panzone in [**2**]. Consider the space $\mathbf{R}^d = \mathbf{R}^n \times \mathbf{R}^m$. Let w be a nonnegative, locally integrable function; we call such a function a *weight*. Let $1 \leq p$, $q < \infty$. We say a measurable function f is in the weighted $L^p(L^q)$-space, $L^p(L^q(w))$, if the norm

$$\|f\|_{L^p(L^q(w))} = \left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\, w(x,y)\ dy\right)^{p/q} dx\right)^{1/p}$$

is finite.

We consider weights that satisfy a condition we call $A_p(A_q)$ that generalizes the A_p condition; see Definition 2. Our condition $A_p(A_q)$ reduces to the well-known A_p condition on two-parameter rectangles $R = Q \times Q'$ when $q = p$. It is interesting to note that the $A_p(A_q)$ spaces do not satisfy the nesting properties that the A_p spaces do, as we discuss below.

The Hardy-Littlewood maximal function is a supremum of averages over cubes. The strong maximal function is an average over oriented rectangles. We consider a second variant, more adapted to mixed-norm spaces, defined in terms of rectangles that are products of cubes. We will call this operator the strong maximal function.

Definition 1. Let f be a measurable function on $\mathbf{R}^n \times \mathbf{R}^m$ and $(x, y) \in \mathbf{R}^n \times \mathbf{R}^m$. Define the *strong maximal function*, $M_S f$, by

$$M_S f\,(x,y) = \sup_{R \ni (x,y)} \frac{1}{|R|} \int_R |f\,(s,t)|\; ds\, dt,$$

where $R = Q \times Q'$ and $Q \subset \mathbf{R}^n$ and $Q' \subset \mathbf{R}^m$ are cubes.

Our main result characterizes the weights w, which can be written as a product of weights $u\,(x)$, $x \in \mathbf{R}^n$, and $v\,(y)$, $y \in \mathbf{R}^m$, for which this maximal function is bounded on $L^p\,(L^q\,(w))$. The following theorem is a weighted version of a result found in [**7**].

Theorem 1. *Let $1 < p,\ q < \infty$ and $w\,(x,y) = u\,(x)\,v\,(y)$. Then there is a constant C, independent of f and depending only on the $A_p\,(A_q)$ norm of w, such that*

$$\left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} |M_S f\,(x,y)|^q\, w\,(x,y)\; dy \right)^{p/q} dx \right)^{1/p} \leq C \left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}^m} |f\,(x,y)|^q\, w\,(x,y)\; dy \right)^{p/q} dx \right)^{1/p}$$

if, and only if, $w \in A_p\,(A_q)$.

We observe that the constant C is bounded below by the $A_p\,(A_q)$ norm of w, an easy consequence of the definitions, and above by a constant that depends only on the $A_p\,(A_q)$ norm of w. However, the techniques employed only show an upper bound that is a power of the $A_p\,(A_q)$ norm of w, and not necessarily the $A_p\,(A_q)$ norm itself, as in the case of the Hardy-Littlewood maximal function.

If $p = q = \infty$, the norm inequality for M_S holds if, and only if, the weight is strictly positive almost everywhere or equal to 0 almost everywhere. If $p = q = 1$, it is known that M_S satisfies a weak-type inequality if, and only if, the weight satisfies an A_1 condition (defined by M_S in place of M). For $p = 1$ and $1 < q < \infty$, a version of inequality (2) of Theorem 1 of [**4**] holds. No such norm inequality holds for $q < p = \infty$ or $1 = q < p$. See [**17**, pp. 51 and 75]. These results remain open when w is not a product weight.

Since the strong maximal function is known to bound the maximal function in the x variable, setting $v = 1$, we obtain $L^p\,(L^q)$ versions of the vector-valued inequalities (1.1) and (1.2). See [**1**] and [**4**].

The proof of Theorem 1 is based on extrapolation techniques developed by Garcia-Cuerva and Rubio de Francia in [**14**] and Chapter IV in [**8**, pp. 433–450]. These techniques allow us to obtain weighted norm inequalities for singular integral operators studied in [**5**] and [**6**]. In particular, we characterize the product weights for which the double Hilbert transform defines a bounded operator on $L^p\,(L^q\,(w))$, generalizing the result in [**7**].

The paper is divided into four sections. In the second section, we discuss the weights in $A_p\,(A_q)$. An extrapolation result is proved in Section 3 and used to prove Theorem 1. Applications to singular integral operators are derived in Section 4.

2. $A_p\,(A_q)$ weights.

Let w be a nonnegative, locally integrable function defined on $\mathbf{R}^d = \mathbf{R}^n \times \mathbf{R}^m$. We will be interested in the following generalization of the A_p condition.

Definition 2. We say that a nonnegative function w is in $A_p\,(A_q)$, $1 < p,\, q < \infty$, if

(2.1)
$$\left(\int_Q \left(\int_{Q'} w\,(x,y)\;dy\right)^{p/q} dx\right)\left(\int_Q \left(\int_{Q'} w\,(x,y)^{1-q'}\;dy\right)^{p'/q'} dx\right)^{p-1} \le C\,|Q \times Q'|^p\,,$$

where $Q \subset \mathbf{R}^n$ and $Q' \subset \mathbf{R}^m$ are cubes (of possibly different edge lengths). We call the smallest such constant the $A_p\,(A_q)$ norm of w, and denote it by $\|w\|_{A_p(A_q)}$.

It follows immediately from the definition that $w \in A_p\,(A_q)$ if, and only if, $w^{1-q'} \in A_{p'}\,(A_{q'})$ and $\|w^{1-q'}\|_{A_{p'}(A_{q'})} = \|w\|_{A_p(A_q)}^{1/(p-1)}$.

Weights in the $A_p\,(A_q)$ spaces satisfy the following characterization. (See, for example, [**8**, p. 400] and [**17**, p. 195].)

Lemma 1. *A weight $w \in A_p\,(A_q)$ if and only if there is a constant C so that*

$$\begin{aligned}(2.2)\qquad &\left(\frac{1}{|Q\times Q'|}\int_Q\int_{Q'} f\right)^p \\ &\qquad \le \frac{C}{\int_Q \left(\int_{Q'} w\,(x,y)\;dy\right)^{p/q} dx}\int_Q\left(\int_{Q'} |f|^q\,w\,dy\right)^{p/q} dx.\end{aligned}$$

for all measurable $f \ge 0$ and $Q\times Q' \subset \mathbf{R}^n\times\mathbf{R}^m$. The smallest C satisfying (2.2) *is equal to $\|w\|_{A_p(A_q)}$.*

Proof. Suppose that $w \in A_p\,(A_q)$. Using the $L^p\,(L^q)$ version of Hölder's inequality [**2**],

$$\begin{aligned}\frac{1}{|Q\times Q'|}\int_Q\int_{Q'} f &= \frac{1}{|Q\times Q'|}\int_Q\int_{Q'} f w^{1/q} w^{-1/q} \\ &\le \frac{1}{|Q\times Q'|}\left(\int_Q\left(\int_{Q'} f^q w\,dy\right)^{p/q} dx\right)^{1/p} \\ &\qquad \times\left(\int_Q\left(\int_{Q'} w^{1-q'}\,dy\right)^{p'/q'} dx\right)^{1/p'}.\end{aligned}$$

Raising both sides to the pth power and applying the $A_p\,(A_q)$ condition yields

$$\begin{aligned}&\left(\frac{1}{|Q\times Q'|}\int_Q\int_{Q'} f\right)^p \\ &\qquad \le \frac{\|w\|_{A_p(A_q)}}{\int_Q\left(\int_{Q'} w\,(x,y)\;dy\right)^{p/q} dx}\left(\int_Q\left(\int_{Q'} f^q w\,dy\right)^{p/q} dx\right).\end{aligned}$$

This shows that $A_p(A_q)$ implies (2.2) with $C \le \|w\|_{A_p(A_q)}$.

To show that (2.2) implies $w \in A_p(A_q)$, we rewrite (2.2) as

$$\left(\int_Q \left(\int_{Q'} w(x,y)\, dy \right)^{p/q} dx \right) \left(\int_Q \int_{Q'} f\, dy\, dx \right)^p \le C\, |Q \times Q'|^p \left(\int_Q \left(\int_{Q'} f^q w\, dy \right)^{p/q} dx \right).$$

Since the function

$$f(x,y) = \chi_Q(x)\, \chi_{Q'}(y)\, w(x,y)^{1-q'} \left\| \chi_{Q'}(\cdot)\, w(x,\cdot)^{1-q'} \right\|_1^{(q-p)/q(p-1)}$$

satisfies

$$\int_Q \int_{Q'} f\, dy\, dx = \int_Q \left(\int_{Q'} f^q w\, dy \right)^{p/q} dx = \int_Q \left(\int_{Q'} w^{1-q'}\, dy \right)^{p'/q'} dx,$$

we see that $w \in A_p(A_q)$ with $\|w\|_{A_p(A_q)} \le C$. □

When $p = q$, the $A_p(A_q)$ condition reduces to the A_p condition over the set of rectangles $\mathcal{R} = \{Q \times Q' : Q \subset \mathbf{R}^n \text{ and } Q' \subset \mathbf{R}^m\}$. We will denote $A_p(A_p)$ by $A_{p,\mathcal{R}}$ when we wish to point out the underlying rectangles. Using the Lebesgue differentiation theorem, such weights satisfy uniform A_p conditions over $\mathbf{R}^n$ and $\mathbf{R}^m$. An analogous result holds for $A_p(A_q)$ weights.

Lemma 2. *If $w \in A_p(A_q)$ then, for almost every $(x,y) \in \mathbf{R}^n \times \mathbf{R}^m$, $w(x,\cdot) \in A_q(\mathbf{R}^m)$ and $w(\cdot,y)^{p/q} \in A_p(\mathbf{R}^n)$. Further, $\|w(\cdot,y)^{p/q}\|_{A_p} \le \|w\|_{A_p(A_q)}$ and $\|w(x,\cdot)\|_{A_q} \le \|w\|_{A_p(A_q)}^{q/p}$.*

Proof. Let $C = \|w\|_{A_p(A_q)}$. Fix a cube $Q \subset \mathbf{R}^n$. We want to show that

$$\left(\int_Q w(x,y)^{p/q}\, dx \right) \left(\int_Q \left(w(x,y)^{p/q} \right)^{1-p'} dx \right)^{(p-1)} \le C\, |Q|^p \tag{2.3}$$

for almost every $y \in \mathbf{R}^m$. Let $Q' \subset \mathbf{R}^m$ be a cube. Since $p = p/q + (p - p/q) = p/q + (q-1)\,p/q$, we can rewrite the $A_p\,(A_q)$-condition as

$$\left(\int_Q \left(\frac{1}{|Q'|}\int_{Q'} w\,(x,y)\;dy\right)^{p/q} dx\right) \times \left(\int_Q \left(\frac{1}{|Q'|}\int_{Q'} w\,(x,y)^{1-q'}\;dy\right)^{p'/q'} dx\right)^{(p-1)} \le C\,|Q|^p\,.$$

By the Lebesgue differentiation theorem, we get (2.3) for almost every y, depending on Q. By considering only cubes with rational vertices and taking limits, we see that for almost every $y \in \mathbf{R}^m$, $w\,(\cdot,y)^{p/q} \in A_p\,(\mathbf{R}^n)$ with a norm bounded by the $A_p\,(A_q)$ norm of w.

A similar argument shows that $w(x,\cdot) \in A_q(\mathbf{R}^m)$ and $\|w(x,\cdot)\|_{A_q} \le \|w\|^{q/p}_{A_p(A_q)}$. Note that, for fixed $y \in \mathbf{R}^m$, $w(\cdot,y)$ and $w(\cdot,y)^{1-q'}$ need not be locally integrable in x, as we mention below. However, both $(\int_{Q'} w\,(x,y)\;dy)^{p/q}$ and $(\int_{Q'} w\,(x,y)^{1-q'}\;dy)^{p'/q'}$ are locally integrable in x, which allows the use of the Lebesgue differentiation theorem. $\square$

Now, suppose that $w\,(x,y) = u\,(x)\,v\,(y)$ with $u^{p/q} \in A_p$ and $v \in A_q$. It then follows that

$$\begin{aligned}\left(\int_Q \left(\int_{Q'} w\,(x,y)\;dy\right)^{p/q} dx\right)\left(\int_Q \left(\int_{Q'} w\,(x,y)^{1-q'}\;dy\right)^{p'/q'} dx\right)^{(p-1)} \\ \le \|u^{p/q}\|_{A_p}\,|Q|^p\,\|v\|^{p/q}_{A_q}\,(|Q'|^q)^{p/q} \\ = \|u^{p/q}\|_{A_p}\,\|v\|^{p/q}_{A_q}\,|Q \times Q'|^p\,.\end{aligned}$$

Thus, $w \in A_p\,(A_q)$. We have

Lemma 3. *The weight $w\,(x,y) = u\,(x)\,v\,(y) \in A_p\,(A_q)$ if and only if $u^{p/q} \in A_p$ and $v \in A_q$. Further, $\|u^{p/q}\|_{A_p} \le \|w\|_{A_p(A_q)}$, $\|v\|_{A_q} \le \|w\|^{q/p}_{A_p(A_q)}$ and $\|w\|_{A_p(A_q)} \le \|u^{p/q}\|_{A_p}\|v\|^{p/q}_{A_q}$.*

Using Hölder's inequality, one sees that $A_p \subset A_{p+\varepsilon}$ for any $\varepsilon > 0$. A deeper result is that given any $w \in A_p$, there is an $\varepsilon > 0$ such that $w \in A_{p-\varepsilon}$. For the $A_p\,(A_q)$ spaces, we have

Proposition 1. *Suppose* $1 < p < \infty$, $1 < q < t < \infty$, *and* $w \in A_p(A_q)$. *Then,* $w \in A_p(A_t)$ *and* $\|w\|_{A_p(A_t)} \le \|w\|_{A_p(A_q)}^{q/t}$.

In fact, if w is a product weight, it follows from results about A_p weights that for $w \in A_p(A_q)$ there is a $t < q$ such that $w \in A_p(A_t)$, though we will not need this result.

While the $A_p(A_q)$ spaces are nested for varying q, no such result holds for the parameter p, as the following example shows.

Example 1. If $1 < s,\ p,\ q < \infty$ and $s \ne p$, then there is a weight $w \in A_p(A_q)$ such that $w \notin A_s(A_q)$. In fact, consider the product weight $w(x,y) = |x|^\alpha$, which is in $A_p(A_q)$ if, and only if, $(|x|^\alpha)^{p/q} \in A_p$ or, equivalently,

$$\frac{-nq}{p} < \alpha < \frac{nq}{p'}.$$

If $s \ne p$, it is easy to see that neither of the intervals $(-(nq/p), (nq/p'))$ and $((-nq/s), (nq/s'))$ is contained in the other.

Further, it should be mentioned that weights in $A_p(A_q)$ need not be locally integrable in one variable with the other variable held fixed. In fact, $w(x,y) = |x|^{-n} \in A_p(A_q)$ if $1 < p < q < \infty$. This shows that it is not necessarily the case that $w(\cdot, y_0) \in A_p$. In fact, in this case, $w(\cdot, y_0) \notin A_t$ for every $y_0 \in \mathbf{R}^m$ and $t > 1$.

3. Extrapolation. The weighted mixed norm inequality is an immediate consequence of the following extrapolation theorem.

Theorem 2. *Let* T *be a sublinear operator. Let* $1 \le s < \infty$ *and* $1 < q, p < \infty$. *Suppose that* T *is bounded on* L_w^s *for every* $w \in A_{s,\mathcal{R}}$, *with a norm that depends only on* $\|w\|_{A_{s,\mathcal{R}}}$. *Then, if* $w(x,y) = u(x)\,v(y)$ *and* $w \in A_p(A_q)$, T *is bounded on* $L^p(L^q(w))$, *with a norm that depends only on* $\|w\|_{A_p(A_q)}$.

The proof of this result relies on the following lemma proved by Rubio de Francia. (See Lemma 5.18 in [**8**, p. 447]).

Lemma 4. *Let $w \in A_\alpha$ for $1 < \alpha < \infty$. Suppose that $1 \le \beta < \infty$ with $\beta \ne \alpha$ and define by γ by $1/\gamma = |1 - (\beta/\alpha)|$. Then, for every nonnegative function $g \in L^\gamma_w$ there exists a $G \in L^\gamma_w$ such that*

(1) $g(x) \le G(x)$;

(2) $\|G\|_{\gamma,w} \le C\,\|g\|_{\gamma,w}$; *the constant C depends only on the exponent β;*

(3) *Either*:

(a) $G\,w \in A_\beta$ *if* $\beta \le \alpha$;

(b) $G^{-1}\,w \in A_\beta$ *if* $\alpha < \beta$.

In either case, the A_β norm of Gw or $G^{-1}w$ depends only on the A_β norm of w, and not on w itself.

We may now prove Theorem 2.

Proof. Observe first that, under the assumptions on T, T is bounded on L^s_w for every $w \in A_{s,\mathcal{R}}$ for every s, $1 < s < \infty$, with a norm that depends only on $\|w\|_{A_{s,\mathcal{R}}}$ by Theorem 5.19 in [**8**, p. 448]. Consequently, the result is true when $1 < p = q < \infty$ (since $A_p(A_p) = A_{p,\mathcal{R}}$).

Suppose that $1 < q < p < \infty$. Let $r = p/q > 1$. Then, there exists a nonnegative function $g \in L^{r'}_{u^{p/q}}(\mathbf{R}^n)$ with $\|g\|_{r',u^{p/q}} = 1$, such that

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|Tf(x,y)|^q\,v(y)\;dy\right)^{p/q}u^{p/q}(x)\;dx\right)^{q/p} = \int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|Tf(x,y)|^q\,v(y)\;dy\right)g(x)\,u^{p/q}(x)\;dx = \Phi.$$

By Lemma 4 with $\alpha = p$, $\beta = q < \alpha$ and $\gamma = r'$, there is a function $G \in L^{r'}_{u^{p/q}}(\mathbf{R}^n)$ such that $g(x) \le G(x)$, $Gu^{p/q} \in A_q(\mathbf{R}^n)$ and the norm of G is bounded by a constant. Since $v \in A_q(\mathbf{R}^m)$, it then follows that the weight W defined by $W(x,y) = G(x)\,u^{p/q}(x)\,v(y)$ is in $A_{q,\mathcal{R}}$. Note that $Gu^{p/q}$ and consequently W have A_q norms that depend only on the $A_p(A_q)$ norm of w. Then, since T is bounded on

L_w^q for every $w \in A_{q,\mathcal{R}}$

$$\begin{aligned}\Phi &= \int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|Tf(x,y)|^q\, v(y)\, dy\right) g(x)\, u^{p/q}(x)\, dx\\ &\le \int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|Tf(x,y)|^q\, G(x)\, u^{p/q}(x)\, v(y)\, dy\, dx\\ &\le C\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|f(x,y)|^q\, G(x)\, u^{p/q}(x)\, v(y)\, dy\, dx\end{aligned}$$

By hypothesis and the comment above, the constant C depends only on $\|w\|_{A_p(A_q)}$. Therefore,

$$\begin{aligned}\Phi &\le C\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|f(x,y)|^q\, G(x)\, u^{p/q}(x)\, v(y)\, dy\, dx\\ &= C\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\, v(y)\, dy\right) G(x)\, u^{p/q}(x)\, dx\\ &\le C\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\, v(y)\, dy\right)^{p/q} u^{p/q}(x)\, dx\right)^{q/p}\|G\|_{r',u^{p/q}}\\ &\le C'\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\, v(y)\, dy\right)^{p/q} u^{p/q}(x)\, dx\right)^{q/p}\end{aligned}$$

which completes the proof when $q < p$.

Now, suppose that $1 < p < q < \infty$. Let $r = p/q < 1$ and define r'' by $1/r'' = q/p - 1$. Then, there exists a nonnegative function $g \in L^{r''}_{u^{p/q}}(\mathbf{R}^n)$ with norm 1 such that

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\, v(y)\, dy\right)^{p/q} u^{p/q}(x)\, dx\right)^{q/p} = \int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\, v(y)\, dy\right)\frac{1}{g(x)}\, u^{p/q}(x)\, dx.$$

By Lemma 4 with $\alpha = p$, $\beta = q > \alpha$ and $\gamma = r''$, there is a function $G \in L^{r''}_{u^{p/q}}(\mathbf{R}^n)$ such that $g(x) \le G(x)$, $u^{p/q}/G \in A_q$ and the norm of G is bounded by a constant. As above, the weight W defined by $W(x,y) = u^{p/q}(x)\, v(y)/G(x)$ is in $A_{q,\mathcal{R}}$ with a norm that depends only on the $A_p\left(A_q\right)$ norm of w. Thus,

$$\begin{aligned}
&\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|Tf(x,y)|^q\,v(y)\;dy\right)^{p/q}u^{p/q}(x)\;dx\right)^{1/p}\\
&=\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|Tf(x,y)|^q\,v(y)\;dy\right)^{p/q}\right.\\
&\qquad\qquad\qquad\qquad\left.\times\,G(x)^{-p/q}\,G(x)^{p/q}\,u^{p/q}(x)\;dx\right)^{1/p}\\
&\leq\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|Tf(x,y)|^q\,v(y)\,dy\right)G(x)^{-1}u^{p/q}(x)\,dx\right)^{1/q}\|G\|_{r'',u^{p/q}}^{1/p}\\
&\leq C\left(\int_{\mathbf{R}^n}\int_{\mathbf{R}^m}|Tf(x,y)|^q\,G(x)^{-1}\,u^{p/q}(x)\,v(y)\;dy\,dx\right)^{1/q}\\
&\leq C\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\,v(y)\;dy\right)G(x)^{-1}\,u^{p/q}(x)\;dx\right)^{1/q}\\
&\leq C\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\,v(y)\;dy\right)\frac{1}{g(x)}\,u^{p/q}(x)\;dx\right)^{1/q}\\
&=C\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}^m}|f(x,y)|^q\,v(y)\;dy\right)^{p/q}u^{p/q}(x)\;dx\right)^{1/p}.
\end{aligned}$$

This inequality completes the proof of the theorem. □

We now consider the proof of Theorem 1. A simple argument shows that if M_S is a bounded operator on $L^p\left(L^q\left(w\right)\right)$ then $w\in A_p\left(A_q\right)$. In fact, suppose $f\geq 0$ and $\operatorname{supp} f\subset Q\times Q'$. If $(x,y)\in Q\times Q'$, then

$$\frac{1}{|Q\times Q'|}\int_Q\int_{Q'}f\leq M_Sf(x,y)\,.$$

This implies that

$$\begin{aligned}
&\left(\frac{1}{|Q\times Q'|}\int_Q\int_{Q'}f\right)^p\int_Q\left(\int_{Q'}w(x,y)\;dy\right)^{p/q}dx\\
&\qquad\qquad\leq\int_Q\left(\int_{Q'}|M_Sf(x,y)|^q\,w(x,y)\;dy\right)^{p/q}dx\\
&\qquad\qquad\leq C\int_Q\left(\int_{Q'}|f(x,y)|^q\,w(x,y)\;dy\right)^{p/q}dx.
\end{aligned}$$

By Lemma 1, $w \in A_p(A_q)$, with $\|w\|_{A_p(A_q)}$ bounded by the operator norm of M_S. An application of Theorem 2 completes the proof of Theorem 1.

The proof above shows that the $A_p(A_q)$ norm of w is bounded by the operator norm of M_S. Whether or not the operator norm of M_S is bounded by the $A_p(A_q)$ norm of w is an open question.

We remark that the proof of Theorem 1 shows that if $\|M_S f\|_{L^p(L^q(w))} \leq C\|f\|_{L^p(L^q(w))}$ then $w \in A_p(A_q)$, without requiring that w be a product weight. Their equivalence, for general weights, is an open question.

4. Double Hilbert transform and singular integral operators. Let $f : \mathbf{R}^2 \to \mathbf{R}$, and define the *double Hilbert transform* of f by

$$Df(x, y) = pv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{uv} f(x - u, y - v)\, du\, dv.$$

Define the *one-variable Hilbert transforms*, H_1 and H_2, by

$$H_1 f(x, y) = pv \int_{-\infty}^{\infty} \frac{1}{u} f(x - u, y)\, du$$

and

$$H_2 f(x, y) = pv \int_{-\infty}^{\infty} \frac{1}{v} f(x, y - v)\, du.$$

It follows that $Df(x, y) = H_2(H_1 f)(x, y)$. Using the fact that $w \in A_{r,\mathcal{R}}$ implies that both $w(\cdot, y_0)$ and $w(x_0, \cdot)$ are in $A_r(\mathbf{R})$, uniformly in x_0 and y_0, and iterating known results for the Hilbert transform, we see that D defines a bounded operator on $L^r_w(\mathbf{R}^2)$ for every $w \in A_{r,\mathcal{R}}$. We have

Theorem 3. *Let $w(x, y) = u(x)\, v(y)$ and $1 < p, q < \infty$. The double Hilbert transform is a bounded operator on $L^p(L^q(w))$ if and only if $w \in A_p(A_q)$.*

Proof. By Theorem 2, D is bounded on $L^p(L^q(w))$ for $1 < p,\, q < \infty$. To prove that the norm inequality implies $w \in A_p(A_q)$, we repeat the argument used to prove Theorem 7 in [**9**, p. 244]. In place of the function $f(\theta) = W(\theta)^{-1/(p-1)}$, we use the function

$$f(x, y) = \chi_I(x)\,\chi_{I'}(y)\,w(x, y)^{1-q'}\,\|\chi_{I'}(\cdot)\,w(x, \cdot)^{1-q'}\|_1^{(q-p)/q(p-1)}$$

employed in the proof of Lemma 1. $\square$

Suppose that the operator T, defined by $Tf(x) = (K * f)(x)$, is a standard Calderon-Zygmund singular integral operator; that is, suppose that:

(1) $|K(x)| \leq C/|x|^{n+m}$,

(2) $\int_{\{a<|x|<b\}} K(x)\, dx = 0$ for $0 < a < b$,

(3) $|\nabla K(x)| \leq C/|x|^{n+m+1}$.

Then, it is well known that T is a bounded operator from $L^p_w(\mathbf{R}^n \times \mathbf{R}^m)$ to itself for $1 < p < \infty$ and $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$, the standard A_p class defined over cubes in $\mathbf{R}^d = \mathbf{R}^n \times \mathbf{R}^m$. See, for example, [**17**]. Since $w \in A_{r,\mathcal{R}}$ implies that $w \in A_p(\mathbf{R}^n \times \mathbf{R}^m)$, it follows from Theorem 2 that T is a bounded operator from $L^p(L^q(w))$ to itself for $1 < p$, $q < \infty$ and $w(x, y) = u(x)\,v(y) \in A_p(A_q)$. However, the spaces $L^p(L^q(w))$ seem better adapted to multiparameter operators like the double Hilbert transform and, like the maximal function considered above, we will consider singular integral operators that conform to this setting.

Let $K(x, y)$ be a function of two variables and set

$$\begin{aligned} \Delta^1_h K(x, y) &= K(x + h, y) - K(x, y) \\ \Delta^2_k K(x, y) &= K(x, y + k) - K(x, y) \\ \Delta^{1,2}_{h,k}(K) &= \Delta^1_h\left(\Delta^2_k(K)\right). \end{aligned}$$

and

$$\begin{aligned} K_1(x) &= \int_{\{\beta_1<|y|<\beta_2\}} K(x, y)\, dy \\ K_2(y) &= \int_{\{\alpha_1<|x|<\alpha_2\}} K(x, y)\, dx, \end{aligned}$$

with α_1, α_2, β_1 and β_2 fixed. Following [**6**], we assume there are fixed $A, \eta > 0$ so that K satisfies the cancelation conditions:

(C1) $\left| \int_{\{\alpha_1 < |x| < \alpha_2, \beta_1 < |y| < \beta_2\}} K(x,y)\, dx\, dy \right| \leq A$,

(C2) $|K_1(x)| \leq A\,|x|^{-n}$, $\left|\Delta_h^1 K_1(x)\right| \leq A\,|h|^{\eta}\,|x|^{-n-\eta}$ for $|x| \geq 2\,|h|$, and similar conditions for $K_2(y)$;

and the size conditions:

(S1) $|K(x,y)| \leq A\,|x|^{-n}\,|y|^{-m}$,

(S2) $\left|\Delta_h^1 K(x,y)\right| \leq A\,|h|^{\eta}\,|x|^{-n-\eta}\,|y|^{-m}$ for $|x| \geq 2\,|h|$, and a similar condition for $\Delta_k^2 K(x,y)$,

(S3) $\left|\Delta_{h,k}^{1,2} K(x,y)\right| \leq A\,(|h|\,|k|)^{\eta}\,|x|^{-n-\eta}\,|y|^{-m-\eta}$ for $|x| \geq 2\,|h|$ and $|y| \geq 2\,|k|$.

Under these assumptions on K, Fefferman and Stein, see [**5**] and [**6**], showed that

$$\|K * f\|_{L^p_w(\mathbf{R}^n \times \mathbf{R}^m)} \leq C\,\|f\|_{L^p_w(\mathbf{R}^n \times \mathbf{R}^m)}$$

for weights w such that $w(\cdot, y_0) \in A_p(\mathbf{R}^n)$ and $w(x_0, \cdot) \in A_p(\mathbf{R}^m)$, uniformly in x_0 and y_0, where C depends only on A, p, and the uniform bounds on w. Since $w \in A_{p,\mathcal{R}}$ implies that both $w(\cdot, y_0) \in A_p(\mathbf{R}^n)$ and $w(x_0, \cdot) \in A_p(\mathbf{R}^m)$, uniformly in x_0 and y_0, we can extrapolate to show that

Theorem 4. *Suppose that K satisfies the cancelation and size conditions above. Then, the operator $K * f$ is a bounded operator on $L^p(L^q(w))$ for $1 < p, q < \infty$ and $w(x,y) = u(x)\,v(y) \in A_p(A_q)$.*

We note that results for many other operators follow from the extrapolation theorem, such as sharp function, multiplier and Littlewood-Paley operators.

Acknowledgments. The author wishes to thank the Mathematics Department of the University of Missouri, Columbia, for its hospitality during his sabbatical stay there, and the referee, for his many helpful comments.

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