The Navier–Stokes equations in the critical Morrey–Campanato space

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Abstract

We shall discuss various points on solutions of the 3D Navier-Stokes equations from the point of view of Morrey-Campanato spaces (global solutions, strong-weak uniqueness, the role of real interpolation, regularity).

The classical *Navier-Stokes equations* describe the motion of a Newtonian fluid; we consider only the case when the fluid is viscous, homogeneous, incompressible and fills the entire space and is not submitted to external forces; then, the equations describing the evolution of the motion $\vec{u}(t, x)$ of the fluid element at time t and position x are given by:

$$
(1) \qquad \begin{cases} \rho \ \partial_t \vec{u} = \mu \ \Delta \vec{u} - \rho \ (\vec{u}.\vec{\nabla}) \ \vec{u} - \vec{\nabla}p \\ \vec{\nabla}.\vec{u} = 0 \end{cases}
$$

ρ is the (constant) *density* of the fluid, and we may assume with no loss of generality that $\rho = 1$. μ is the *viscosity* coefficient, and we may assume as well that $\mu = 1$. p is the (unknown) *pressure*, whose action is to maintain the divergence of $\vec{u}$ to be 0 (this *divergence free* condition expresses the incompressibility of the fluid).

We shall use the scaling property of equations (1). When $(\vec{u}, p)$ is a solution on $(0, T) \times \mathbb{R}^3$ of the Cauchy problem associated to equations (1) and initial value $\vec{u}_0$, then, for every $\lambda > 0$ and every $x_0 \in \mathbb{R}^3$,

$$
(\lambda\vec{u}(\lambda^2 t, \lambda(x - x_0)), \lambda^2 p(\lambda^2 t, \lambda(x - x_0)))
$$

is a solution on $(0, \lambda^{-2}T) \times \mathbb{R}^3$ of the Cauchy problem with initial value $\lambda\vec{u}_0(\lambda(x - x_0))$. Therefore, we shall consider the Cauchy problem with initial

value in a critical shift-invariant Banach space: we shall require that $\vec{u}_0 \in B^3$ where B is a Banach space of distributions such that, for every $\lambda > 0$ and every $x_0 \in \mathbb{R}^3$ and for every $f \in B$, we have $\|f(x - x_0)\|_B = \|f\|_B$ and $\lambda\|f(\lambda x)\|_B = \|f\|_B$. If we require moreover that B is continuously embedded in the space of locally square integrable functions, then B must be embedded into the homogeneous Morrey–Campanato space $\dot{M}^{2,3}$ which will play a prominent part throughout the paper.

1. The Navier-Stokes equations in the critical Morrey-Campanato space

In order to solve equations (1), we use the Leray–Hopf operator $\mathbb{P}$ which is the orthogonal projection operator on divergence-free vector fields. We thus consider the following Navier–Stokes equations on $\vec{u}(t, x)$, $t \in (0, \infty)$, $x \in \mathbb{R}^3$:

$$(2) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla} \cdot (\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

(Every solution of (2) is a solution of (1). Conversely, under the restriction that $\vec{u}$ vanishes at infinity in a weak sense, every solution of (1) is a solution of (2); see [6] or [18]). Solving the Cauchy problem associated to the initial value $\vec{u}_0$ then amounts to solve the integral equation

$$(3) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds.$$

In order to solve (3), we define the bilinear operator B by

$$(4) \qquad B(\vec{u}, \vec{v})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v}) \, ds.$$

We are going to describe the solutions of (3) when $\vec{u}_0$ belongs to the homogeneous Morrey–Campanato space $\dot{M}^{2,3}$.

Definition 1. *For $1 < p \le q < \infty$, the homogeneous Morrey–Campanato space $\dot{M}^{p,q}(\mathbb{R}^3)$ is defined as the space of locally p-integrable functions f such that*

$$(5) \qquad \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/q - 1/p)}\left(\int_{|x-x_0|<R} |f(x)|^p \, dx\right)^{1/p} < \infty;$$

the predual of $\dot{M}^{p,q}$ is then the space of functions f which may be decomposed as a series $\sum_{n\in\mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$ with

$$R_n > 0, \quad f_n \in L^{p/(p-1)}, \quad \|f_n\|_{p/(p-1)} \le R_n^{3(1/q-1/p)} \quad and \quad \sum_{n\in\mathbb{N}} |\lambda_n| < \infty.$$

For $p = 1 \leq q < \infty$, the homogeneous Morrey–Campanato space $\dot{M}^{1,q}(\mathbb{R}^3)$ is defined as the space of locally bounded measures μ such that

$$(6) \qquad \sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(\frac{1}{q}-1)} |\mu|(B(x_0, R)) < \infty;$$

the predual of $\dot{M}^{p,q}$ is then the space of functions f which may be decomposed as a series $\sum_{n \in \mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$ with

$$R_n > 0, \quad f_n \text{ continuous}, \quad \|f_n\|_\infty \leq R_n^{3(1/q-1)} \quad and \quad \sum_{n \in \mathbb{N}} |\lambda_n| < \infty.$$

When the initial value $\vec{u}_0$ belongs to $(\dot{M}^{2,3})^3$, we may search for a solution in two ways: either we use the formalism of mild solutions introduced by Kato in the study of solutions on Lebesgue spaces [11], or we use a mollification of the equations and then construct a weak solution through energy estimates and compactnes criteria (a process introduced by Leray in the study of solutions in L^2 [21]).

We then have the following result:

Theorem 1. *Let $\vec{u}_0 \in (\dot{M}^{2,3}(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Then the fixed-point problem*

$$(7) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds$$

can be solved in the following three cases:

(A) Local mild solution for a regular initial value:

If $\vec{u}_0$ belongs more precisely to $(\tilde{M}^{2,3})^3$, where $\tilde{M}^{2,3}$ is the closure of $\mathcal{D}(\mathbb{R}^3)$ in $\dot{M}^{2,3}$, then there exists a positive $T = T(\vec{u}_0)$ such that the sequence $\vec{u}^{(n)}$ defined by

$$(8) \qquad \vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

remains bounded in the space $(\mathcal{E}_T)^3$ where $\mathcal{E}_T$ is defined as

$$f \in \mathcal{E}_T \iff \begin{cases} f \in L^2_{loc}((0,T) \times \mathbb{R}^3), \\ \sup_{0<t<T} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} < \infty \quad and \\ \sup_{0<t<T} t^{1/2}\|f(t,.)\|_\infty < \infty \end{cases}$$

and normed by

$$\|f\|_{\mathcal{E}_T} = \sup_{0<t<T} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} + \sup_{0<t<T} t^{1/2}\|f(t,.)\|_\infty.$$

Moreover, the sequence $\vec{u}^{(n)}$ converges in $(\mathcal{E}_T)^3$ to a solution $\vec{u}$ of (7) which belongs to $\mathcal{C}([0,T], (\dot{M}^{2,3})^3)$.

(B) Global mild solution for a small initial value:

There exists a positive constant ϵ_0 such that when $\vec{u}_0$ belongs to $(\dot{M}^{2,3})^3$ with

$$\|\vec{u}_0\|_{\dot{M}^{2,3}} \leq \epsilon_0,$$

then the sequence $\vec{u}^{(n)}$ defined by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

remains bounded in the space $(\mathcal{E}_\infty)^3$ where $\mathcal{E}_\infty$ is defined as

$$f \in \mathcal{E}_\infty \Longleftrightarrow \begin{cases} f \in L^2_{loc}((0, +\infty) \times \mathbb{R}^3), \\ \sup_{0<t} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} < \infty \quad \text{and} \\ \sup_{0<t} t^{1/2}\|f(t,.)\|_\infty < \infty \end{cases}$$

and normed by

$$\|f\|_{\mathcal{E}_\infty} = \sup_{0<t} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} + \sup_{0<t} t^{1/2}\|f(t,.)\|_\infty.$$

Moreover, the sequence $\vec{u}^{(n)}$ converges in $(\mathcal{E}_\infty)^3$ to a solution $\vec{u}$ of (7) which satisfies $\sup_{0<t}\|\vec{u}(t,.)\|_{\dot{M}^{2,3}} < \infty$.

(C) Global weak solution for a general initial value:

Let $\omega \in \mathcal{D}(\mathbb{R}^3)$ with $\omega \geq 0$ and $\int_{\mathbb{R}^3} \omega \, dx = 1$; then the mollified equations are given for $\epsilon > 0$ by

$$(9) \qquad \begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla}.((\vec{u} * \omega_\epsilon) \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \\ \vec{u}(0,.) = \vec{u}_0 \end{cases}$$

where $\omega_\epsilon = \frac{1}{\epsilon^3}\omega(\frac{x}{\epsilon})$. The equations (9) have a unique global solution $\vec{u}_\epsilon$ such that

$$(10) \qquad \sup_{x_0 \in \mathbb{R}^3, \ R>0, \ t>0} \frac{1}{R + \sqrt{t}} \int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \, dx < \infty.$$

We have moreover

$$(11) \qquad \sup_{\epsilon>0} \sup_{x_0 \in \mathbb{R}^3, \ R>0, \ t>0} \frac{1}{R + \sqrt{t}} \int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \, dx < \infty.$$

and

$$(12) \qquad \sup_{\epsilon>0} \sup_{x_0 \in \mathbb{R}^3, \ t>0} \frac{1}{\sqrt{t}} \int_0^t \int_{\|x-x_0\|<\sqrt{t}} |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x)|^2 \, ds \, dx < \infty.$$

There exists a sequence $(\epsilon_k)_{k\in\mathbb{N}}$ (depending on $\vec{u}_0$) such that ϵ_k decreases to 0 and $\vec{u}_{\epsilon_k}$ converges strongly in $L^2_{loc}((0,\infty)\times\mathbb{R}^3)$ to a solution $\vec{u}$ of (7). Moreover, this solution satisfies the local energy inequality: the pressure p such that

$$\partial_t \vec{u} = \Delta\vec{u} - \vec{\nabla}.(\vec{u}\otimes\ \vec{u}) - \vec{\nabla}p \ \text{for all } \phi\in\mathcal{D}(Q)$$

may be chosen in $L^{3/2}_{loc}((0,\infty)\times\mathbb{R}^3))$ and such that for all $\phi\in\mathcal{D}((0,\infty)\times\mathbb{R}^3)$ with $\phi\geq 0$ we have

$$2\iint |\vec{\nabla}\otimes\vec{u}|^2\phi\ dx\ dt \leq \iint |\vec{u}|^2(\partial_t\phi + \Delta\phi)\ dx\ dt$$

(13)
$$+ \iint (|\vec{u}|^2 + 2p)(\vec{u}.\vec{\nabla})\phi\ \ dx\ dt.$$

We discuss the proof of points (A) and (B) in Section 2, and the proof of point (C) in Section 3. Whereas points (A) and (B) are classical (they were first proved, in the setting of Morrey–Campanato spaces, by Kato [12]; see also [30], [4] and [2]), point (C) may appear new (it is however a straightforward consequence of the theory of weak solutions in L^2_{uloc} developed in [18]).

2. Mild solutions for the Navier–Stokes equations

In the formalism of mild solutions, we try to solve (7) by the fixed-point algorithm:

$$\vec{u} = \lim_{n\to\infty}\vec{u}^{(n)}$$

with

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \qquad \text{and} \qquad \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}).$$

The resolution of this fixed-point problem is based on a general tool for multilinear equations in a Banach space:

Lemma 1. *Let E be a Banach space and T a bounded bilinear operator on E*

(14)
$$\|T(x,y)\|_E \leq C_0\|x\|_E\|y\|_E.$$

Let $x_0\in E$ with $\|x_0\|_E < \frac{1}{4C_0}$. Then, the equation $x = x_0 + T(x,x)$ has at least one solution. More precisely, it has one unique solution $x\in E$ such that $\|x\|_E \leq \frac{1}{2C_0}$.

This lemma is straightfoward, since the mapping $x \mapsto x_0 + T(x,x)$ is then a contraction on the closed ball $\bar{B}(0, \|x_0\|_E + \frac{1}{4C_0})$. We may remark that the existence and uniqueness result holds in the closed ball [1], [18], [19] (even though the mapping $x \mapsto x_0 + T(x,x)$ is no longer a contraction) and can be extended to more general multilinear operators [19]:

Lemma 2. *Let $k \geq 2$. Let E be a Banach space and T_k be a bounded k-linear operator on E*

$$\|T_k(x_1, \ldots, x_k)\|_E \leq C_0 \|x_1\|_E \ldots \|x_k\|_E.$$

Let

(15)
$$r_k = (C_0 k)^{-\frac{1}{k-1}} \frac{k-1}{k} \ \text{ and } \ R_k = \frac{k}{k-1} r_k.$$

Let B_k be the closed ball $B_k = \bar{B}(0, R_k)$. If $x_0 \in E$ with $\|x_0\|_E \leq r_k$, then the equation $x = x_0 + T_k(x, \ldots, x)$ has at least one solution. More precisely, it has one unique solution x_∞ in B_k.

We may moreover precise the speed of convergence of the Picard-Duhamel approximation of x_∞ [19]:

Lemma 3. *Under the assumptions of Lemma 2, let x_n be defined from x_0 by $x_{n+1} = x_0 + T_k(x_n, \ldots, x_n)$. If $\|x_0\|_E \leq r_k$, then we have*

(16)
$$\limsup_{n \to \infty} \ n^2 \|x_{n+1} - x_n\|_E \leq \frac{2R_k}{k-1}.$$

Of course, if $\|x_0\|_E < r_k$, the rate of convergence is much better (exponentially decreasing, due to contractivity: $\|x_{n+1} - x_n\|_E \leq \left(\frac{\|x_0\|_E}{r_k}\right)^{(n+1)(k-1)} \|x_0\|_E$).

We now come back to the Navier–Stokes equations. The construction of mild solutions relies on the fact that the operator $e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}$ is a matrix of convolutions operators (in the x variable) whose kernels $K_{i,j}(t-s, x)$ are controlled by

$$|K_{i,j}(t-s, x)| \leq C \frac{1}{(\sqrt{t-s} + \|x\|)^4}.$$

In 1984, Kato [11] proved the existence of mild solutions in L^p, $p \geq 3$. For $p > 3$, he used the estimate

(17)
$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{v})\|_p \leq C_p (t-s)^{-\frac{1}{2} - \frac{3}{2p}} \|\vec{u}\|_p \|\vec{v}\|_p$$

to prove the boundedness of B on $L^\infty([0, T], (L^p)^3)$:

(18)
$$\|B(\vec{u}, \vec{v})(t, .)\|_p \leq C_p t^{\frac{1}{2} - \frac{3}{2p}} \sup_{0 < s < t} \|\vec{u}(s, .)\|_p \sup_{0 < s < t} \|\vec{v}(s, .)\|_p.$$

For the critical case $p = 3$, inequality (17) becomes

(19)
$$\|e^{(t-s)\Delta} \mathbb{P} \vec{\nabla}.(\vec{u} \otimes \vec{v})\|_3 \leq C \frac{1}{(t-s)} \|\vec{u}\|_3 \|\vec{v}\|_3.$$

This is a very unconvenient estimate for dealing with $\vec{u}, \vec{v} \in L^{\infty}([0,T], (L^3)^3)$, since $\int_0^t \frac{ds}{t-s}$ diverges both at the endpoints $s = 0$ and $s = t$. Kato then used an idea of Weissler [32], namely to use the smoothing properties of the heat kernel (when applied to $\vec{u}_0 \in (L^3)^3$) to search for the existence of a solution in a smaller space of mild solutions ; indeed, whereas the bilinear operator B is unbounded on $\mathcal{C}([0,T], (L^3(\mathbb{R}^3))^3)$ [28], it becomes bounded on the smaller space $\{\vec{f} \in \mathcal{C}([0,T], (L^3(\mathbb{R}^3))^3) \ / \ \sup_{0<t<T} \sqrt{t}\|\vec{f}(t,.)\|_{\infty} < \infty\}$. Thus, we replace the estimate (19) (which leads to a divergent integral) by the estimates

$$(20) \qquad \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_3 \leq C\frac{1}{\sqrt{t-s}\sqrt{s}}\|\vec{u}\|_3\sqrt{s}\|\vec{v}\|_{\infty}$$

and

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_{\infty}$$
$$(21) \qquad \leq C\frac{1}{\sqrt{t-s}}\min\left(\frac{1}{t-s}\|\vec{u}\|_3\|\vec{v}\|_3, \frac{1}{s}\sqrt{s}\|\vec{u}\|_{\infty}\sqrt{s}\|\vec{v}\|_{\infty}\right)$$

which lead to two convergent integrals.

In the same way, in order to construct mild solutions in $\dot{M}^{2,3}$, one uses the smoothing properties of the heat kernel:

$$(22) \qquad \sup_{t>0} \sqrt{t} \ \|e^{t\Delta}\vec{u}_0\|_{\infty} \leq C\|\vec{u}_0\|_{\dot{M}^{2,3}}.$$

From (22), we find as well that

$$(23) \qquad \|e^{t\Delta}\vec{u}_0\|_{\dot{M}^{4,6}} \leq \sqrt{\|e^{t\Delta}\vec{u}_0\|_{\dot{M}^{2,3}}\|e^{t\Delta}\vec{u}_0\|_{\infty}} \leq \sqrt{C}t^{-1/4}\|\vec{u}_0\|_{\dot{M}^{2,3}}.$$

Thus, $\left(e^{t\Delta}\vec{u}_0\right)_{0<t<T}$ belongs to the space $(\mathcal{E}_T)^3$ defined in Theorem 1, point (A) $(T < \infty)$ or point (B) $(T = \infty)$. Then, the proof of points (A) and (B) relies on the estimates

$$(24) \qquad \|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_{\dot{M}^{4,6}} \leq C\frac{1}{(t-s)^{1/2} \ s^{3/4}}s^{1/4}\|\vec{u}\|_{\dot{M}^{4,6}} \ s^{1/2}\|\vec{v}\|_{\infty}$$

and

$$\|e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v})\|_{\infty}$$
$$(25) \qquad \leq C\frac{1}{\sqrt{t-s}}\min\left(\frac{s^{1/4}\|\vec{u}\|_{\dot{M}^{4,6}}s^{1/4}\|\vec{v}\|_{\dot{M}^{4,6}}}{\sqrt{t-s}\sqrt{s}}, \frac{\sqrt{s}\|\vec{u}\|_{\infty}\sqrt{s}\|\vec{v}\|_{\infty}}{s}\right)$$

which prove that the bilinear operator B is bounded on $(\mathcal{E}_T)^3$:

$$(26) \qquad \|B(\vec{u}, \vec{v})\|_{\mathcal{E}_T} \leq C_0 \|\vec{u}\|_{\mathcal{E}_T} \|\vec{v}\|_{\mathcal{E}_T}$$

where C_0 does not depend on $T \in (0, +\infty]$. Since we have

$$(27) \qquad \|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \leq C \|\vec{u}_0\|_{\dot{M}^{2,3}}$$

and

$$(28) \qquad \lim_{T \to 0} \|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} = 0 \text{ when } \vec{u}_0 \in (\tilde{M}^{2,3})^3,$$

we get the convergence of $\vec{u}^{(n)}$ in $(\mathcal{E}_T)^3$ when $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \leq \frac{1}{4C_0}$, i.e. when $\vec{u}_0$ is small ($T = +\infty$) or when $\vec{u}_0$ is regular and T is small enough. Moreover, B is obviously bounded from $(\mathcal{E}_T)^3 \times (\mathcal{E}_T)^3$ to $(L^\infty((0,T), \dot{M}^{2,3})^3$. Thus, the convergence of $\vec{u}^{(n)}$ to a solution $\vec{u}$ holds as well in the norm of $(L^\infty(\dot{M}^{2,3})^3$ as in the norm of $(\mathcal{E}_T)^3$. Moreover, when $\vec{u}_0$ is regular, it is easy to check by induction on n that $\vec{u}^{(n)}$ belongs to the space

$$\{\vec{f} \in (\mathcal{E}_T)^3 \ / \ \lim_{t \to 0} t^{1/4}\|\vec{f}\|_{\dot{M}^{4,6}} = \lim_{t \to 0} t^{1/2}\|\vec{f}\|_\infty = 0\} \cap \mathcal{C}([0,T], (\tilde{M}^{2,3})^3)$$

which is closed in $(\mathcal{E}_T)^3 \cap (L^\infty(\dot{M}^{2,3})^3$. Thus, one gets the proof of points (A) and (B) of Theorem 1.

3. Maximal solutions

The Navier–Stokes equations are locally well-posed in L^∞, since the bilinear operator B is bounded on $(\mathcal{F}_T)^3$, where $\mathcal{F}_T$ is defined as

$$f \in \mathcal{F}_T \Leftrightarrow f \in L^2_{loc}((0,T) \times \mathbb{R}^3) \text{ and } \sup_{0<t<T} \|f(t,.)\|_\infty < \infty$$

and normed by

$$\|f\|_{\mathcal{F}_T} = \sup_{0<t<T} \|f(t,.)\|_\infty.$$

We easily can check that

$$(29) \qquad \|B(\vec{u}, \vec{v})(t,.)\|_\infty \leq C_0 t^{\frac{1}{2}} \sup_{0<s<t} \|\vec{u}(s,.)\|_\infty \sup_{0<s<t} \|\vec{v}(s,.)\|_\infty.$$

so that, by lemma 1 or 2, we may conclude that the Navier–Stokes equations associated to $\vec{u}_0 \in (L^\infty)^3$

$$(30) \qquad \vec{u} = e^{t\Delta}\vec{u}_0 - \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u}) \, ds$$

has a (unique) mild solution in $(\mathcal{F}_T)^3$ with

$$(31) \qquad T = \frac{1}{(4C_0\|\vec{u}_0\|_\infty)^2}.$$

Thus, the solution described in point (A) of Theorem 1 can be continued on a maximal interval $[0, T^*)$, with

$$(32) \qquad T^* = +\infty \text{ or } \liminf_{t\to T^*} \sqrt{T^* - t}\|\vec{u}(t,.)\|_\infty > \frac{1}{4C_0}.$$

This maximal solution remains in $\dot{M}^{2,3}$:

Proposition 1. *Let $\vec{u}_0 \in (\tilde{M}^{2,3})^3$ and let $\vec{u}$ be the maximal continuation in $L^\infty_{loc}((0,T^*),(L^\infty)^3)$ of the mild solution associated to $\vec{u}_0$ by Theorem 1. Then, $\vec{u} \in \mathcal{C}([0,T^*),(\tilde{M}^{2,3})^3)$.*

This is a direct consequence of Theorem 1. Indeed, if $T < T^*$ is such that $\vec{u} \in \mathcal{C}([0,T],(\tilde{M}^{2,3})^3)$ and if $\delta \in (0,T)$, then $\vec{u}$ is uniformly bounded on $[\delta, T]$; moreover, we have, for $t \in [\delta, T]$,

$$(33) \qquad \vec{u}(t,.) = e^{(t-\delta)\Delta}\vec{u}(\delta,.) - \int_\delta^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{u})\ ds,$$

hence

$$\|\vec{u}(t,.)\|_{\dot{M}^{2,3}} \le \|\vec{u}(\delta,.)\|_{\dot{M}^{2,3}}$$
$$(34) \qquad\qquad + C \sup_{\delta \le s \le T} \|\vec{u}(s,.)\|_\infty \int_\delta^t \frac{1}{(t-s)}\|\vec{u}(s,.)\|_{\dot{M}^{2,3}}\ ds$$

and the Gronwall lemma shows that the norm $\|\vec{u}(t,.)\|_{\dot{M}^{2,3}}$ remains bounded as $t \to T$. If $0 < t_0 < T$, the Kato algorithm provides a mild solution in $\mathcal{C}([t_0, t_0 + t_1],(\tilde{M}^{2,3})^3)$ where t_1 is such that, for a positive constant ϵ_0,

$$\sup_{0<t<t_1} t^{1/4}\|e^{t\Delta}\vec{u}(t_0,.)\|_{\dot{M}^{4,6}} + t^{1/2}\|e^{t\Delta}\vec{u}(t_0,.)\|_\infty \le \epsilon_0,$$

hence, at least, for t_1 such that

$$t_1^{1/4}\sqrt{\|\vec{u}(t_0,.)\|_{\dot{M}^{2,3}}\|\vec{u}(t_0,.)\|_\infty} + t_1^{1/2}\|\vec{u}(t_0,.)\|_\infty \le \epsilon_0,$$

while the same Kato algorithm provides a mild solution in $L^\infty([t_0, t_0 + t_2],(L^\infty)^3)$ where t_2 is such that, for a positive constant ϵ_1,

$$t_2^{1/2}\|\vec{u}(t_0,.)\|_\infty \le \epsilon_1.$$

Moreover, by uniqueness in $L^\infty([t_0, t_0 + t_2],(L^\infty)^3)$, this mild solution coincides with the maximal solution $\vec{u}$. Thus, we may conclude that $\vec{u}$ remains in $\mathcal{C}([0, t_0 + \inf(t_1,t_2)),(\tilde{M}^{2,3})^3)$ and that we may choose t_0 such that $t_0 + \inf(t_1,t_2) > T$. This proves Proposition 1.

4. Global weak solutions for the Navier–Stokes equations

The proof of Point (C) in Theorem 1 is based on some local energy estimates for the solution of the mollified equations

$$(35 - \epsilon) \qquad \begin{cases} \partial_t \vec{u}_\epsilon = \Delta \vec{u}_\epsilon - \mathbb{P}\vec{\nabla}.((\vec{u}_\epsilon * \omega_\epsilon) \otimes \vec{u}_\epsilon) \\ \vec{\nabla}.\vec{u}_\epsilon = 0 \\ \vec{u}_\epsilon(0,.) = \vec{u}_0 \end{cases}$$

These estimates are described in [18] in the study of the Navier–Stokes equations in L^2_{uloc}, where L^2_{uloc} is the space of uniformly locally square integrable functions:

$$f \in L^2_{uloc} \Leftrightarrow \sup_{x_0 \in \mathbb{R}^3} \int_{\|x - x_0\| < 1} |f(x)|^2 \, dx < \infty$$

normed with

$$\|f\|_{L^2_{uloc}} = \sup_{x_0 \in \mathbb{R}^3} \sqrt{\int_{\|x - x_0\| < 1} |f(x)|^2 \, dx}.$$

We recall the main result proved in [18]:

Proposition 2. *Let $\vec{u}_0 \in (L^2_{uloc}(\mathbb{R}^3))^3$ be such that $\vec{\nabla}.\vec{u}_0 = 0$. Define $\alpha_0 = \|\vec{u}_0\|_{L^2_{uloc}}$ and $\alpha_1 = \min(1, \alpha_0)$. Then, there exists a positive constant C_0 (which does not depend on $\vec{u}$ nor on ϵ) such that the equations $(35 - \epsilon)$ have a solution $\vec{u}_\epsilon$ on $(0, T_0) \times \mathbb{R}^3$ with $T_0 = \min(1, \frac{\alpha_1^2}{\alpha_0^2 C_0^4})$ and such that for all $0 < t < T_0$ we have*

$$(36) \qquad \|\vec{u}_\epsilon(t,.)\|_{L^2_{uloc}} \le \sqrt{C_0}\|\vec{u}_0\|_{L^2_{uloc}}\left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2}t\right)^{-1/4}$$

and

$$(37) \quad \sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{\|x - x_0\| < 1} |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x))|^2 dx \, ds \le C_0\|\vec{u}_0\|^2_{L^2_{uloc}}\left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2}t\right)^{-1/2}.$$

We now use the scaling property of the Navier–Stokes equations. When $\vec{v}_\epsilon$ is a solution on $(0, T) \times \mathbb{R}^3$ of the Cauchy problem associated to equations $(35 - \epsilon)$ and initial value $\vec{v}_0$, then, for every $\lambda > 0$, $\lambda\vec{v}_\epsilon(\lambda^2 t, \lambda x)$ is a solution on $(0, \lambda^{-2}T) \times \mathbb{R}^3$ of the Cauchy problem associated to equations $(35 - \frac{\epsilon}{\lambda})$ with initial value $\lambda\vec{v}_0(\lambda x)$.

We now use the following points:

- we have uniqueness of the solutions of $(35 - \epsilon)$ in the space $L^\infty((L^2_{uloc})^3)$
- the constant C_0 in Proposition 2 does not depend on ϵ
- when $u \in \dot{M}^{2,3}$, then $\sup_{\lambda > 0} \|\lambda u(\lambda x)\|_{L^2_{uloc}} = \|u\|_{\dot{M}^{2,3}}$.

Thus, applying Proposition 2 to $\lambda \vec{u}_0(\lambda x)$ and to equations $(35 - \frac{\epsilon}{\lambda})$, we find a solution $\vec{v}_{\frac{\epsilon}{\lambda}}$ defined on $(0, T_0)$ where T_0 depends only on $\|\vec{u}_0\|_{\dot{M}^{2,3}}$ and not on ϵ nor λ, hence the solution $\vec{u}_\epsilon$ of $(35 - \epsilon)$ associated to $\vec{u}_0$ satisfies $\vec{u}_\epsilon = \frac{1}{\lambda} \vec{v}_{\frac{\epsilon}{\lambda}}(\frac{t}{\lambda^2}, \frac{x}{\lambda})$, hence is defined on $(0, \lambda^2 T_0)$ and satisfies for all $0 < t < \lambda^2 T_0$

$$(38) \quad \sup_{x_0 \in \mathbb{R}^3} \int_{\|x-x_0\|<\lambda} |\vec{u}_\epsilon(t,x)|^2 \, dx \leq C_0 \lambda \|\lambda u_0(\lambda x)\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2 \lambda^2} t\right)^{-1/2}$$

and

$$\sup_{x_0 \in \mathbb{R}^3} \int_0^t \int_{\|x-x_0\|<\lambda} |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x))|^2 \, dx \, ds$$

$$(39) \qquad\qquad \leq C_0 \lambda \|\lambda u_0(\lambda x)\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2 \lambda^2} t\right)^{-1/2}.$$

Since $\vec{u}_\epsilon$ is defined on $(0, \lambda^2 T_0)$ for every positive λ, $\vec{u}_\epsilon$ is defined on $(0, +\infty)$. Moreover, we may estimate $\int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \, dx$ by using (38): if $t \leq \frac{1}{4} R^2 T_0$, then

$$\int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \, dx \leq C_0 R \|R u_0(Rx)\|_{L^2_{uloc}}^2 \left(1 - \frac{\alpha_0^2 C_0^4}{\alpha_1^2 R^2} t\right)^{-1/2}$$

$$\leq 2 C_0 R \|u_0\|_{\dot{M}^{2,3}}^2 \,,$$

while if $t \geq \frac{1}{4} R^2 T_0$ we have

$$\int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \, dx \leq \int_{\|x-x_0\|<\sqrt{\frac{4t}{T_0}}} |\vec{u}_\epsilon(t,x)|^2 \, dx \leq 2 C_0 \sqrt{\frac{4t}{T_0}} \|u_0\|_{\dot{M}^{2,3}}^2.$$

Thus, we have proved

$$(40) \qquad \sup_{\epsilon>0} \sup_{x_0 \in \mathbb{R}^3, \, R>0, \, t>0} \frac{1}{R + \sqrt{t}} \int_{\|x-x_0\|<R} |\vec{u}_\epsilon(t,x)|^2 \, dx < \infty.$$

and we get similarly

$$(41) \qquad \sup_{\epsilon>0} \sup_{x_0 \in \mathbb{R}^3, \, t>0} \frac{1}{\sqrt{t}} \int_0^t \int_{\|x-x_0\|<\sqrt{t}} |\vec{\nabla} \otimes \vec{u}_\epsilon(s,x)|^2 \, ds \, dx < \infty.$$

Those estimates then allows one to use the limiting process of Leray [21] to extract a subsequence $\vec{u}_{\epsilon_n}$ that is convergent to a solution $\vec{u}$ of the Navier-Stokes equations associated to $\vec{u}_0$. More precisely, when ϵ_n converges to 0, we have for all $\phi \in \mathcal{D}((0,T_0) \times \mathbb{R}^3)$ strong convergence of $\phi \vec{u}_\epsilon$ in $L^p((0,T_0), (L^2)^3)$ for all $p < \infty$ and weak convergence in $L^2((0,T_0), (H^1)^3))$. The details of the proof (and the proof of the local energy inequality) are exactly similar to the case of weak solutions in L^2_{uloc} [18].

5. Comparison of weak and mild solutions

If $\vec{u}_0 \in (\tilde{M}^{2,3})^3$, we have a maximal regular solution which is described by Point (A) of Theorem 1 and Proposition 1, and global weak solutions which are described by Point (C) if Theorem 1. As a matter of fact, those solutions coincide on the domain of definition of the regular solution. We have more precisely the following convergence theorem:

Theorem 2. *Let* $\vec{u}_0 \in (\tilde{M}^{2,3})^3$ *and let* $\vec{u}$ *be the maximal continuation in* $L^\infty_{loc}((0,T^*),(L^\infty)^3)$ *of the mild solution associated to* $\vec{u}_0$ *by Theorem 1, and let* $\vec{u}_\epsilon$ *be the solution of the mollified equations* (9).

Then, for every $T \in (0,T^*)$, *there exists a positive* ϵ_T *such that, for every* $\epsilon \in (0,\epsilon_T)$, $\vec{u}_\epsilon \in \mathcal{C}([0,T],(\tilde{M}^{2,3})^3)$ *and moreover, we have*

$$(42) \qquad \lim_{\epsilon \to 0} \|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{C}([0,T],(\tilde{M}^{2,3})^3)} = 0.$$

Indeed, we have, for any positive θ,

$$(43) \qquad \|B(\vec{u},\vec{v})\|_{\mathcal{E}_\theta} \leq C_0 \|\vec{u}\|_{\mathcal{E}_\theta} \|\vec{v}\|_{\mathcal{E}_\theta}$$

and

$$(44) \qquad \|B(\vec{u}*\omega_\epsilon,\vec{v})\|_{\mathcal{E}_\theta} \leq C_0 \|\vec{u}*\omega_\epsilon\|_{\mathcal{E}_\theta} \|\vec{v}\|_{\mathcal{E}_\theta} \leq C_0 \|\vec{u}\|_{\mathcal{E}_\theta} \|\vec{v}\|_{\mathcal{E}_\theta}$$

where the positive constant C_0 does not depend on θ. If $\vec{v}_0 \in (\tilde{M}^{2,3})^3$ satisfies the inequality

$$\|e^{t\Delta}\vec{v}_0\|_{\mathcal{E}_\theta} \leq \frac{1}{4C_0},$$

then the Kato algorithm provides a mild solution in $\mathcal{C}([0,\theta],(\tilde{M}^{2,3})^3)$ of the Navier-Stokes equations or of the mollified Navier–Stokes equations associated to the initial value $\vec{v}_0$. By compactness of $[0,T]$, hence of the set $\{\vec{u}(t,.) \ / \ 0 \leq t \leq T\}$ in $(\tilde{M}^{2,3})^3$, there exists a positive θ such that

$$(45) \qquad \sup_{0 \leq t \leq T} \|e^{s\Delta}\vec{u}(t,.)\|_{\mathcal{E}_\theta} \leq \frac{1}{16C_0}$$

Now, we are going to show that, if $t_0 \in [0,T]$, if $\vec{u}_\epsilon(t_0,.) \in (\tilde{M}^{2,3})^3$ for $0 < \epsilon < \epsilon(t_0)$ and $\lim_{\epsilon \to 0} \|\vec{u}(t_0,.) - \vec{u}_\epsilon(t_0,.)\|_{\dot{M}^{2,3}} = 0$, then there exists a positive $\eta(t_0)$ such that, for $0 < \epsilon < \eta(t_0)$, $\vec{u}_\epsilon \in \mathcal{C}([t_0,t_0+\theta],(\tilde{M}^{2,3})^3)$ and

$$\lim_{\epsilon \to 0} \|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{C}([t_0,t_0+\theta],(\tilde{M}^{2,3})^3)} = 0.$$

Since θ does not depend on t_0, we shall have proved the theorem.

Since $\lim_{\epsilon \to 0} \|\vec{u}(t_0, .) - \vec{u}_\epsilon(t_0, .)\|_{\dot{M}^{2,3}} = 0$, we find that

$$\lim_{\epsilon \to 0} \|e^{t\Delta}\vec{u}(t_0, .) - e^{t\Delta}\vec{u}_\epsilon(t_0, .)\|_{\mathcal{E}_\theta} = 0.$$

Thus, there exists a positive $\eta(t_0)$ such that, for $0 < \epsilon < \eta(t_0)$,

$$\|e^{t\Delta}\vec{u}_\epsilon(t_0, .)\|_{\mathcal{E}_\theta} \leq \frac{1}{8C_0}.$$

This proves that, for $0 < \epsilon < \eta(t_0)$, $\vec{u}_\epsilon \in \mathcal{C}([t_0, t_0 + \theta], (\tilde{M}^{2,3})^3)$, with

$$\|\vec{u}_\epsilon(t_0 + t, .)\|_{\mathcal{E}_\theta} \leq \frac{1}{4C_0}.$$

Now, for $0 \leq t \leq \theta$, we define $\vec{w}(t, .) = \vec{u}(t_0 + t, .)$ and $\vec{w}_\epsilon(t, .) = \vec{u}_\epsilon(t_0 + t, .)$; we have

$$\vec{w}_\epsilon(t, .) - \vec{w}(t, .) = e^{t\Delta}(\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .)) + B(\vec{w}, \vec{w} - \vec{w}_\epsilon)$$
$$+ B((\vec{w} - \vec{w}_\epsilon) * \omega_\epsilon, \vec{w}_\epsilon) + B(\vec{w} - (\vec{w} * \omega_\epsilon), \vec{w}_\epsilon)$$

which gives

$$\|\vec{w}_\epsilon(t, .) - \vec{w}(t, .)\|_{\mathcal{E}_\theta} \leq \|e^{t\Delta}(\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .))\|_{\mathcal{E}_\theta}$$
$$+ C_0\|\vec{w}\|_{\mathcal{E}_\theta}\|\vec{w} - \vec{w}_\epsilon\|_{\mathcal{E}_\theta} + C_0\|(\vec{w} - \vec{w}_\epsilon) * \omega_\epsilon\|_{\mathcal{E}_\theta}\|\vec{w}_\epsilon\|_{\mathcal{E}_\theta}$$
$$+ C_0\|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta}\|\vec{w}_\epsilon\|_{\mathcal{E}_\theta}$$
$$\leq C\|\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .)\|_{\dot{M}^{2,3}}$$
$$+ C_0\frac{1}{4C_0}\|\vec{w} - \vec{w}_\epsilon\|_{\mathcal{E}_\theta} + C_0\|\vec{w} - \vec{w}_\epsilon\|_{\mathcal{E}_\theta}\frac{1}{4C_0}$$
$$+ C_0\|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta}\frac{1}{4C_0}$$

and finally

$$(46) \quad \|\vec{w}_\epsilon(t, .) - \vec{w}(t, .)\|_{\mathcal{E}_\theta} \leq 2C\|\vec{u}_\epsilon(t_0, .) - \vec{u}(t_0, .)\|_{\dot{M}^{2,3}} + \frac{1}{2}\|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta}.$$

The operators $f \mapsto f * \omega_\epsilon$ are equicontinuous on $\mathcal{E}_\theta$ and when $\varphi \in \mathcal{D}([0, \theta] \times \mathbb{R}^3)$ the functions $\varphi * \omega_\epsilon$ converge to φ in $\mathcal{D}$ (hence in $\mathcal{E}_\theta$) as ϵ goes to O. Moreover, $\vec{w}$ belongs to the closure of $(\mathcal{D}([0, \theta] \times \mathbb{R}^3))^3$ in $(\mathcal{E}_\theta)^3$. Thus, we have $\lim_{\epsilon \to 0} \|\vec{w} - \vec{w} * \omega_\epsilon\|_{\mathcal{E}_\theta} = 0$. Finally, we get $\lim_{\epsilon \to 0} \|\vec{w}_\epsilon(t, .) - \vec{w}(t, .)\|_{\mathcal{E}_\theta} = 0$, which proves that $\lim_{\epsilon \to 0} \|\vec{u} - \vec{u}_\epsilon\|_{\mathcal{C}([t_0, t_0 + \theta], (\tilde{M}^{2,3})^3)} = 0$.

6. The condition $\lim_{t\to 0} \sqrt{t}\|\vec{u}\|_\infty = 0$

The results described above may be partially extended to the case when $\vec{u}_0 \notin (\tilde{M}^{2,3})^3$, provided that we assume that $\vec{u}_0 \in (\dot{M}^{2,3})^3$ and $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$. Under those assumptions, we have the following results:

i) *local existence* : there exists a positive $T = T(\vec{u}_0)$ such that the sequence $\vec{u}^{(n)}$ defined by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

remains bounded in the space $(\mathcal{E}_T)^3$ where $\mathcal{E}_T$ is defined as

$$f \in \mathcal{E}_T \iff \begin{cases} f \in L^2_{loc}((0,T)\times\mathbb{R}^3), \\ \sup_{0<t<T} t^{1/4}\|f(t,.)\|_{\dot{M}^{4,6}} < \infty \text{ and} \\ \sup_{0<t<T} t^{1/2}\|f(t,.)\|_\infty < \infty \end{cases}$$

Moreover, the sequence $\vec{u}^{(n)}$ converges in $(\mathcal{E}_T)^3$ to a solution $\vec{u}$ of (7) which belongs to $\mathcal{C}([0,T], (\dot{M}^{2,3})^3)$ and satisfies $\lim_{t\to 0} \sqrt{t}\|\vec{u}\|_\infty = 0$.

ii) *uniqueness* : if $\vec{u}_1$ and $\vec{u}_2$ are two solutions of (7) (associated to the same initial value $\vec{u}_0$) which satisfy (for $j = 1, 2$)

$$\vec{u}_j \in L^\infty([0,T], (\dot{M}^{2,3})^3) \text{ and } \lim_{t\to 0} \sqrt{t}\|\vec{u}_j\|_\infty = 0,$$

then $\vec{u}_1 = \vec{u}_2$.

iii) *maximal solutions* : let $\vec{u}$ be the maximal continuation in $L^\infty_{loc}((0,T^*), (L^\infty)^3)$ of the mild solution associated to $\vec{u}_0$ by Point i). Then, $\vec{u} \in \mathcal{C}((0,T^*), (\tilde{M}^{2,3})^3)$. More precisely, $B(\vec{u},\vec{u})$ is locally Hölderian (of exponent 1/2) [27] [18] and is continuous at $t = 0$. However, $t \mapsto e^{t\Delta}\vec{u}_0$ may be not (strongly) continuous at $t = 0$.

iv) *weak solutions* : we cannot identify the weak solutions given by Point C) in Theorem 1 to the maximal solution given by Point iii). However, if $\vec{u}$ is such a weak solution and if moreover $\vec{u} \in L^\infty_{loc}((0,T^*), (L^\infty)^3)$ with $\lim_{t\to 0} \sqrt{t}\|\vec{u}\|_\infty = 0$, then $\vec{u}$ is the maximal solution given by Point iii). This is a direct consequence of Point ii), since in that case $\vec{u}$ obviously belongs to $L^\infty([0,T], (\dot{M}^{2,3})^3)$ for every $T < T^*$: indeed, we already know that $\int_{B(x_0,R)} |\vec{u}(t,x)|^2 \, dx \le C(R+\sqrt{t})$; moreover, we have $|\vec{u}(t,x)| \le C_T t^{-1/2}$, hence $\int_{B(x_0,R)} |\vec{u}(t,x)|^2 \, dx \le C'_T R^3 t^{-1}$, and we conclude since $\min(R+\sqrt{t}, 2R^3 t^{-1}) = 2R$.

7. Comparison of mild solutions

When $\vec{u}_0 \in (\dot{M}^{2,3})^3$, Kato's algorithm may converge to a solution of the Navier–Stokes equations on other norms than the norm of $\mathcal{E}_T$. Indeed, we have seen that there exists a positive constant C_0 such that

$$(47) \qquad \|B(\vec{u}, \vec{v})\|_{\mathcal{E}_T} \le C_0 \|\vec{u}\|_{\mathcal{E}_T} \|\vec{v}\|_{\mathcal{E}_T};$$

Thus, the sequence $\vec{u}^{(n)}$ defined by

$$\vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \text{ and } \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)})$$

will converge in $\mathcal{E}_T^3$ to a solution $\vec{u}$ of (7), as soon as $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{E}_T} \le \frac{1}{4C_0}$.

On the other hand, we may use the embedding $\dot{M}^{2,3} \subset BMO^{(-1)}$. Koch and Tataru [13] have proved that there exists a positive constant C_1 such that

$$(48) \qquad \|B(\vec{u}, \vec{v})\|_{\mathcal{G}_T} \le C_0 \|\vec{u}\|_{\mathcal{G}_T} \|\vec{v}\|_{\mathcal{G}_T}$$

where $\mathcal{G}_T$ is defined as

$$f \in \mathcal{G}_T \iff \begin{cases} f \in L^2_{loc}((0,T) \times \mathbb{R}^3) \\ \sup_{0<t<T, \ x_0 \in \mathbb{R}^3} t^{-3/2} \int_0^t \int_{B(x_0,\sqrt{t})} |f(s,x)|^2 \ dx \ ds < \infty \\ \sup_{0<t<T} t^{1/2} \|f(t,.)\|_\infty < \infty \end{cases}$$

Thus, the sequence $\vec{u}^{(n)}$ will converge in $\mathcal{G}_T^3$ to a solution $\vec{u}$ of (7), as soon as $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{G}_T} \le \frac{1}{4C_1}$. Due to Lemma 3, this sequence will then satisfy

$$(49) \qquad \sup_{0<t<T} \sqrt{t}\|\vec{u}^{(n+1)} - \vec{u}^{(n)}\|_\infty \le C\frac{1}{(n+1)^2}.$$

We may as well use the embedding $L^{3,\infty} \subset \dot{M}^{2,3}$. Meyer [27] has proved that there exists a positive constant C_2 such that

$$(50) \qquad \|B(\vec{u}, \vec{v})\|_{\mathcal{H}_T} \le C_2 \|\vec{u}\|_{\mathcal{H}_T} \|\vec{v}\|_{\mathcal{H}_T}$$

where $\mathcal{H}_T$ is defined as $\mathcal{H}_T = \mathcal{C}_w([0,T], L^{3,\infty}(\mathbb{R}^3))$ ($f \in \mathcal{H}_T$ means that f is continuous and bounded from $(0,T]$ to $L^{3,\infty}$ and is weakly continuous at $t = 0$). Thus, the sequence $\vec{u}^{(n)}$ will converge in $\mathcal{H}_T^3$ to a solution $\vec{u}$ of (7), as soon as $\|e^{t\Delta}\vec{u}_0\|_{\mathcal{H}_T} \le \frac{1}{4C_2}$. Due to Lemma 3, this sequence will then satisfy

$$(51) \qquad \sup_{0<t<T} \|\vec{u}^{(n+1)} - \vec{u}^{(n)}\|_{\dot{M}^{2,3}} \le C\frac{1}{(n+1)^2}.$$

A frequently asked question on mild solutions is the regularity of those solutions. Since Meyer's solutions do not use the smoothing properties of the heat kernel, one may wonder if the solution $\vec{u}$ obtained in $(\mathcal{H}_T)^3$ will satisfy $\sup_{0<t<T} \sqrt{t}\|\vec{u}(t,.)\|_\infty < \infty$. Such a question is raised for instance in [9]. As a matter of fact, the answer is positive, due to the persistency formalism developed in [7], [18], [19] which gives a more precise answer:

Theorem 3. *Let* $\vec{u}_0 \in (\dot{M}^{2,3}(\mathbb{R}^3))^3$ *with* $\vec{\nabla}.\vec{u}_0 = 0$. *Let the sequence* $\vec{u}^{(n)}$ *be defined by*

$$(52) \qquad \vec{u}^{(0)} = e^{t\Delta}\vec{u}_0 \ and \ \vec{u}^{(n+1)} = e^{t\Delta}\vec{u}_0 - B(\vec{u}^{(n)}, \vec{u}^{(n)}).$$

Let $T \in (0, +\infty]$. *Then the following assertions are equivalent:*

$$(A) \qquad \sum_{n\in\mathbb{N}} \sup_{0<t<T} \|\vec{u}^{(n+1)}(t,.) - \vec{u}^{(n)}(t,.)\|_{\dot{M}^{2,3}} < \infty$$

$$(B) \qquad \sum_{n\in\mathbb{N}} \sup_{0<t<T} \sqrt{t}\|\vec{u}^{(n+1)}(t,.) - \vec{u}^{(n)}(t,.)\|_\infty < \infty$$

Let A_T and B_T be the norms

$$A_T(f) = \sup_{0<t<T} \|f(t,.)\|_{\dot{M}^{2,3}} \ \text{and} \ B_T(f) = \sqrt{t}\|f(t,.)\|_\infty.$$

We easily get

$$A_T(B(\vec{u}, \vec{v})) \leq C \min(A_T(\vec{u})B_T(\vec{v}), B_T(\vec{u})A_T(\vec{v}))$$
$$(53) \qquad\qquad\qquad \leq C\sqrt{A_T(\vec{u})B_T(\vec{v})B_T(\vec{u})A_T(\vec{v})}.$$

On the other hand, we have, for $0 < \tau < t < T$

$$\|B(\vec{u}, \vec{v}))\|_\infty \leq C \int_0^\tau \frac{ds}{(t-s)^{3/2}} A_T(\vec{u})A_T(\vec{v}) + C \int_\tau^t \frac{ds}{s\sqrt{t-s}} B_T(\vec{u})B_T(\vec{v}).$$

Hence, we have

$$\|B(\vec{u}, \vec{v}))\|_\infty \leq C(t-\tau)^{-1/2}A_T(\vec{u})A_T(\vec{v}) + C\tau^{-1}\sqrt{t-\tau}B_T(\vec{u})B_T(\vec{v}).$$

For

$$\tau = t\frac{B_T(\vec{u})B_T(\vec{v})}{A_T(\vec{u})A_T(\vec{v}) + B_T(\vec{u})B_T(\vec{v})}$$

we find

$$(54) \qquad B_T(B(\vec{u}, \vec{v})) \leq C\sqrt{A_T(\vec{u})A_T(\vec{v})(A_T(\vec{u})A_T(\vec{v}) + B_T(\vec{u})B_T(\vec{v}))}.$$

Inequalities (53) and (54) then give

$$
\begin{aligned}
A_T(B(\vec{u}, \vec{v})) &+ B_T(B(\vec{u}, \vec{v})) \\
&\leq C\sqrt{A_T(\vec{u})A_T(\vec{v})(A_T(\vec{u}) + B_T(\vec{u}))(A_T(\vec{v}) + B_T(\vec{v}))}.
\end{aligned}
$$
(55)

From (55) (to get $(A) \Rightarrow (B)$) and (53) (to get $(B) \Rightarrow (A)$), we see that Theorem 3 is a direct consequence of the following lemma:

Lemma 4. *Let N_1 and N_2 be two norms on a vector space E and T a bilinear operator on E such that there exists a positive constant C_0 such that*

$$
(56) \qquad \forall x, y \in E \quad N_2(T(x,y)) \leq C_0\sqrt{N_1(x)N_2(x)N_1(y)N_2(y)}.
$$

Let $x_0 \in E$ and let the sequence (x_n) be defined by

$$
(57) \qquad\qquad\qquad x_{n+1} = T(x_n, x_n).
$$

If $\sum_{n\in\mathbb{N}} N_1(x_{n+1} - x_n) < \infty$, then $\sum_{n\in\mathbb{N}} N_2(x_{n+1} - x_n) < \infty$.

Indeed, let $\alpha_0 = N_1(x_0)$, $\alpha_{n+1} = N_1(x_{n+1} - x_n)$ and, similarly, $\beta_0 = N_2(x_0)$, $\beta_{n+1} = N_2(x_{n+1} - x_n)$. We write

$$
\begin{aligned}
x_{n+2} - x_{n+1} &= T(x_{n+1}, x_{n+1}) - T(x_n, x_n) \\
&= T(x_{n+1} - x_n, x_{n+1}) + T(x_n, x_{n+1} - x_n),
\end{aligned}
$$

hence

$$
(58) \qquad
\begin{aligned}
\beta_{n+2} &\leq C_0\sqrt{2\alpha_{n+1}\beta_{n+1}\Big(\sum_{k=0}^{n+1}\alpha_k\Big)\Big(\sum_{j=0}^{n+1}\beta_j\Big)} \\
&\leq \frac{1}{2}\beta_{n+1} + C_0^2\alpha_{n+1}\Big(\sum_{k=0}^{n+1}\alpha_k\Big)\Big(\sum_{j=0}^{n+1}\beta_j\Big).
\end{aligned}
$$

Now, if N_0 is big enough to grant that

$$
C_0^2\Big(\sum_{k=N_0+1}^{+\infty}\alpha_{k+1}\Big)\Big(\sum_{j=0}^{+\infty}\alpha_j\Big) \leq \frac{1}{4}
$$

we get that, for $N \geq N_0 + 2$ we have

$$
\sum_{j=0}^{N}\beta_j \leq \beta_0 + \beta_1 + \frac{1}{2}\sum_{j=1}^{N-1}\beta_j + \sum_{j=1}^{N_0}C_0^2\alpha_j\Big(\sum_{k=0}^{j}\alpha_k\Big)\Big(\sum_{l=0}^{j}\beta_l\Big) + \frac{1}{4}\sum_{j=0}^{N-1}\beta_j
$$

and thus

$$
(59) \qquad \sum_{j=0}^{\infty}\beta_j \leq 4\Big(\beta_0 + \beta_1 + \sum_{j=1}^{N_0}C_0^2\alpha_j\Big(\sum_{k=0}^{j}\alpha_k\Big)\Big(\sum_{l=0}^{j}\beta_l\Big)\Big)
$$

8. Serrin's uniqueness criterion

Recall that we consider the Navier-Stokes equations on the whole space $\mathbb{R}^3$:

$$(60) \qquad \begin{cases} \exists p \in \mathcal{D}'((0,T) \times \mathbb{R}^3) & \partial_t \vec{u} = \Delta \vec{u} - \vec{\nabla}.(\vec{u} \otimes \vec{u}) - \vec{\nabla} p \\ \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

We shall speak of weak solutions when the derivatives in (60) are taken in the sense of distributions theory.

Leray [21] studied the Cauchy initial value problem for equations (60) with a square-integrable initial value. He proved the existence of weak solutions, which satisfy moreover an energy inequality:

Definition 2 (Leray solutions). *A Leray solution on $(0,T)$ for the Navier-Stokes equations with initial value $\vec{u}_0 \in (L^2)^3$ is a solution $\vec{u}$ such that*

   i) *$t \mapsto \vec{u}(t,.)$ is weakly continuous from $(0,T)$ to $(L^2)^3$*

   ii) *$\vec{u}(t,.)$ converges weakly to $\vec{u}_0$ as $t \to 0^+$,*

   iii) *$\vec{u} \in L^\infty((0,T),(L^2)^3) \cap L^2((0,T),(\dot{H}^1)^3)$,*

   iv) *$\vec{u}$ satisfies the Leray energy inequality*

$$(61) \qquad \textit{for all } t \in (0,T), \ \|\vec{u}(t,.)\|_2^2 + 2 \int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes \vec{u}|^2 \ dx \ ds \leq \|\vec{u}_0\|_2^2.$$

Weak continuity of (a representant of) $\vec{u}$ is a consequence of the Navier–Stokes equations and of the hypothesis iii). An easy consequence of inequality (61) is then the strong continuity at $t = 0$:

$$\lim_{t \to 0^+} \|\vec{u} - \vec{u}_0\|_2 = 0.$$

But it is still not known whether we have continuity for all time t and whether we have uniqueness in the class of Leray solutions. Serrin's theorem [29] gives a criterion for uniqueness:

Proposition 3 (Serrin's uniqueness theorem). *Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0,+\infty]$) with initial value $\vec{u}_0$ such that :*

   i) *$\vec{u} \in L^\infty((0,T),(L^2(\mathbb{R}^3)^3)$;*

   ii) *$\vec{u} \in L^2((0,T),(\dot{H}^1(\mathbb{R}^3)^3)$;*

   iii) *For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T), L^{3/r}))^3$ with $2/\sigma = 1 - r$.*

Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

The limit case $r = 1$ is dealt with Von Wahl's theorem [31]:

Proposition 4. [Sohr and Von Wahl's uniqueness Theorem] *Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0, T) \times \mathbb{R}^3$ (for some $T \in (0, +\infty]$) with initial value $\vec{u}_0$ such that:*

i) *$\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^3)^3)$;*

ii) *$\vec{u} \in L^2((0,T), (H^1(\mathbb{R}^3)^3)$;*

iii) *$\vec{u}$ belongs to $(\mathcal{C}([0,T], L^3))^3$.*

Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0, T)$.

The theorem of Sohr and Von Wahl has been generalized to the case of a solution $\vec{u} \in (L^\infty([0,T], L^3))^3$ (instead of $(\mathcal{C}([0,T], L^3))^3$) by Kozono and Sohr [14].

We sketch the proof of those well-known propositions. Let $\vec{v}$ be another solution associated to $\vec{u}_0$ on $(0,T)$ (with associated pressure q) such that $\vec{v} \in L^\infty((0,T), (L^2(\mathbb{R}^3)^3) \cap L^2((0,T), (\dot{H}^1(\mathbb{R}^3)^3)$. The main point is to check the validity of the formula

$$(62)\quad \partial_t \int \vec{u}.\vec{v}\,dx = -2\int \vec{\nabla}\otimes\vec{u}.\vec{\nabla}\otimes\vec{v}\,dx + \int \vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\,dx - \int \vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\,dx.$$

This is checked by regularizing $\vec{u}$ and $\vec{v}$: we use a smoothing function $\theta(t,x) = \alpha(t)\beta(x) \in \mathcal{D}(\mathbb{R}^{3+1})$, where α is supported in $[-1,1]$, with $\iint \theta\,dx\,dt = 1$, and define, for $\epsilon > 0$,

$$\theta_\epsilon(t,x) = \frac{1}{\epsilon^{3+1}}\,\theta\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right).$$

Then, $\theta_\epsilon * \vec{u}$ and $\theta_\epsilon * \vec{v}$ are smooth functions on $(\epsilon, T-\epsilon) \times \mathbb{R}^3$ and we may write $\partial_t\big((\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\big) = (\theta_\epsilon * \partial_t\vec{u}).(\theta_\epsilon * \vec{v}) + (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \partial_t\vec{v})$. We then get by an integration with respect to x :

$$(63)\quad \partial_t \int (\theta_\epsilon * \vec{u}).(\theta_\epsilon * \vec{v})\,dx = -2\int (\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\,dx$$

$$+ \int (\theta_\epsilon * [\vec{u}\otimes\vec{u}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{v}])\,dx + \int (\theta_\epsilon * [\vec{v}\otimes\vec{v}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}])\,dx$$

We may rewrite the last summand in

$$(64)\quad \int (\theta_\epsilon * [\vec{v}\otimes\vec{v}]).(\theta_\epsilon * [\vec{\nabla}\otimes\vec{u}])\,dx = -\int (\theta_\epsilon * [\vec{\nabla}.(\vec{v}\otimes\vec{v})]).(\theta_\epsilon * \vec{u})\,dx$$

$$= -\int (\theta_\epsilon * [\{\vec{v}.\vec{\nabla}\}\vec{v}]).(\theta_\epsilon * \vec{u})\,dx$$

To deal with $\theta_\epsilon * [\vec{u} \otimes \vec{u}]$, we write that the pointwise product maps $L^{2/r} \dot{H}^r \times L^{2/(1-r)} L^{3/r}$ to $L^2 L^2$, hence $\theta_\epsilon * [\vec{u} \otimes \vec{u}]$ converges strongly to $\vec{u} \otimes \vec{u}$ in $(L^2((0,T) \times \mathbb{R}^3))^{3\times3}$. To deal with $\theta_\epsilon * [\{\vec{v}.\vec{\nabla}\}\vec{v}]$, we write that the pointwise product maps $\dot{H}^r \times L^2$ to the pre-dual $L^{\frac{3}{3-r}}$ of $L^{3/r}$ and that smooth functions are dense in $L^{3/r}$; thus, $\theta_\epsilon * [(\vec{v}.\vec{\nabla})\vec{v}]$ converges strongly to $(\vec{v}.\vec{\nabla})\vec{v}$ in $(L^{\frac{2}{1+r}} L^{\frac{3}{3-r}})^3$ while $\theta_\epsilon * \vec{u}$ converges weakly to $\vec{u}$ in $(L^{\frac{2}{1-r}} L^{3/r})^3$. This proves (62).

Since $\vec{u} \otimes \vec{u} \in (L^2((0,T) \times \mathbb{R}^3))^{3\times3}$ and $\vec{u} = e^{t\Delta}\vec{u}(0) - \mathbb{P} \int_0^t e^{(t-s)\Delta}\vec{\nabla}.(\vec{u} \otimes \vec{u})\, ds$, $t \mapsto \vec{u}$ is continuous from $[0,T]$ to $(L^2(dx))^3$ and (since $t \mapsto \vec{v}$ is weakly continuous from $[0,T]$ to $(L^2(dx))^3$) $t \mapsto \int \vec{u}.\vec{v}\, dx$ is continuous. Thus, we may integrate equality (62) and obtain

$$(65) \quad \begin{aligned} \int \vec{u}(t,x).\vec{v}(t,x)\, dx &+ 2 \int_0^t \int_{\mathbb{R}^3} \vec{\nabla} \otimes \vec{u}.\vec{\nabla} \otimes \vec{v}\, dx\, ds \\ &= \|\vec{u}_0\|_2^2 + \int_0^t \int_{\mathbb{R}^3} \vec{u}.(\vec{u}.\vec{\nabla})\vec{v}\, dx\, ds - \int_0^t \int_{\mathbb{R}^3} \vec{u}.(\vec{v}.\vec{\nabla})\vec{v}\, dx\, ds \end{aligned}$$

Of course, this equality holds as well for $\vec{v} = \vec{u}$.

Now, if we assume moreover that $\vec{v}$ satisfies the Leray inequality

$$(66) \quad \|\vec{v}(t)\|_2^2 + 2 \int_0^t \|\vec{\nabla} \otimes \vec{v}\|_2^2\, ds \le \|\vec{u}_0\|_2^2,$$

we get the following inequality for $\vec{u} - \vec{v}$:

$$(67) \quad \begin{aligned} \|\vec{u}(t,.) - \vec{v}(t,.)\|_2^2 \le &- 2 \int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes (\vec{u} - \vec{v})|^2\, dx\, ds \\ &- 2 \int_0^t \int_{\mathbb{R}^3} \vec{u}.\big((\vec{u} - \vec{v}).\vec{\nabla}\big)\vec{v}\, dx\, ds \end{aligned}$$

Moreover, we have the antisymmetry property $\int_0^t\int_{\mathbb{R}^3} \vec{u}.\big((\vec{u} - \vec{v}).\vec{\nabla}\big)\vec{u}\, dx\, ds = 0$.

We then write

$$(68) \quad \begin{aligned} &\left| \int_\tau^t \int_{\mathbb{R}^3} \vec{u}.\big((\vec{u} - \vec{v}).\vec{\nabla}\big)(\vec{v} - \vec{u})\, dx\, ds \right| \\ &\le C_r \left(\int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}}\, ds \right)^{\frac{1-r}{2}} \left(\int_0^t \|\vec{v} - \vec{u}\|_{\dot{H}^1}^2\, ds \right)^{\frac{1}{2}} \left(\int_\tau^t \|\vec{v} - \vec{u}\|_{\dot{H}^r}^{\frac{2}{r}}\, ds \right)^{\frac{r}{2}} \\ &\le C_r' \left(\int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}}\, ds \right)^{\frac{1-r}{2}} \left(\int_0^t \|\vec{v} - \vec{u}\|_{\dot{H}^1}^2\, ds \right)^{\frac{1+r}{2}} \left(\sup_{\tau<s<t} \|\vec{v} - \vec{u}\|_2^2 \right)^{\frac{1-r}{2}} \end{aligned}$$

If $r < 1$, we may easily conclude: we write with help of the Young inequality

$$(69) \quad C_r' a^{(1-r)/2} b^{(1+r)/2} \le \frac{1-r}{2}\, C_r'^{2/(1-r)} a + \frac{1+r}{2}\, b$$

Thus, if $\vec{u} = \vec{v}$ on $[0, \tau]$, we find from (67) and (68) that

$$(70) \qquad \sup_{0 < s \leq t} \|\vec{u} - \vec{v}\|_2^2 \leq C_r'' \left(\int_\tau^t \|\vec{u}\|_{L^{3/r}}^{\frac{2}{1-r}} \, ds \right)^{1/2} \sup_{0 < s \leq t} \|\vec{u} - \vec{v}\|_2^2$$

and uniqueness is valid on a bigger interval. By weak continuity of $t \mapsto \vec{v}$, we find $\vec{u} = \vec{v}$.

If $r = 1$, $\vec{u}$ belongs to $(\mathcal{C}([0, T], L^3))^3$; if $T_0 < T$, then for each $\epsilon > 0$ we may split $\vec{u}$ on $[0, T_0]$ in $\vec{u} = \vec{\alpha} + \vec{\beta}$ with $\|\vec{\alpha}\|_{L^\infty L^3} < \epsilon$ and $\vec{\beta} \in (L^\infty((0, T_0) \times \mathbb{R}^3))^3$. Then we write

$$\left| \int_0^t \int_{\mathbb{R}^3} \vec{u}.\big((\vec{u} - \vec{v}).\vec{\nabla}\big)(\vec{v} - \vec{u}) \, dx \, ds \right|$$

$$\leq C\|\vec{\alpha}\|_{L^\infty L^3} \int_0^t \|\vec{v} - \vec{u}\|_{\dot{H}^1}^2 \, ds$$

$$(71) \qquad + \|\vec{\beta}\|_\infty \left(\int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes (\vec{v} - \vec{u})|^2 \, dx \, ds \right)^{\frac{1}{2}} \left(\int_0^t \int_{\mathbb{R}^3} |\vec{v} - \vec{u}|^2 \, dx \, ds \right)^{\frac{1}{2}}$$

$$\leq 2C\epsilon \int_0^t \int_{\mathbb{R}^3} |\vec{\nabla} \otimes (\vec{v} - \vec{u})|^2 \, dx \, ds$$

$$+ \left(\frac{4}{C\epsilon} \|\vec{\beta}\|_\infty^2 + C\epsilon \right) \int_0^t \int_{\mathbb{R}^3} |\vec{v} - \vec{u}|^2 \, dx \, ds.$$

Choosing ϵ such that $2C\epsilon < 1$, we get that

$$(72) \qquad \|\vec{v}(t, .) - \vec{u}(t, .)\|_2^2 \leq \left(\frac{4}{C\epsilon} \|\vec{\beta}\|_\infty^2 + C\epsilon \right) \int_0^t \|\vec{v}(s, .) - \vec{u}(s, .)\|_2^2 \, ds.$$

The Gronwall lemma gives then that $\vec{u} = \vec{v}$.

Thus, the main tool in proving Propositions 3 and 4 is the facts that when $f \in L^\infty L^2 \cap L^2 \dot{H}^1$, then f belongs to $L^{2/r} \dot{H}^r$ and that the pointwise product is bounded from $\dot{H}^r \times L^{3/r}$ to L^2. Considering the space $\dot{X}^r$ of pointwise multipliers from $\dot{H}^r$ to L^2 then gives a direct generalization of Propositions 3 and 4, as it has been observed in [18] [20]:

Definition 3 (Pointwise multipliers of negative order). *For $0 \leq r < \frac{3}{2}$, we define the space $\dot{X}_r(\mathbb{R}^3)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Sobolev space $\dot{H}^r(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\dot{X}_r$ is given by the operator norm of pointwise multiplication:*

$$(73) \qquad \|f\|_{\dot{X}_r} = \sup\{ \|fg\|_2 \ / \ \|g\|_{\dot{H}^r} \leq 1 \}.$$

The closure of the space $\mathcal{D}$ of smooth test functions in $\dot{X}_r$ will be denoted by $\tilde{X}_r$.

The spaces $\dot{X}_r$ have been characterized by Maz'ya [23] in terms of Sobolev capacities. A weaker result establishes a comparison between the spaces $\dot{X}_r$ and the Morrey–Campanato spaces $\dot{M}^{2,p}$ [5] [18].

Lemma 5 (Comparison theorem). *For $2 < p \le 3/r$ and $0 < r$ we have* $\dot{M}^{p,3/r} \subset \dot{X}_r \subset \dot{M}^{2,3/r}$.

Another easy result is the embedding $L^{3/r,\infty} \subset \dot{X}_r$ for $r < 3/2$.

We may now state the generalization of Propositions 3 and 4:

Proposition 5. *Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0, +\infty]$) with initial value $\vec{u}_0$ such that:*

i) $\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^3)^3)$;

ii) $\vec{u} \in L^2((0,T), (\dot{H}^1(\mathbb{R}^3)^3)$;

iii) *For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T), \dot{X}_r))^3$ with $2/\sigma = 1 - r$.*

Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

A similar results holds for $r = 1$ when iii) is replaced by

iii') $\vec{u}$ belongs to $(\mathcal{C}([0,T], \tilde{X}_1))^3$.

The structure of the multiplier spaces $\dot{X}_r$ is not easy to describe. However, when $r < 1$, we may replace the space $\dot{X}_r$ by the (greater) Morrey–Campanato space $\dot{M}^{2,3/r}$:

Theorem 4. *Let $\vec{u}_0 \in (L^2(\mathbb{R}^3))^3$ with $\vec{\nabla}.\vec{u}_0 = 0$. Assume that there exists a solution $\vec{u}$ of the Navier-Stokes equations on $(0,T) \times \mathbb{R}^3$ (for some $T \in (0, +\infty]$) with initial value $\vec{u}_0$ such that:*

i) $\vec{u} \in L^\infty((0,T), (L^2(\mathbb{R}^3)^3)$;

ii) $\vec{u} \in L^2((0,T), (\dot{H}^1(\mathbb{R}^3)^3)$;

iii) *For some $r \in [0,1)$, $\vec{u}$ belongs to $(L^\sigma((0,T), \dot{M}^{2,3/r}))^3$ with $2/\sigma = 1-r$.*

Then, $\vec{u}$ satisfies the Leray energy inequality and it is the unique Leray solution associated to $\vec{u}_0$ on $(0,T)$.

For $r = 0$, we have $\dot{M}^{2,\infty} = \dot{X}_0 = L^\infty$, and this is Serrin's theorem. When $0 < r < 1$, we use the fact that $L^2 \cap \dot{H}^1 \subset \dot{B}_2^{r,1} \subset \dot{H}^r$. Thus, in generalizing Serrin's theorem, we may replace the pointwise multipliers from $\dot{H}^r$ to L^2 by the pointwise multipliers from the Besov space $\dot{B}_2^{r,1}$ to L^2. We then conclude with the following lemma:

Lemma 6. *For $0 \leq r \leq 3/2$, we define the space $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)$ as the space of functions which are locally square integrable on $\mathbb{R}^3$ and such that pointwise multiplication with these functions maps boundedly the Besov space $\dot{B}_2^{r,1}(\mathbb{R}^3)$ to $L^2(\mathbb{R}^3)$. The norm in $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)$ is given by the operator norm of pointwise multiplication:*

$$
(74) \qquad \|f\|_{\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)} = \sup\{ \|fg\|_2 \ / \ \|g\|_{\dot{B}_2^{r,1}} \leq 1\}.
$$

Then, f belongs to $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)$ if and only if f belongs to $\dot{M}^{2,3/r}$ (with equivalence of norms).

The embedding $\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2) \subset \dot{M}^{2,3/r}$ is obvious: if $\omega \in \mathcal{D}(\mathbb{R}^3)$ is equal to 1 on $B(0,1)$, then we find that

$$
\int_{B(x_0,R)} |f(x)|^2 \, dx \leq \|f\|_{\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)}^2 \left\| \omega\left(\frac{x-x_0}{R}\right) \right\|_{\dot{B}_1^{r,2}}^2
$$
$$
= R^{3-2r} \|f\|_{\mathcal{M}(\dot{B}_2^{r,1} \mapsto L^2)}^2 \|\omega\|_{\dot{B}_1^{r,2}}^2.
$$

Conversely, if $f \in \dot{M}^{2,3/r}$ and $g \in \dot{B}_1^{2,3/r}$, then we use the decomposition of g in a regular enough Daubechies basis of compactly supported wavelets [26], [10], [18]. The wavelet basis is an orthonormal basis of $L^2(\mathbb{R}^3)$ which is given as a family of functions $(\psi_{\epsilon,j,k})_{1 \leq \epsilon \leq 7, j \in \mathbb{Z}, k \in \mathbb{Z}^3}$ derived through dyadic dilations and translations from a finite set of functions $(\psi_\epsilon)_{1 \leq \epsilon \leq 7}$:

$$
(75) \qquad \psi_{\epsilon,j,k}(x) = 2^{3j/2} \psi_\epsilon(2^j x - k)
$$

where the functions ψ_ϵ are compactly supported and of class $\mathcal{C}^2$. Then for $0 \leq r < 3/2$, the family $(\psi_{\epsilon,j,k})_{1 \leq \epsilon \leq 7, j \in \mathbb{Z}, k \in \mathbb{Z}^3}$ is a Riesz basis of $\dot{B}_2^{r,1}(\mathbb{R}^3)$; more precisely, there exists two positive constants $0 < A_r \leq B_r < \infty$ such that for all $g \in \dot{B}_2^{r,1}(\mathbb{R}^3)$ we have

$$
(76) \qquad A_r \|g\|_{\dot{B}_2^{r,1}} \leq N_r(g) \leq B_r \|g\|_{\dot{B}_2^{r,1}(\mathbb{R}^3)}
$$

with

$$
(77) \qquad N_r(g) = \sum_{j \in \mathbb{Z}} 2^{jr} \sqrt{\sum_{k \in \mathbb{Z}^3} \sum_{1 \leq \epsilon \leq 7} |\langle g|\psi_{\epsilon,j,k}\rangle|^2}.
$$

Now, we write

$$
(78) \qquad \|fg\|_2 \leq \sum_{j \in \mathbb{Z}} \left\| \sum_{k \in \mathbb{Z}^3} \sum_{1 \leq \epsilon \leq 7} \langle g|\psi_{\epsilon,j,k}\rangle \, \psi_{\epsilon,j,k} f \right\|_2 ;
$$

since the family $(\psi_{\epsilon,j,k}f)_{k\in\mathbb{Z}^3,\ 1\le\epsilon\le 7}$ is (uniformly) locally finite, we find that

$$(79)\quad \Big\|\sum_{k\in\mathbb{Z}^3}\sum_{1\le\epsilon\le 7}\langle g|\psi_{\epsilon,j,k}\rangle\,\psi_{\epsilon,j,k}f\Big\|_2^2 \le C\sum_{k\in\mathbb{Z}^3}\sum_{1\le\epsilon\le 7}\big|\langle g|\psi_{\epsilon,j,k}\rangle\big|^2\|\psi_{\epsilon,j,k}f\|_2^2$$

and, since ψ_ϵ is bounded and compactly supported,

$$(80)\qquad\qquad \|\psi_{\epsilon,j,k}f\|_2^2 \le C 2^{3j}\|f\|_{\dot M^{2,3/r}}^2 2^{-3j(1-\frac{2r}{3})}.$$

Thus, we get

$$(81)\quad \|fg\|_2 \le C\sum_{j\in\mathbb{Z}}\sqrt{\sum_{k\in\mathbb{Z}^3}\sum_{1\le\epsilon\le 7}|\langle g|\psi_{\epsilon,j,k}\rangle|^2\|f\|_{\dot M^{2,3/r}}^2 2^{2jr}} \le C'\|g\|_{\dot B_2^{r,1}}\|f\|_{\dot M^{2,3/r}}.$$

Thus, Lemma 6 is proved.

Theorem 4 does not include the limit case $r = 1$, which is still an open question:

Open question 1. *In Theorem 4, does a similar results holds for $r = 1$ when* iii) *is replaced by*

iii') $\vec u$ *belongs to* $(\mathcal{C}([0,T], \tilde M^{2,3}))^3$?

We end this section with two further remarks:

i) the condition $\vec u \in (L^2((0,T), L^\infty))^3$ in the limit case $r = 0$ may be modified in $\vec u \in (L^2((0,T), BMO))^3$ [15]. In order to prove this, one replaces poinwise multiplication with the paraproduct operator. Using paramultipliers instead of multipliers, Germain has recently extended Proposition 5 to negative values of r [8].

ii) If $\vec u$ is a solution to the Navier–Stokes equations such that

$$\vec u \in (L^\sigma((0,\infty), \dot M^{2,3/r}))^3$$

with $2/\sigma = 1 - r$ and $0 < r < 1$, then it is easy to see that $\vec u_0$ belongs to the Besov space $(\dot B_{\dot M^{2,3/r}}^{r-1,\sigma})^3$ based on the Morrey–Campanato space $\dot M^{2,3/r}$. Conversely, if $\vec u_0$ belongs to the Besov space $(\dot B_{\dot M^{2,3/r}}^{r-1,\sigma})^3$ and has a small enough norm in this space, then one can construct a solution $\vec u \in (L^\sigma((0,\infty), \dot M^{2,3/r}))^3$ [16] [18].

9. Uniqueness theorems

In 1997, Furioli, Lemarié-Rieusset and Terraneo [6] proved uniqueness of mild solutions in $\mathcal{C}([0, T^*), (L^3)^3)$. They extended their proof to the case of Morrey-Campana to spaces by using the Besov spaces over Morrey-Campanato spaces described by Kozono and Yamazaki [16] and found that uniqueness holds as well in the class $\mathcal{C}([0, T^*), (\tilde{M}^{p,3})^3)$ for $p > 2$, where $\tilde{M}^{p,3}$ is the closure of the smooth compactly supported functions in the Morrey-Campanato space $\dot{M}^{p,3}$. In his thesis dissertation, May [22] [20] proved a slightly more general result by extending the approach of Monniaux (i.e. by using the maximal $L^p L^q$ property of the heat kernel):

Proposition 6. *If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0, T^*) \times \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0, T^*), (\tilde{X}_1)^3)$ and have the same initial value, then $\vec{u} = \vec{v}$.*

May's result generalizes the results of Furioli, Lemarié–Rieusset and Terraneo, but leaves open the limit case of $\dot{M}^{2,3}$:

Open question 2. *Does uniqueness holds in $(\mathcal{C}([0, T^*), \tilde{M}^{2,3}))^3$?*

The problem of uniqueness we may consider in a more general approach is the following one:

Definition 4 (Regular critical space). *A regular critical space is a Banach space X such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^3) \subset X \subset L^2_{loc}(\mathbb{R}^3)$ and such that moreover:*

(a) *for all $x_0 \in \mathbb{R}^3$ and for all $f \in X$, $f(x - x_0) \in X$ and $\|f\|_X = \|f(x - x_0)\|_X$.*

(b) *for all $\lambda > 0$ and for all $f \in X$, $f(\lambda x) \in X$ and $\lambda \|f(\lambda x)\|_X = \|f\|_X$.*

(c) *$\mathcal{D}(\mathbb{R}^3)$ is dense in X.*

We have the obvious result:

Lemma 7. *Let X be a regular critical space. Then X is continuously embedded in $\tilde{M}^{2,3}$.*

We shall then consider the problem of uniqueness in $\mathcal{C}([0, T^*), X^3)$:

Uniqueness problem:

Let X be a regular critical space. If $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0, T^*) \times \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0, T^*), X^3)$ and have the same initial value, then do we have $\vec{u} = \vec{v}$?

We may do the following remarks on this problem:

i) We can write the Navier–Stokes equations as

(82)
$$\begin{cases} \partial_t \vec{u} = \Delta \vec{u} - \mathbb{P}\vec{\nabla} \cdot (\vec{u} \otimes \vec{u}) \\ \vec{\nabla}.\vec{u} = 0 \end{cases}$$

since a solution $\vec{u} \in \mathcal{C}([0, T^*), X^3)$ vanishes at infinity in the sense of [6] (see also [18]). Another way to write the equations is then

(83)
$$\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u})$$

where B is the bilinear operator

(84)
$$B(\vec{u}, \vec{v})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.(\vec{u} \otimes \vec{v}) \, ds.$$

ii) It is easy to check that, in a regular critical space, local uniqueness implies global uniqueness. Local uniqueness means that, if $T^* > 0$ and if $\vec{u}$ and $\vec{v}$ are two weak solutions of the Navier-Stokes equations on $(0, T^*) \times \mathbb{R}^3$ such that $\vec{u}$ and $\vec{v}$ belong to $\mathcal{C}([0, T^*), X^3)$ and have the same initial value, then there exists a positive ϵ such that we have $\vec{u} = \vec{v}$ on $[0, \epsilon] \times \mathbb{R}^3$. Global uniqueness then means that we must have $\vec{u} = \vec{v}$ on $[0, T^*)$.

iii) The basic idea in Furioli, Lemarié-Rieusset and Terraneo [6] is to split the solutions in tendency and fluctuation, and to use different estimates on each term. More precisely, we consider two mild solutions $\vec{u} = e^{t\Delta}\vec{u}_0 - B(\vec{u}, \vec{u}) = e^{t\Delta}\vec{u}_0 - \vec{w}_1$ and $\vec{v} = e^{t\Delta}\vec{u}_0 - B(\vec{v}, \vec{v}) = e^{t\Delta}\vec{u}_0 - \vec{w}_2$ in $\mathcal{C}([0, T^*), X^3)$ and write $\vec{w} = \vec{u} - \vec{v} = \vec{w}_2 - \vec{w}_1 = -B(\vec{w}, \vec{v}) - B(\vec{u}, \vec{w})$, and finally

(85) $$\vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0).$$

Thus we see clearly the role of the fluctuations: they control the behaviour of $\vec{w}$. We shall use the regularization properties of the heat kernel for the term $e^{t\Delta}\vec{u}_0$ (mainly, that $\lim_{t\to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$), while we shall use the fact that, for $i = 1$ or 2, we have

$$\lim_{\epsilon \to 0} \sup_{0 \le t \le \epsilon} \|\vec{w}_i(t)\|_X = 0,$$

thus we shall assume that the norm of $\vec{w}_i$ is very small.

Those remarks give us a simple way for proving uniqueness of some regular solutions. First, we define fully adapted critical Banach spaces for the Navier–Stokes equations. The notion of adapted spaces was introduced by Cannone in his book [2]: Cannone studied Banach spaces X such that the bilinear operator B defined by

$$B(\vec{f},\vec{g})(t) = \int_0^t e^{(t-s)\Delta}\mathbb{P}\vec{\nabla}.\vec{f}\otimes\vec{g}\ ds$$

is bounded from $L^\infty((0,T),X^3)\times L^\infty((0,T),X^3)$ to $L^\infty((0,T),X^3)$. According to Cannone, a Banach space X is *adapted* to the Navier-Stokes equations if the following assertions are satisfied:

1) X is a shift-invariant Banach space of distributions
2) the pointwise product between two elements of X is still well defined as a tempered distribution
3) there is a sequence of real numbers $\eta_j > 0$, $j \in \mathbb{Z}$, such that

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\,\eta_j < \infty$$

and such that

$$\forall j \in \mathbb{Z}, \forall f \in X, \forall g \in X\ \|\Delta_j(fg)\|_X \leq \eta_j\,\|f\|_X\,\|g\|_X$$

If X is a Banach space adapted (according to Cannone) to the Navier-Stokes equations, then the bilinear transform B is continuous on $L^\infty((0,T),X^3)$. But this definition doesn't work in the case of critical spaces: if the norm of X is invariant under the dilations $f \mapsto \lambda f(\lambda x)$ and if we have the inequalities $\|\Delta_j(fg)\|_X \leq \eta_j\,\|f\|_X\,\|g\|_X$, then we find that $\eta_j = 2^j\eta_0$ and thus

$$\sum_{j\in\mathbb{Z}} 2^{-|j|}\,\eta_j = \infty\,.$$

Other definitions of adapted spaces have been proposed by Meyer and Muschetti [27] or Auscher and Tchamitchian [1]. While those definitions are introduced to deal with critical spaces, they don't allow to prove the boundedness of B on $L^\infty((0,T),X^3)$, but on a smaller space of smooth trajectories (as in the case of Theorem 1, points (A) and (B)).

However, there are some critical shift–invariant spaces E for which the boundedness of B on $L^\infty((0,T),E^3)$ is known : the first instance was given by Le Jan and Sznitman [17] and is known since the works of Cannone as the Besov space $\dot{B}_{PM}^{2,\infty}$ based on pseudo-measures [3] [18]; another instance was then given by Yves Meyer [27]: the Lorentz space $L^{3,\infty}$. All those examples can be dealt with with the following notion of fully adapted Banach spaces:

Definition 5 (Fully adapted critical space). *A fully adapted critical Banach space for the Navier–Stokes equations is a Banach space E such that we have the continuous embeddings $\mathcal{D}(\mathbb{R}^3) \subset E \subset L^2_{loc}(\mathbb{R}^3)$ and such that moreover:*

(a) *for all $x_0 \in \mathbb{R}^3$ and for all $f \in E$, $f(x - x_0) \in E$ and $\|f\|_E = \|f(x - x_0)\|_E$.*

(b) *for all $\lambda > 0$ and for all $f \in E$, $f(\lambda x) \in E$ and $\lambda\|f(\lambda x)\|_E = \|f\|_E$.*

(c) *The closed unit ball of E is a metrizable compact subset of $\mathcal{S}'(\mathbb{R}^3)$.*

(d) *e^{Δ} maps boundedly E to the space $\mathcal{M}$ of pointwise multipliers of E*

(e) *Let F be the Banach space*
$$F = \left\{ f \in L^1_{loc} \ / \ \exists (f_n), (g_n) \in E^{\mathbb{N}} \right.$$
$$\left. s.t. \ f = \sum_{n \in \mathbb{N}} f_n g_n \ and \ \sum_{n \in \mathbb{N}} \|f_n\|_E \|g_n\|_E < \infty \right\}$$

(normed with $\|f\|_F = \min_{f = \sum_{n \in \mathbb{N}} f_n g_n} \sum_{n \in \mathbb{N}} \|f_n\|_E \|g_n\|_E$). There exists a Banach space of tempered distributions G such that

i) *e^{Δ} maps boundedly F to G*

ii) *the real interpolation space $[F, G]_{1/2, \infty}$ is continuously embedded into E*

iii) *for all $\lambda > 0$ and for all $f \in G$, $f(\lambda x) \in G$ and $\|f(\lambda x)\|_G = \|f\|_G$.*

Hypothesis (c) (together with (a)) shows that E is invariant under convolution with an integrable kernel:

(86) $\quad \forall f \in E \ \forall g \in L^1 \ f * g \in E$ and $\|f * g\|_E \leq \|f\|_E \|g\|_1$.

This hypothesis (c) is fulfilled in the case where E is the dual space of a separable Banach space containing $\mathcal{S}$ as a dense subspace.

The following proposition shows why those spaces are called adapted to the Navier–Stokes equations:

Proposition 7. *Let E be a fully adapted critical space and let $\mathcal{M} = \mathcal{M}(E \mapsto E)$ be the space of pointwise multipliers of E. For $T \in (0, +\infty)$, let $\mathcal{A}_T$ and $\mathcal{B}_T$ be the spaces defined by*

$$f \in \mathcal{A}_T \Leftrightarrow f \in L^2_{loc}((0, T) \times \mathbb{R}^3), \ \sup_{0 < t < T} \|f(t, .)\|_E < \infty$$

and
$$f \in \mathcal{B}_T \Leftrightarrow f \in L^1_{loc}((0, T) \times \mathbb{R}^3), \ \sup_{0 < t < T} t^{1/2} \|f(t, .)\|_{\mathcal{M}} < \infty.$$

Then B is bounded from $(\mathcal{A}_T)^3 \times (\mathcal{A}_T)^3$ to $(\mathcal{A}_T)^3$ and from $(\mathcal{A}_T)^3 \times (\mathcal{B}_T)^3$ or $(\mathcal{B}_T)^3 \times (\mathcal{A}_T)^3$ to $(\mathcal{A}_T)^3$. More precisely, there exists a constant C_E such that, for all $T \in (0, +\infty]$, all $\vec{u}_0 \in E^3$, all $\vec{f}, \vec{g} \in (\mathcal{A}_T)^3$ and all $\vec{h} \in (\mathcal{A}_T)^3$ we have

$$(87) \qquad \sup_{t>0} \sqrt{t} \|e^{t\Delta} \vec{u}_0\|_{\mathcal{M}} \leq C_E \|\vec{u}_0\|_E$$

$$(88) \qquad \|B(\vec{f}, \vec{g})\|_{\mathcal{A}_T} \leq C_E \|\vec{f}\|_{\mathcal{A}_T} \|\vec{g}\|_{\mathcal{A}_T}$$

and

$$(89) \qquad \|B(\vec{f}, \vec{h})\|_{\mathcal{A}_T} + \|B(\vec{h}, \vec{f})\|_{\mathcal{A}_T} \leq C_E \|\vec{f}\|_{\mathcal{A}_T} \|\vec{h}\|_{\mathcal{B}_T}.$$

Since e^{Δ} maps E to $\mathcal{M}$, (87) is a direct consequence of the homogeneity of the norm of E (and therefore of the norm of $\mathcal{M}$). (89) is a direct consequence of the convolution inequality (86). We now prove (88). We want to estimate the norm $\|B(\vec{f}, \vec{g})\|_E$. We split the integral $I = \int_0^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{f}) \, ds$ into $G_A + H_A$ with

$$\begin{cases} G_A = \displaystyle\int_0^A e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g}) \, ds \\[2mm] H_A = \displaystyle\int_A^t e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g}) \, ds \end{cases}$$

We then use the inequalities

$$\begin{cases} \|e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_F \leq C \dfrac{1}{\sqrt{t-s}} \|\vec{f}\|_E \|\vec{g}\|_E \\[3mm] \|e^{(t-s)\Delta} \mathbb{P}\vec{\nabla}.(\vec{f} \otimes \vec{g})\|_G \leq C \dfrac{1}{(t-s)^{\frac{3}{2}}} \|\vec{f}\|_E \|\vec{g}\|_E \end{cases}$$

We obtain

$$\|H_A\|_F \leq C\sqrt{t-A} \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E$$

and

$$\|G_A\|_G \leq C \frac{1}{\sqrt{t-A}} \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E.$$

If $\lambda \geq \sqrt{t}$, we define $A(\lambda) = 0$ so that $I = H_A$ and $G_A = 0$; if $0 < \lambda < \sqrt{t}$, we define $A(\lambda) = t - \lambda^2$. Thus, we find that, for every $\lambda > 0$, $I = G_{A(\lambda)} + H_{A(\lambda)}$ with

$$\|H_{A(\lambda)}\|_F \leq C\lambda \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E$$

and

$$\|G_{A(\lambda)}\|_G \leq C \frac{1}{\lambda} \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E.$$

Since such a splitting may be done for any positive λ, we obtain

$$\|I\|_{[F,G]_{1/2,\infty}} \leq C \sup_{0<s<t} \|\vec{f}\|_E \sup_{0<s<t} \|\vec{g}\|_E.$$

Then, one easily finishes the proof of Proposition 7.

Combining (85) and Proposition 7, we easily get the following uniqueness result:

Theorem 5. *If X is a regular critical space such that X is boundedly embedded into a a fully adapted critical space E, then uniqueness holds in $(\mathcal{C}([0,T^*),X))^3$.*

Indeed, we have, with the notations of Proposition 7 and of formula (85)

$$\|\vec{w}\|_{\mathcal{A}_T} \leq C_E \|\vec{w}\|_{\mathcal{A}_T} (2\|e^{t\Delta}\vec{u}_0\|_{\mathcal{B}_T} + \|\vec{w}_1\|_{\mathcal{A}_T} + \|\vec{w}_2\|_{\mathcal{A}_T})$$

with

$$\lim_{T\to 0} \|e^{t\Delta}\vec{u}_0\|_{\mathcal{B}_T} = \lim_{T\to 0} \|\vec{w}_1\|_{\mathcal{A}_T} = \lim_{T\to 0} \|\vec{w}_2\|_{\mathcal{A}_T} = 0$$

so that $\vec{w} = 0$ on $(0,T)$ for T small enough. Thus, local (hence global) uniqueness holds in $(\mathcal{C}([0,T^*),X))^3$.

Examples of fully adapted spaces

i) the space of Le Jan and Sznitman

$$E = \dot{B}_{PM}^{2,\infty} = \{f \in \mathcal{S}'(\mathbb{R}^3) \ / \ \hat{f} \in L^1_{loc} \text{ and } \xi^2 \hat{f}(\xi) \in L^\infty\}$$

with $F \subset \dot{B}_{PM}^{1,\infty}$ and $G = \dot{B}_{PM}^{3,1}$

ii) the homogeneous Besov space

$$E = \dot{B}_p^{3/p-1,\infty} \text{ where } 1 \leq p < 3$$

with $F \subset \dot{B}_p^{3/p-2,\infty}$ and $G = \dot{B}_p^{3/p,1}$

iii) the Lorentz space

$$E = L^{3,\infty}$$

with $F = L^{3/2,\infty}$ and $G = L^\infty$

iv) the homogeneous Morrey–Campanato spaces based on Lorentz spaces:

$$E = \dot{M}_*^{p,3} \text{ where } 2 < p \leq 3$$

with $F = \dot{M}_*^{p/2,3/2}$ and $G = L^\infty$. The space $\dot{M}_*^{p,q}(\mathbb{R}^3)$ is defined for $1 < p \leq q < \infty$ as the space of locally integrable functions f such that

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{3(1/q - 1/p)} \|\mathbb{1}_{B(x_0,R)} f\|_{L^{p,\infty}} < \infty;$$

the predual of $\dot{M}_*^{p,q}(\mathbb{R}^3)$ is then the space of functions f which may be decomposed as a series $\sum_{n \in \mathbb{N}} \lambda_n f_n$ with f_n supported by a ball $B(x_n, R_n)$ with $R_n > 0$, $f_n \in L^{p/(p-1),1}$, $\|f_n\|_{L^{p/(p-1),1}} \leq R_n^{3(1/q-1/p)}$ and $\sum_{n \in \mathbb{N}} |\lambda_n| < \infty$.

All those examples however give no new information on the uniqueness problem, since we have the embeddings (for $2 \leq p < 3$ and $2 < q \leq 3$)

$$(90) \qquad \dot{B}_{PM}^{2,\infty} \subset \dot{B}_p^{3/p-1,\infty} \subset L^{3,\infty} \subset \dot{M}_*^{q,3} \subset \dot{X}_1$$

and thus uniqueness may be dealt with by using May's theorem (Proposition 6).

We finish this section with an example of a regular space where uniqueness holds but which cannot be dealt with by using either Theorem 5 or Proposition 6:

Theorem 6. *Let X be defined as the space of locally integrable functions f such that*

$$\sup_{x_0 \in \mathbb{R}^3} \sup_{0 < R < \infty} R^{-1/2} \|\mathbb{1}_{B(x_0,R)} f\|_{L^{2,1}} < \infty$$

and let $\tilde{X}$ be the closure of $\mathcal{D}$ in X. Then

   a) *Uniqueness holds in $(\mathcal{C}([0,T^*], \tilde{X}))^3$.*
   b) *$\tilde{X}$ is not included in the multiplier space $\dot{X}_1 = \mathcal{M}(\dot{H}^1 \mapsto L^2)$*
   c) *there is no fully adapted critical space E such that $\tilde{X} \subset E$.*

The proof of (b) and (c) is easy. Indeed, let β be the bilinear operator $\beta(u,v) = \frac{1}{\sqrt{-\Delta}}(uv)$. Then β is continuous from $\dot{X}_1 \times \dot{X}_1$ to $\dot{X}_1$ (this is a direct consequence of the characterization by Maz'ya and Verbitsky of $\mathcal{M}(\dot{H}^1 \mapsto H^{-1})$ [25]) and from $E \times E$ to E for a fully adapted critical space E (as is checked by writing $\frac{1}{\sqrt{-\Delta}} = \int_0^\infty e^{t\Delta} \sqrt{-\Delta} \, dt$ and adapting the proof of Proposition 7). Thus, we shall prove (b) and (c) by checking that β is not bounded from $\tilde{X} \times \tilde{X}$ to $\dot{M}^{2,3}$. Let $\omega \in \mathcal{D}(\mathbb{R})$ with $\omega = 1$ on $[-1/4, 1/4]$ and $\int_{\mathbb{R}} \omega(t) \, dt = 1$. We define

$$(91) \qquad u_R(x) = R\omega(Rx_1)\omega(Rx_2)\omega(x_3/R)$$

$u_R \in \tilde{X}$ and $\|u_R\|_X = \|u_1\|_X$.

If β were bounded from $\tilde{X} \times \tilde{X}$ to $\dot{M}^{2,3}$, we would have that

$$\lim_{R \to \infty} \beta(u_R, u_R) \in \dot{M}^{2,3} \subset L^2_{loc}$$

(the limit being taken in the sense of distributions). But

$$(92) \qquad \lim_{R \to \infty} \beta(u_R, u_R) = \frac{c_0}{|x|^2} * (\delta(x_1) \otimes \delta(x_2) \otimes 1) = \frac{\pi c_0}{\sqrt{x_1^2 + x_2^2}}$$

for a positive constant c_0. Since $(\sqrt{x_1^2 + x_2^2})^{-1}$ is not locally square integrable, we proved (b) and (c).

In order to prove (a), we use again the equality

$$(93) \qquad \vec{w} = B(\vec{w}_1, \vec{w}) + B(\vec{w}, \vec{w}_2) - B(e^{t\Delta}\vec{u}_0, \vec{w}) - B(\vec{w}, e^{t\Delta}\vec{u}_0).$$

where we have $\vec{w} \in \mathcal{C}([0, T^*), (\tilde{X})^3)$, $\vec{w}_i \in \mathcal{C}([0, T^*), (\tilde{X})^3)$ with $\vec{w}_i(0, .) = 0$ for $i \in \{1, 2\}$, $\sup \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty < \infty$ and $\lim_{t \to 0} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty = 0$. We then use the inclusion $X \subset \dot{M}^{2,3}_*$. Since the pointwise product maps boundedly $X \times \dot{M}^{2,3}_*$ to $\dot{M}^{1,3/2}$ and since we have

$$(94) \qquad \qquad \dot{M}^{2,3}_* = [\dot{M}^{1,3/2}, L^\infty]_{1/2,\infty}$$

we find that

$$(95) \quad \sup_{0 < t < T} \|\vec{w}\|_{\dot{M}^{2,3}_*} \leq$$
$$\leq C \sup_{0 < t < T} \|\vec{w}\|_{\dot{M}^{2,3}_*} \Big(\sup_{0 < t < T} \|\vec{w}_1\|_X + \sup_{0 < t < T} \|\vec{w}_2\|_X + \sup_{0 < t < T} \sqrt{t}\|e^{t\Delta}\vec{u}_0\|_\infty \Big).$$

Thus, Theorem 6 is proved.

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