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A SIMPLE PROOF OF THE DERIVATIVE OF THE INDEFINITE RIEMANN-COMPLETE INTEGRAL

Abstract

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The purpose of this paper is to give a simple and completely elementary proof that the derivative of the indefinite Riemann-complete integral (RC integral) equals the integrand almost everywhere. (Elementary in this case means using the least amount of measure theory possible). Since the RC integral can be defined without the use of any measure theory, this seems to be desirable. This result is not only important in its own right but is also useful in proving that the special Denjoy integral is equivalent to the RC integral, see [7]. Some of the proofs I have seen of this result use either the Vitali Covering Theorem or measure theory beyond sets of measure zero, see [5], [6] and [7]. On the other hand the referee has called my attention to the proof given in [8] that does not make much use of measure theory. Finally, Bartle [1] gives an elementary proof by using the Vitali Covering Theorem with very little measure theory involved. Because of the covering lemma used in the present paper, however, the proof found here is different than the others and perhaps more elementary as well. Let us begin by reviewing the definition of the RC integral which is presented by Ralph Henstock in [3].

Let f be a real-valued function defined on $[a, b]$ and let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$. For each $k = 1, 2, \dots, n$, let ξ_k be an arbitrary point of $[x_{k-1}, x_k]$. If $\xi = \{\xi_1, \xi_2, \dots, \xi_n\}$, then we call (P, ξ) a tagged partition of $[a, b]$ and

$$S(f, P, \xi) = \sum_{k=1}^n f(\xi_k)(x_k - x_{k-1})$$

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the Riemann sum of f with respect to P and ξ . If δ is a positive function defined on $[a, b]$, then we will say that (P, ξ) is compatible with δ if $x_k - x_{k-1} < \delta(\xi_k)$ for each k .

Definition. The function f is said to be RC integrable on $[a, b]$, with integral I , if to each $\epsilon > 0$ there corresponds δ , a positive function defined on $[a, b]$, with $|S(f, P, \xi) - I| < \epsilon$ for each (P, ξ) a tagged partition of $[a, b]$ compatible with δ . We shall use $\int_a^b f$ to denote the RC integral of f on $[a, b]$.

Please see Henstock's paper or any of the works in the references, especially [4], for the various properties of the RC integral. We will, however, need the following well known result whose proof is given in [2, p. 161].

Henstock's Lemma. Let f be RC integrable on $[a, b]$ and, given $\epsilon > 0$, let δ be a positive function defined on $[a, b]$ such that $|S(f, P, \xi) - \int_a^b f| < \epsilon$ for each (P, ξ) a tagged partition of $[a, b]$ compatible with δ . If p is a (partial) sum of terms

$$\left\{ \left| f(\xi_k)(x_k - x_{k-1}) - \int_{x_{k-1}}^{x_k} f \right| \right\},$$

for any number of distinct intervals $[x_{k-1}, x_k]$ of a tagged partition of $[a, b]$ compatible with δ , then $p \leq 2\epsilon$.

So that our proof will depend on no measure theory beyond sets of measure zero, we need the following elementary covering lemma. Its proof uses nothing more than the Heine-Borel Theorem and the fact that an open set can be expressed as the countable disjoint union of open intervals.

Remark. A set of real numbers E is not of measure zero if there exists $\epsilon_0 > 0$ such that $\sum_{n=1}^{\infty} |J_n| \geq \epsilon_0$ for $\{J_n\}$ any sequence of open intervals that covers E where $|J_n|$ denotes the length of J_n .

Lemma. Let $E \subseteq (a, b)$ be a set which is not of measure zero and let ϵ_0 be as in the above Remark. If \mathbb{C} is any collection of open subintervals of $[a, b]$ which covers E , then there exists $\{I_1, I_2, \dots, I_N\}$, a finite disjoint subcollection of \mathbb{C} , such that $\sum_{k=1}^N |I_k| > \frac{\epsilon_0}{3}$.

PROOF. Since $\cup_{I \in \mathbb{C}} I$ is an open set there exists $\{(a_n, b_n)\}$, a disjoint sequence of open intervals, such that $\cup_{I \in \mathbb{C}} I = \cup_{n=1}^{\infty} (a_n, b_n)$. Since $E \subseteq \cup_{n=1}^{\infty} (a_n, b_n)$, $\sum_{n=1}^{\infty} (b_n - a_n) \geq \epsilon_0$. Now for each (a_n, b_n) choose $[a'_n, b'_n]$, a closed subinterval of (a_n, b_n) , such that $b'_n - a'_n = \frac{3}{4}(b_n - a_n)$. Thus

$$\sum_{n=1}^{\infty} (b'_n - a'_n) \geq \frac{3\epsilon_0}{4}.$$

Now let n be a fixed positive integer. For each $x \in [a'_n, b'_n]$ there exists $J_x \in \mathbb{C}$ such that $x \in J_x \subseteq (a_n, b_n)$. Thus $\{J_x : x \in [a'_n, b'_n]\}$ is an open cover of $[a'_n, b'_n]$. By the Heine-Borel Theorem there exists $\{J_1, J_2, \dots, J_p\}$, a finite number of sets from this open cover, such that $[a'_n, b'_n] \subseteq \cup_{k=1}^p J_k$. (A familiar argument shows that $\sum_{k=1}^p |J_k| \geq b'_n - a'_n$.) Furthermore we may assume, by discarding some of the intervals if necessary, that no interval in $\{J_k\}_{k=1}^p$ is a subset of the union of the remaining intervals in $\{J_k\}_{k=1}^p$. Thus each J_i contains a point $x_i \notin \cup_{k \neq i} J_k$ and we may assume by renumbering the J_k 's if necessary, that $x_1 < x_2 < \dots < x_p$. Because of this a little thought shows that both $\{J_1, J_3, J_5, \dots\}$ and $\{J_2, J_4, J_6, \dots\}$ are finite disjoint subcollections of \mathbb{C} . Clearly either

$$\sum_k |J_{2k-1}| \geq \frac{1}{2} \sum_{k=1}^p |J_k| \quad \text{or} \quad \sum_k |J_{2k}| \geq \frac{1}{2} \sum_{k=1}^p |J_k|.$$

Thus depending on which of the two previous inequalities holds, we have found \mathbb{C}_n , a finite disjoint subcollection of \mathbb{C} , such that

$$\sum_{I \in \mathbb{C}_n} |I| \geq \frac{1}{2} \sum_{k=1}^p |J_k| \geq \frac{1}{2} (b'_n - a'_n).$$

Thus for each positive integer n there exists \mathbb{C}_n , a finite disjoint subcollection of \mathbb{C} , each of whose open intervals is a subset of (a_n, b_n) and such that

$$\sum_{I \in \mathbb{C}_n} |I| \geq \frac{1}{2} (b'_n - a'_n).$$

Summing both sides of the previous inequality, we have

$$\sum_{n=1}^{\infty} \sum_{I \in \mathbb{C}_n} |I| \geq \frac{1}{2} \sum_{n=1}^{\infty} (b'_n - a'_n) \geq \frac{3\epsilon_0}{8}.$$

Therefore $\cup_{n=1}^{\infty} \mathbb{C}_n = \{I_1, I_2, I_3, \dots\}$ is a countable disjoint subcollection of \mathbb{C} for which $\sum_{n=1}^{\infty} |I_n| \geq \frac{3\epsilon_0}{8}$. Finally choose N so large that $\sum_{k=1}^N |I_k| > \frac{\epsilon_0}{3}$ and the proof is complete. \square

We are now ready to give our proof of the main result of this paper. In what follows, $F(I)$ denotes $F(d) - F(c)$ and $|I|$ denotes the length of I where $I = (c, d)$.

Theorem. *Let f be RC integrable on $[a, b]$ and let $F(x) = \int_a^x f$. Then $F' = f$ almost everywhere on $[a, b]$.*

PROOF. We let $E = \{x : x \in (a, b) \text{ and either } F'(x) \text{ does not exist or } F'(x) \neq f(x)\}$ and show that E has measure zero. It is easy to prove that F is continuous on $[a, b]$, see [2, p. 163]. For each $x \in E$ there exists $\epsilon(x) > 0$ such that, for any positive number δ , there exists v in (a, b) such that

$$0 < |v - x| < \delta \text{ and } \left| \frac{F(v) - F(x)}{v - x} - f(x) \right| > \epsilon(x).$$

Suppose for definiteness that $v > x$. Because of the above inequality and since

$$\frac{F(v) - F(t)}{v - t}$$

is continuous at x , there exists u in (a, b) such that $u < x$, $v - u < \delta$, and

$$\left| \frac{F(v) - F(u)}{v - u} - f(x) \right| > \epsilon(x)$$

so that

$$|F(v) - F(u) - f(x)(v - u)| > \epsilon(x)(v - u).$$

Let $E_n = \{x : x \in E \text{ and } \epsilon(x) > \frac{1}{n}\}$ so that $E = \cup_{n=1}^{\infty} E_n$. Therefore it is sufficient to show that each E_n has measure zero. Suppose E_n is not of measure zero for some n and let ϵ_0 be a positive number such that $\sum_{n=1}^{\infty} |J_n| \geq \epsilon_0$ for $\{J_n\}$ any sequence of open intervals that covers E_n . For $\frac{\epsilon_0}{6n} > 0$ there exists δ , a positive function defined on $[a, b]$, such that

$$\left| S(f, P, \xi) - \int_a^b f \right| < \frac{\epsilon_0}{6n}$$

whenever (P, ξ) is a tagged partition of $[a, b]$ compatible with δ . Let $x \in E_n$ which implies the existence of u_x and v_x in (a, b) such that $u_x < x < v_x$, $v_x - u_x < \delta(x)$, and

$$\left| F(v_x) - F(u_x) - f(x)(v_x - u_x) \right| > \epsilon(x)(v_x - u_x) > \frac{1}{n}(v_x - u_x). \quad (1)$$

Note that $\{(u_x, v_x) : x \in E_n\}$ is a collection of open subintervals of $[a, b]$ that covers E_n . By the Lemma, there exists I_1, I_2, \dots, I_N , a finite disjoint subcollection of these open intervals, such that $\sum_{k=1}^N |I_k| > \frac{\epsilon_0}{3}$. Now since each I_k contains x_k for which $|I_k| < \delta(x_k)$, by Henstock's Lemma we have

$$\sum_{k=1}^N |F(I_k) - f(x_k)| |I_k| \leq 2 \cdot \frac{\epsilon_0}{6n} = \frac{\epsilon_0}{3n}. \quad (2)$$

(Please note the meaning of $F(I_k)$ that was given right before the statement of this theorem.)

But, using (1), we have

$$\sum_{k=1}^N |F(I_k) - f(x_k)| I_k| > \frac{1}{n} \cdot \sum_{k=1}^N |I_k| > \frac{\epsilon_0}{3n}$$

which contradicts (2) and completes the proof. \square

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