INFINITE DETERMINANTS ASSOCIATED WITH HILL'S EQUATION

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1. Introduction and Summary. Hill's equation is the differential equation for a one-dimensional linear oscillator with a periodic potential. In most applications, the question of the existence of a periodic solution arises. The main purpose of this investigation is to examine the analytic character of the transcendental function, whose zeros determine the periodic solutions. For the special case of Mathieu's equation the results obtained here have previously been used for solving the inhomogeneous equation, and the cases where Hill's equation has two periodic solutions have been discussed in detail and applied to the construction of "transparent layers" [1].

We consider the differential equation of Hill's type:

$$y''+4(\omega^2+q(x))y=0\ , \tag{1.1}$$

where $q(x)$ is an even function of period π which can be expanded in a Fourier series

$$q(x)=2\sum_{n=1}^{\infty} t_n \cos 2nx\ . \tag{1.2}$$

We shall assume that the constants t_n satisfy

$$\sum_{n=1}^{\infty} |t_n|<\infty\ . \tag{1.3}$$

The most widely investigated problem connected with (1.1) is the question of the existence of solutions with period π or 2π. Let $y_1(x)$, $y_2(x)$ denote the solutions of (1.1) which satisfy the initial conditions

$$y_1(0)=1,\ y_1'(0)=0\ ;\quad y_2(0)=0,\ y_2'(1)=1. \tag{1.4}$$

Then the following elementary statements hold (see for instance Schaefke [5]: Equation (1.1) has

(α) an even solution of period π if and only if $y_1'(\pi/2)=0$
(α') an odd solution of period π if and only if $y_2(\pi/2)=0$
(β) an even solution of period 2π if and only if $y_1(\pi/2)=0$
(β') an odd solution of period 2π if and only if $y_2'(\pi/2)=0$.

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The conditions (α), (α') and (β), (β') can be reduced to two single ones because

(1.5) $$y_1(\pi)-1=2y_1'(\pi/2)y_2(\pi/2)\ ,$$

(1.6) $$y_1(\pi)+1=2y_1(\pi/2)y_2'(\pi/2)\ .$$

In order to find directly a solution of (1.1) which has a period π, we put

(1.7) $$y=\sum_{n=-\infty}^{\infty} c_n \exp(2nxi)\ ,$$

where

(1.8) $$\bar{c}_n=c_{-n}$$

for a real function $y(x)$. (As usual, a bar denotes the conjugate complex quantity). By substituting (1.7) into (1.2) we obtain an infinite system of homogeneous linear equations for the c_n. The determinant of this system can be written in the form

(1.9) $$\sin^2 \pi\omega\, D_0(\omega)$$

where $D_0(\omega)$ is an infinite determinant of the type

(1.10) $$D_0(\omega)=|d_{n,m}|, \qquad n,\ m=0,\ \pm 1,\ \pm 2, \cdots .$$

Here

(1.11) $$d_{n,m}=\delta_{n,m}+\left(\frac{t_{n-m}}{\omega^2-n^2}\right)\ ,$$

(1.12) $$t_{n-m}=t_{m-n}=t_{|n-m|}\ , \quad t_0=0\ .$$

As usual, $\delta_{n,m}=1$ if $n=m$ and $\delta_{n,m}=0$ if $n\neq m$.

The vanishing of the expression (1.9) is a necessary and sufficient condition for (1.1) to have a solution with period π. According to Whittaker and Watson [7]

(1.13) $$y_1(\pi)-1=-2D_0(\omega)\sin^2\pi\omega\ .$$

This shows that the vanishing of (1.5) is an immediate consequence of the vanishing of the term (1.9) and vice versa. Also, it provides two alternative ways of approximating the eigenvalues ω for which $y_1(\pi)=1$. If we compute $y_1(\pi)$ approximately by applying the Picard iteration to (1.1), we arrive at trigonometric polynomials or series. If we use the principal minors of D_0, we obtain algebraic equations for the approximate values of ω which will be particularly suitable for large ω.

To obtain even or odd solutions of (1.1) which are of period π we may put

$$y=\left(\frac{c_0}{\sqrt{2}}\right)+\sum_{n=1}^{\infty} c_n \cos 2nx \tag{1.14}$$

or

$$y=\sum_{n=1}^{\infty} c_n \sin 2nx \tag{1.15}$$

respectively. By substituting (1.14) or (1.15) into (1.1) we obtain an infinite system of homogeneous linear equations for the c_n. After an appropriate normalization of these equations, we can write the determinants of the resulting systems in the form $\omega \sin(\pi\omega)C_+$ and $\omega^{-1}\sin(\pi\omega)S_+$, where the infinite determinants C_+ and S_+ can be defined as follows: Let

$$\varepsilon_m=2 \text{ for } m=\pm 1,\ \pm 2,\ \pm 3,\cdots;\ \varepsilon_0=1 \tag{1.16}$$

$$\begin{cases} \operatorname{sgn} m=1 \text{ for } m=1,\ 2,\ 3,\cdots;\ \operatorname{sgn} 0=0 \\ \operatorname{sgn} m=-1 \text{ for } m=-1,\ -2,\ -3,\cdots. \end{cases} \tag{1.17}$$

Let the t_n be defined by (1.2) and (1.12). Then

$$C_+=|(\varepsilon_n\varepsilon_m)^{-1/2}(1+\operatorname{sgn} n \operatorname{sgn} m)[\delta_{n,m}+(t_{n-m}+t_{n+m})(\omega^2-n^2)^{-1}]| \qquad (n,\ m=0,\ 1,\ 2,\cdots), \tag{1.18}$$

$$S_+=|\delta_{n,m}+(t_{n-m}-t_{n+m})(\omega^2-n^2)^{-1}| \qquad (n,\ m=1,\ 2,3,\cdots), \tag{1.19}$$

where n denotes the rows and m denotes the columns of the infinite determinants C_+ and S_+.

We shall prove the following extension of Equation (1.13):

THEOREM 1. *The infinite determinants C_+ and S_+ can be expressed in terms of $y_1'(\pi/2)$ and $y_2(\pi/2)$ as*

$$2\omega \sin(\pi\omega)C_+=-y_1'(\pi/2)\,, \tag{1.20}$$

$$\omega^{-1}\sin(\pi\omega)S_+=2y_2(\pi/2)\,. \tag{1.21}$$

They are related to the infinite determinant D_0 by

$$D_0=C_+S_+\,. \tag{1.22}$$

A similar factorization theorem can be proved for the infinite determinant arising in the problem of determing whether (1.1) has a solution of period 2π.

Equations (1.19) and (1.21) show that S_+ and $y_2(\pi/2)$ depend in a special way on ω. We shall write $S_+(\omega)$ for S_+ and $y_2(\pi/2, \omega)$ for $y_2(\pi/2)$ if we wish to emphasize the dependency on ω. $S_+(\omega)$ has poles of the first order (at most) at $\omega=\pm 1, \pm 2, \cdots$. Since the individual terms in the determinant $S_+(\omega)$ tend to $\delta_{n,m}$ as $|\omega|\to\infty$, we may expect that $S_+(\omega)\to 1$ as $|\omega|\to\infty$. Therefore we may expect that (1.21) will lead to a formula of the type

$$y_2(\pi/2, \omega)=\sum_{n=0}^{\infty} g_n \frac{\sin \pi\omega}{\omega-n}, \tag{1.23}$$

where g_n are constant coefficients. Now the form of the infinite series on the right-hand side of (1.23) suggests that it can also be written as

$$\int_{-\pi/2}^{\pi/2} G(\theta) \exp(2i\omega\theta)\, d\theta, \tag{1.24}$$

which would imply the existence of a formula of the type

$$\int_{-\infty}^{\infty} y_2(\pi/2, \omega) \exp(-2i\omega\theta)\, d\omega=0 \quad \text{for} \quad |\theta|>\frac{\pi}{2}. \tag{1.25}$$

Actually, a result more general than (1.25) is true. We shall prove the following formula for the Fourier transformation with respect to ω.

THEOREM 2. *Let the t_n in* (1.12) *be real constants satisfying*

$$\sum_{n=1}^{\infty} n^2 |t_n| < \infty,$$

and let $y(x, \omega)$ be the solution of (1.1) *for a real value of ω which satisfies the initial conditions*

$$y(0, \omega)=a, \; y'(0, \omega)=b. \tag{1.26}$$

Then there exists a function $G(x, \theta)$ of the real variables x and θ which is defined in the region $-|x|\le\theta\le|x|$ such that

$$y(x, \omega)=a \cos 2\omega x+\int_{-x}^{x} G(x, \theta) e^{2i\omega\theta} d\theta, \tag{1.27}$$

$$\frac{\partial^2 G}{\partial x^2}-\frac{\partial^2 G}{\partial \theta^2}+4q(x)G=0, \tag{1.28}$$

$$G(x, x)=G(x, -x)=\frac{b}{2}-a\sum_{n=1}^{\infty} \frac{t_n}{n} \sin 2\pi x, \tag{1.29}$$

$$G_\theta(x,\ x)=-G_\theta(x,\ -x)=2\sum_{n=1}^{\infty} t_n \sin nx\left\{a \sin nx+\frac{b}{n}\cos nx\right\}$$
$$+a\sum_{n,\ m=1}^{\infty}\frac{t_n t_m}{nm}\sin 2nx \sin 2mx\ . \tag{1.30}$$

Here G_θ stands for $\partial G/\partial\theta$.

2. Proof of Theorem 1. Since Theorem 1 involves the determinants of infinite matrices, it is important to know something about their finite "sections". We shall define these sections as follows: Let N be a nonnegative integer, and let (M) be an infinite matrix. If the rows and columns of (M) are labeled by subscripts running from one to infinity, we denote by (M_N) the square matrix of order N which results if we let the subscripts in (M) run from one to N only. If the rows and columns in (M) are labeled by the subscripts $0,\ 1,\ 2,\cdots$, we define (M_N) by the rows and columns of (M) for which the subscripts run from zero to N. Finally, if the subscripts in (M) run from $-\infty$ to $+\infty$, then in (M_N) we let them run from $-N$ to $+N$ only. In each case, (M_N) is called the Nth section of (M). The determinant of (M) is defined as the limit of the determinants of (M_N) as $N\to\infty$.

We shall denote by (D), (C), (S) the matrices whose elements are given respectively by the elements of the infinite determinants D_0, C_+, and S_+. In addition, we shall introduce the matrix (T) with the general element $\tau_{n,\,m}(n,\ m=0,\ \pm1,\ \pm2,\ \cdots)$, where

$$\tau_{n,\,m}=(\delta_{n,\,m}+\operatorname{sgn} n\,\delta_{-n,\,m})(\varepsilon_n)^{1/2}. \tag{2.1}$$

As usual the first subscript n in $\tau_{n,\,m}$ denotes the rows of (T) and the second subscript denotes the columns. The matrix (T) has a formal inverse (T^{-1}), whose general element is given by

$$(\delta_{n,\,m}+\operatorname{sgn} m\,\delta_{-n,\,m})(\varepsilon_m)^{-1/2}. \tag{2.2}$$

In fact it follows from an easy computation that the general element of $(T)(T^{-1})$ is

$$\{\delta_{n,\,m}(1+\operatorname{sgn} n \operatorname{sgn} m)+\delta_{-n,\,m}(\operatorname{sgn} n+\operatorname{sgn} m)\}(\varepsilon_n\varepsilon_m)^{-1/2}=\delta_{n,\,m}\ . \tag{2.3}$$

It is important to observe that the Nth section (T_N^{-1}) of (T^{-1}) is the inverse of the Nth section (T_N) of T .

Now we shall compute, in a purely formal way, the elements of the matrix

$$(D^*)=(T)(D)(T^{-1})\ . \tag{2.4}$$

By a simple computation we find from (1.11), (1.12), (2.1) and (2.2) that the general element $d^*_{n,m}$ of (D^*) is given by

$$(2.5)\quad (\varepsilon_n\varepsilon_m)^{1/2}\, d^*_{n,m}=\delta_{n,m}(1+\operatorname{sgn} n \operatorname{sgn} m)+\delta_{n,-m}(\operatorname{sgn} n+\operatorname{sgn} m)$$
$$+\frac{t_{n-m}}{\omega^2-n^2}(1+\operatorname{sgn} n \operatorname{sgn} m)+\frac{t_{n+m}}{\omega^2-n^2}(\operatorname{sgn} n+\operatorname{sgn} m)\,.$$

Equation (2.5) shows that $d^*_{n,m}=0$ if n and m are both different from zero and of different sign. It also shows that for n, $m=0, 1, 2, 3, \cdots$ the elements of (D^*) are exactly those of (C). In fact, for $n\geq 0$, $m\geq 0$, we always have $\delta_{n,-m}(\operatorname{sgn} n+\operatorname{sgn} m)=0$, and $\operatorname{sgn} n+\operatorname{sgn} m=1+\operatorname{sgn} n \operatorname{sgn} m$, unless $n=m=0$. But in this case, $t_{n-m}=t_{n+m}=0$, and again $d^*_{n,m}$ is equal to the corresponding element of C_+ in (1.18). Similarly, we find that for n, $m=-1, -2, -3, \cdots$, the elements of (D^*) are exactly those of (S) if we "invert" the labeling of the elements of (S) by substituting for every subscript its opposite (negative) value.

Therefore (1.22) would be proven if we could deal with infinite determinants in the same way as with finite ones. In the particular problem under consideration this is actually the case. If we form the matrix $(T_N)(D_N)(T_N^{-1})$ we obtain (D^*_N) for all N and we find that its determinant actually equals the product of the determinants of (S_N) and (C_N) because its elements are those of (S_N) and (C_N) respectively. Equation (1.22), namely $D_0=C_+S_+$, follows if we simply let N tend towards infinity.

Next we must prove equations (1.20) and (1.21). It suffices to do this for arbitrary but fixed real values of $t_1, t_2, t_3, \cdots$. Indeed, it is not difficult to show that both sides in (1.20) and (1.21) depend analytically on any particular parameter t_ν ($\nu=1, 2, \cdots$). Then the only variable which matters is ω. As mentioned above, we shall write $y_2(\pi/2, \omega)$ and $y_1'(\pi/2, \omega)$ for $y_2(\pi/2)$ and $y_1'(\pi/2)$ whenever we wish to exhibit the dependency on ω of these quantities; similarly, we shall write $C_+(\omega)$ and $S_+(\omega)$ for C_+ and S_+. It is easily seen that both sides in (1.20) and (1.21) are entire functions of ω and also entire functions of $\lambda=\omega^2$.

Now we can prove (1.20) and (1.21) by proving the following lemmas:

LEMMA 1. *The quotients*

$$(2.6)\qquad \frac{2\omega \sin \pi\omega\, C_+(\omega)}{y_1'(\pi/2,\ \omega)}\,,\qquad \frac{\omega^{-1} \sin \pi\omega S(\omega)}{2y_2(\pi/2,\ \omega)}$$

are entire functions of $\omega^2=\lambda$.

Proof. It has been mentioned in the introduction that the numerator and denominator of (2.6) vanish for the same values of $\lambda=\omega^2$. It remains merely to be shown that the denominators have simple zeros only. We observe first that these zeros are real, because any solution or derivative of a solution of (1.1) that vanishes at $x=0$ and $x=\pi/2$ is a solution of a Sturm-Liouville problem. Since

$$\frac{\partial}{\partial\lambda} y_2(\pi/2)=4\{y_2'(\pi/2)\}^{-1}\int_0^{\pi/2}\{y_2(x)\}^2dx \tag{2.7}$$

$$\frac{\partial}{\partial\lambda} y_1'(\pi/2)=-4\{y_1(\pi/2)\}^{-1}\int_0^{\pi/2}\{y_1(x)\}^2dx\,, \tag{2.8}$$

the right-hand sides of (2.7) and (2.8) are different from zero and therefore the denominator in (2.6) has simple zeros. This completes the proof of Lemma 1.

LEMMA 2. *The quotients* (2.6) *are entire functions without zeros.*

Proof. From (1.5), (1.13), (1.22) we see that the product of the quotients (2.6) equals -1.

LEMMA 3. *The quotients* (2.6) *are independent of* $\lambda=\omega^2$.

Proof. This lemma follows from the fact that for both the numerators and the denominators of the quotients (2.6) the order of growth with respect to λ does not exceed 1/2. For $y_2(\pi/2,\ \omega)$ we can show this by solving (1.1) with the help of Picard's iteration method. Putting

$$u_0(x,\ \omega)=(\sin 2\omega x)/(2\omega)\,, \tag{2.9}$$

$$u_n(x,\ \omega)=-\frac{2}{\omega}\int_0^x \sin 2\omega(x-\xi)q(\xi)u_{n-1}(\xi,\ \omega)d\xi\,,\quad (n=1,2,\cdots), \tag{2.10}$$

we have

$$y_2(x,\ \omega)=\sum_{n=0}^{\infty}u_n(x,\ \omega)\,. \tag{2.11}$$

In order to estimate $|y_2|$ for large values of $|\omega|$, let Q be a positive constant such that

$$|q(\xi)|\leq Q \tag{2.12}$$

for all real values of ξ. Let $|\omega|\geq 2$. Then obviously $|u_0|\leq \exp(2|\omega|x)$ for real positive x. From this it follows by induction and by using

(2.10) that for real positive values of x

$$|u_n(x,\ \omega)| \leq x^n Q^n e^{2|\omega|x} (n!)^{-1} (\omega/2)^{-n-1}. \tag{2.13}$$

Therefore we have from (2.11) for $|\omega| \geq 2$:

$$|y_2(\pi/2,\ \omega)| \leq \exp(\pi|\omega| + Q\pi/2)\ . \tag{2.14}$$

A similar estimate can be derived for $y_1'(\pi/2,\ \omega)$. Since the right-hand side of (2.14) is of order of growth unity with respect to ω, its order of growth with respect to λ is 1/2.

The corresponding statement for the numerators in (2.6) can be derived from Hadamard's inequality for determinants. If we write

$$(\pi/2)\sum_{n=1}^{\infty}\left(1-\frac{\omega^2}{n^2}\right)\ , \tag{2.15}$$

for $(\sin \pi\omega)/(2\omega)$, and if we multiply each row of S_+ by the corresponding factor of (2.15), the numerator involving S_+ in (2.6) becomes a determinant for which the sum of the squares of the absolute values of the nth row is at most σ_n, where

$$\sigma_n = \{1+(|\omega|^2+|t_{2n}|)n^{-2}\}^2 + \sum_{m=1}^{\infty}(|t_{n-m}|+|t_{n+m}|)^2 n^{-4}. \tag{2.16}$$

We have from Hadamard's inequality

$$|(2\omega)^{-1}(\sin \pi\omega) S_+(\omega)| \leq 2\pi^{-1} \prod_{n=1}^{\infty} \{\sigma_n\}^{1/2}. \tag{2.17}$$

Now we wish to estimate $|\sigma_n|$. From (1.2) we find that there exists a constant M such that for all $n=1,\ 2,\ 3, \cdots$

$$|t_{2n}| \leq 2M,\ \sum_{m=1}^{\infty}(|t_{n-m}|+|t_{n+m}|)^2 \leq M^2. \tag{2.18}$$

Therefore

$$|\sigma_n| \leq \{1+(|\omega|^2+M)n^{-2}\}^2 \tag{2.19}$$

and

$$\prod_{n=1}^{\infty}\{\sigma_n\}^{1/2} \leq \{\sinh \pi(|\omega^2|+M)^{1/2}\}\pi^{-1/2}(|\omega|^2+M)^{-1/2}. \tag{2.20}$$

Together with (2.17), this shows that the left-hand side of (2.17) is of order of growth $\leq 1/2$ with respect to $\lambda=\omega^2$. An analogous proof can be given for $|2\omega \sin \pi\omega\, C_+(\omega)|$.

Now we can prove Lemma 3 by using a known theorem about factorization of functions of an order of growth <1 (See Nevanlinna

[2, pp. 205–213] or Titchmarsh [6, Chap. VIII]. According to this theorem we have for both the numerators and the denominators of the quotients (2.6) a representation of the form

$$A\,\omega^{2a}\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{a_n}\right),$$

where the a_n are the simple roots common to the numerator and denominator if both are considered as functions of $\lambda=\omega^2$. Therefore, the quotients in (2.6) are independent of ω, as stated in Lemma 3.

Now we can prove (1.20) and (1.21) by computing the value of the quotients in (2.6) for $\omega\to i\infty$. It is easily seen that for $\omega\to i\infty$ both S_+ and C_+ tend toward unity. From (2.9), (2.10) and (2.11) we can show that $y_2(\pi/2,\ \omega)/u_0(\omega)$ tends also towards unity as $\omega\to i\infty$, regardless of the particular nature of $q(x)$. The behavior of $y_1'(\pi/2,\ \omega)/(2\omega\sin\pi\omega)$ can be described in a similar manner, and this completes the proof of Theorem 1.

3. **Proof of Theorem 2.** In this section, we shall use a theorem given by Paley and Wiener [3, Theorem X, p. 13]. According to this theorem, *the following two classes of functions are identical*:

(I) *The class of all entire functions $F(\omega)$ satisfying*

$$|F(\omega)|=o(e^{2A|\omega|})\qquad(|\omega|\to\infty)\tag{3.1}$$

for a positive real value of A; and

(II) *The class of all entire functions of the form*

$$F(\omega)=\int_{-A}^{A}f(\theta)e^{2i\omega\theta}\,d\theta\,,\tag{3.2}$$

where $f(\theta)$ belongs to L_2 over $(-A,\ A)$.

In proving Theorem 2 we shall confine ourselves to the case where $a=0$, $y=y_2(x,\ \omega)$. If we construct y_2 in the manner described by (2.9), (2.10), (2.11), we find from (2.13) that for $x>0$ and $|\omega|\to\infty$:

$$\left|y_2(x,\ \omega)-\{u_0(x,\ \omega)+\cdots+u_n(x,\ \omega)\}\right|=O(|\omega|^{-n-2}e^{2|\omega|x})\tag{3.3}$$

and

$$|u_n(x,\ \omega)|=O(|\omega|^{-n-1}e^{2|\omega|x}).\tag{3.4}$$

Now it follows from an application of Paley and Wiener's theorem that

$$y_2(x,\ \omega)=\int_{-x}^{x}e^{2i\omega\theta}G(x,\ \theta)\,d\theta,\tag{3.5}$$

where

$$G(x,\ \theta)=\sum_{n=0}^{\infty} g_n(x,\ \theta)\ , \tag{3.6}$$

$$g_n(x,\ \theta)=\pi^{-1}\int_{-\infty}^{\infty} e^{-2i\omega\theta}u_n(x,\ \omega)\,d\omega\ . \tag{3.7}$$

It follows from (3.4) that for $n>0$, $g_n(x,\ \theta)$ is $(n-1)$ times differentiable with respect to θ, with a continuous $(n-1)^{\text{st}}$ derivative. Outside the interval $-x\leq\theta\leq x$, all of the $g_n(x,\ \theta)$ vanish identically. Therefore at $\theta=\pm x$ only $g_0(x,\ \theta)$ and $g_1(x,\ \theta)$ contribute to the value of $G(x,\ \theta)$ and to its first derivative with respect to θ. These contributions can be found by a direct computation. In the same way, it can be verified that g_0, g_1, g_2 are twice differentiable within the region $-x<\theta<x$, having one-sided continuous derivatives at $\theta=\pm x$, provided that $\sum_{n=1}^{\infty} n^2|t_n|<\infty$.

The only part of Theorem 2 that now remains to be proved is equation (1.28). If we substitute the expression (3.6) for G into (1.28), it will suffice to prove that for $n=1,\ 2,\ 3,\cdots$,

$$\frac{\partial^2 g_n}{\partial x^2}-\frac{\partial^2 g_n}{\partial\theta^2}+4q(x)g_{n-1}=0 \tag{3.8}$$

and for $n=0$

$$\frac{\partial^2 g_0}{\partial x^2}-\frac{\partial^2 g_0}{\partial\theta^2}=0\ . \tag{3.9}$$

Since $g_0=1/2$ for $-x<\theta<x$, it is trivial to show that (3.9) holds. Equation (3.8) may be verified for $n=1$ directly by observing that

$$g_1(x,\ \theta)=\sum_{n=1}^{\infty}\frac{2t_n}{n^2}\cos nx\ (\cos nx-\cos n\theta)\ . \tag{3.10}$$

For $n\geq 2$ we may proceed as follows. It suffices to prove, instead of (3.8), that

$$\int_{-x}^{x}\left(\frac{\partial^2 g_n}{\partial x^2}-\frac{\partial^2 g_n}{\partial\theta^2}+q(x)g_{n-1}\right)e^{2i\omega\theta}d\theta=0 \tag{3.11}$$

for all values of ω. Since the left-hand side of (3.11) is an analytic function of ω, it suffices to show that it vanishes for all real values of ω. We shall prove this by expressing the left-hand side of (3.11) in terms of the $u_n(x,\ \omega)$ which satisfy the recurrence relations

$$\frac{\partial^2 u_n}{\partial x^2}+4\omega^2 u_n+4q(x)u_{n-1}=0. \tag{3.12}$$

((3.12) can be derived easily from (2.9) and (2.10)). It follows from (3.5) and (3.7) that

$$u_n(x, \omega)=\int_{-x}^{x} g_n(x, \theta)e^{2i\omega\theta}\,d\theta\,. \tag{3.13}$$

Therefore we have for $n\geq 2$:

$$\frac{\partial^2 u_n}{\partial x^2}=\int_{-x}^{x}\frac{\partial^2 g_n}{\partial x^2}e^{2i\omega\theta}\,d\theta\,, \tag{3.14}$$

since any term derived by differentiating the integral in (3.13) with respect to its limits vanishes for $n\geq 2$. For the same reason we find from an integration by parts that

$$-\int_{-x}^{x}\frac{\partial^2 g_n}{\partial x^2}\exp\,(2i\omega\theta)\,d\theta=4\omega^2 u_n(x, \theta)\,. \tag{3.15}$$

Equations (3.15), (3.13), (3.12) show that (3.11) and (3.12) are equivalent. Since (3.12) is true, the proof of Theorem 2 has been completed.

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