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TWO-GROUPS AND JORDAN ALGEBRAS

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Stroud and Paige have introduced an important class of central simple Jordan algebras $B(2^n)$ of characteristic two. This paper determines the automorphism groups of the algebras $B(2^n)$ and, in so doing, produces an infinite family of finite 2-groups. This is accomplished by characterizing the automorphisms of $B(2^n)$ as matrices operating on the natural basis for the underlying vector space of $B(2^n)$ and then using this characterization to obtain generators and commuting relations for the automorphism groups.

Throughout the paper let $q = 2^{n-2}$, $r = 2^{n-1}$, $s = 2^n$, and $t = 2^{n+1}$. $\delta_{i,j}$ is the Kronecker delta.

1. The algebras.

In 1965 J. B. Stroud [3], pursuing some earlier work of E. C. Paige [2], defined the following class of vector spaces and proved that they are central simple Jordan Algebras of characteristic two:

DEFINITION. Let $B(2^n)$ for $n \geqq 2$ be the vector space over the field Z_2 of two elements with basis $u_{-1}, u_0, u_1, \cdots, u_{s-2}, v_1, v_2, \cdots, v_s$ and with multiplication in $B(2^n)$ defined inductively as follows:

The products u_iu_j for $-1 \leqq i,\ j \leqq s-2$ are defined by:

(1) $u_0u_i = u_i$ for $-1 \leqq i \leqq s-2$,

(2) $u_{-1}^2 = 0$, $u_{-1}u_i = u_{i-1}$ for $0 \leqq i \leqq s-2$,

(3) $u_iu_j = u_ju_i$ for $-1 \leqq i,\ j \leqq s-2$,

(4) $u_1^2 = u_2^2 = u_1u_2 = 0$.

Assuming the products u_iu_j are defined for $-1 \leqq i,\ j \leqq 2^k - 2$ where $2 \leqq k \leqq n-1$, let $p = 2^k$ and define

(5) $u_{p+i}u_j = H_{i,j}u_{p+i+j}$ when $u_iu_j = H_{i,j}u_{i+j}$, $j \neq -1$ with $H_{i,j}$ in Z_2,

(6) $u_{p+i}u_{p+j} = 0$.

For $k \geqq 1$, $m \geqq 0$, $p = 2^k$, define the products u_iv_j by:

(7) $u_iv_j = v_ju_i$ for $-1 \leqq i \leqq s-2$, $1 \leqq j \leqq s$,

(8) $u_0v_j = v_j$ for $1 \leqq j \leqq s$,

(9) $u_{-1}v_j = v_j + v_{j-1}$, $v_0 = 0$ for $1 \leqq j \leqq s$,

(10) $u_{p-1}v_{(2m)p+j} = v_{(2m+1)p+j-1} + v_{(2m+1)p+j}$ for $1 \leqq j \leqq p$,

(11) $u_{p+i}v_{(2m)p+j} = d_{i,j}v_{(2m+1)p+i+j} + e_{i,j}v_{(2m+1)p+i+j+1}$

when

(12) $u_iv_j = d_{i,j}v_{i+j} + e_{i,j}v_{i+j+1}$ for $0 \leqq i \leqq p-2$, $1 \leqq j \leqq p$ and $d_{i,j}$, $e_{i,j}$ in Z_2,

(13) $u_{p+i}v_{(2m+1)p+j} = 0$ for $-1 \leqq i \leqq p-2$, $1 \leqq j \leqq p$.

Finally, define the products v_iv_j by:

(14) $v_iv_j = v_jv_i$ for $1 \leqq i,\ j \leqq s$,

(15) $v_2^2 = u_0$,

(16) $v_1v_j = \begin{cases} u_{j-3} + u_{j-2}, & j \text{ even} \\ u_{j-3}, & j \text{ odd} \end{cases}$
where $u_j = 0$ for $j < -1$,

(17) $v_{p+i}v_j = D_{i,j}u_{p+i+j-4} + E_{i,j}u_{p+i+j-3}$
when

(18) $v_iv_j = D_{i,j}u_{i+j-4} + E_{i,j}u_{i+j-3}$ for $1 \leqq i \leqq p,\ 2 \leqq j \leqq p$ and $D_{i,j}$, $E_{i,j}$ in Z_2,

(19) $v_{p+i}v_{p+j} = 0,\ 1 \leqq i,\ j \leqq p$.

From this definition it is clear that $B(2^n)$ is commutative ((3), (7), (14)) and that u_0 is its identity element ((1), (8)). Moreover, for all i, j, a, b (i.e., $-1 \leqq i,\ j \leqq s-2$ and $1 \leqq a,\ b \leqq s$),

$$u_iu_j = h_{i,j}u_{i+j} \text{ where } h_{i,j} = 0 \text{ or } 1,$$

$$u_iv_a = \begin{cases} g_{i,a}(v_{i+a} + v_{i+a+1}),\ v_0 = 0, & \text{when } i \equiv 1 \pmod 2 \\ g_{i,a}v_{i+a} & \text{when } i \equiv 0 \pmod 2 \end{cases}$$
where $g_{i,a} = 0$ or 1, and

$$v_av_b = \begin{cases} f_{a,b}u_{a+b-4} & \text{when } a \equiv b \pmod 2 \\ f_{a,b}(u_{a+b-4} + u_{a+b-3}) & \text{when } a \not\equiv b \pmod 2 \end{cases}$$
where $f_{a,b} = 0$ or 1.

The multiplication table for $B(4)$ is easily computed to be

	u_{-1}	u_0	u_1	u_2	v_1	v_2	v_3	v_4
u_{-1}	0	u_{-1}	u_0	u_1	v_1	$v_1 + v_2$	$v_2 + v_3$	$v_3 + v_4$
u_0	u_{-1}	u_0	u_1	u_2	v_1	v_2	v_3	v_4
u_1	u_0	u_1	0	0	$v_2 + v_3$	$v_3 + v_4$	0	0
u_2	u_1	u_2	0	0	v_3	v_4	0	0
v_1	v_1	v_1	$v_2 + v_3$	v_3	0	$u_{-1} + u_0$	u_0	$u_1 + u_2$
v_2	$v_1 + v_2$	v_2	$v_3 + v_4$	v_4	$u_{-1} + u_0$	u_0	$u_1 + u_2$	u_2
v_3	$v_2 + v_3$	v_3	0	0	u_0	$u_1 + u_2$	0	0
v_4	$v_3 + v_4$	v_4	0	0	$u_1 + u_2$	u_2	0	0

and the following table summarizes the inductive definition of $B(2p)$ if the multiplication table of $B(p)$ is known:

<table>
<tr><th></th><th>$u_{-1} \cdots u_{p-2}$</th><th>$u_{p-1} \cdots u_{2p-2}$</th><th>$v_1 \cdots v_p$</th><th>$v_{p+1} \cdots v_{2p}$</th></tr>
<tr><td>u_{-1}</td><td rowspan="3">KNOWN</td><td>(2)</td><td rowspan="3">KNOWN</td><td>(9)</td></tr>
<tr><td>u_0</td><td>(1)</td><td>(8)</td></tr>
<tr><td>$\vdots$
u_{p-2}</td><td>(5)</td><td>(10), (11), (13) with smaller values of p</td></tr>
<tr><td rowspan="2">u_{p-1}
$\vdots$
u_{2p-2}</td><td rowspan="2">COMMUTATIVITY
(3)</td><td rowspan="2">ZERO
(6)</td><td>(10)</td><td rowspan="2">ZERO
(13)</td></tr>
<tr><td>(11)</td></tr>
<tr><td rowspan="2">v_1
$\vdots$
v_p</td><td rowspan="2">KNOWN</td><td rowspan="2">COMMUTATIVITY
(7)</td><td rowspan="2">KNOWN</td><td>(16)</td></tr>
<tr><td>(17) with (14)</td></tr>
<tr><td>v_{p+1}
$\vdots$
v_{2p}</td><td>COMMUTATIVITY
(7)</td><td>ZERO
((13) with (7))</td><td>COMMUTATIVITY
(14)</td><td>ZERO
(19)</td></tr>
</table>

The numbers indicate the equation used to determine the particular block of the multiplication table, COMMUTATIVITY in a block means that the block in question is determined from a corresponding block by the commutativity of $B(2^n)$, and ZERO denotes a block all of whose entries are zero.

2. The automorphisms.

DEFINITION. Define a set of $t \times t$ matrix forms A_n for $n \geqq 2$ inductively as follows: Let

$$A_2 = \begin{bmatrix} 1 & a_0 & a_1 & a_2 & 0 & a_0 & 0 & b_4 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & a_0 & 0 & 0 & 0 & a_0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & a_0 & 0 & b_4 & 1 & a_0 & a_1 & a_1 + a_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & a_0 & 0 & 0 & 1 & a_0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

where a_0, a_1, a_2 and b_4 are in Z_2.

Assuming that the matrix $A_n = (c_{i,j})$ where $-1 \leqq i, j \leqq t-2$ is defined in terms of elements a_e and b_m of Z_2 with $0 \leqq e \leqq s-2$ and

$0 \leqq m \leqq s$, we define the matrix $A_{n+1} = (d_{i,j})$ for $-1 \leqq i,\ j \leqq 2t - 2$ in terms of elements a_e and b_m of Z_2 with $0 \leqq e \leqq t - 2$ and $0 \leqq m \leqq t$ as follows: For $-1 \leqq i,\ j \leqq s - 2$,

(i) $d_{i,j} = d_{i+s,j+s} = c_{i,j}$.

(ii) $d_{i,j+t} = d_{i+s,j+3s} = c_{i,j+s}$.

(iii) $d_{i+t,j} = d_{i+3s,j+s} = c_{i+s,j}$.

(iv) $d_{i+t,j+t} = d_{i+3s,j+3s} = c_{i+s,j+s}$.

(v) $d_{i+s,j} = d_{i+s,j+t} = d_{i+3s,j} = d_{i+3s,j+t} = 0$.

(vi) $d_{i,j+s} = \begin{cases} 0 \text{ if } i = j \\ p_{i,j}a_{e+s} \text{ if } i \neq j \end{cases}$ where $p_{i,j}$ is in Z_2 and $c_{i,j} = p_{i,j}a_e$.

(vii) $d_{i,j+3s} = p_{i,j+s}b_{m+s}$ where $p_{i,j+s}$ is in Z_2 and $c_{i,j+s} = p_{i,j+s}b_m$ with $a_0 = b_2$.

(viii) $d_{i+t,j+s} = p_{i+s,j}b_{m+s}$ where $p_{i+s,j}$ is in Z_2 and $c_{i+s,j} = p_{i+s,j}b_m$ with $a_0 = b_2$.

(ix) $d_{i+t,j+3s} = \begin{cases} 0 \text{ if } i = j \\ p_{i+s,j+s}a_{e+s} + q_{i+s,j+s}a_{e+s+1} \text{ if } i \neq j \end{cases}$

where $p_{i+s,j+s}$ and $q_{i+s,j+s}$ are in Z_2 and $c_{i+s,j+s} = p_{i+s,j+s}a_e + q_{i+s,j+s}a_{e+1}$ with the convention that if $c_{i+s,j+s} = a_w$, then $e = w$.

The matrices A_n can be summarized by the following figure. If the $2^k \times 2^k$ matrix A_k is known, then A_{k+1} is the $2^{k+1} \times 2^{k+1}$ matrix given in block form by

$$A_{k+1} = \left[\begin{array}{c|c|c} & UR(A_k) + 2^{k-1} & UR(A_k) + 2^k \\ \cline{2-3} A_k & 0_{2^{k-1}} & UR(A_k) + 2^{k-1} \\ \hline 0_{2^k} & \multicolumn{2}{c}{A_k} \end{array}\right]$$

where, for $m = 2^{k-1}$ or 2^k, 0_m is the $m \times m$ zero matrix and $UR(A_k) + m$ is the $2^{k-1} \times 2^{k-1}$ matrix obtained by adding m to each subscript of the $2^{k-1} \times 2^{k-1}$ block in the upper right hand corner of A_k under the convention that if an entry in the upper right hand block of A_k is zero, then the corresponding entry in $UR(A_k) + m$ is also zero.

We now prove the characterization theorem.

THEOREM. *A linear transformation A of the Jordan algebra $B(2^n)$ over Z_2 with $n \geqq 2$ is an automorphism of $B(2^n)$ if and only* *if its matrix relative to the canonical basis of* $B(2^n)$ *is of the form* A_n.

Proof. The proof is outlined by several lemmas.

First we establish some general results about the automorphisms of $B(2^n)$. If A is an automorphism of $B(2^n)$ with matrix $(c_{i,j})$, $-1 \leqq i,\ j \leqq t-2$, $c_{i,j}$ in Z_2, relative to the canonical basis of $B(2^n)$ then we can show:

LEMMA 1. $c_{i,j} = c_{i,s+j} = c_{s+i,j} = c_{s+i,s+j} = 0$ *for* $-1 \leqq j < i \leqq s-2$.

LEMMA 2. $c_{i,i} = 1$ *for* $-1 \leqq i \leqq t-2$ *and* $c_{i,s+i} = c_{s+i,i} = 0$ *for* $-1 \leqq i \leqq s-2$.

LEMMA 3. $c_{i,j} = \delta_{i,j}$ *for* $-1 \leqq i,\ j \leqq t-2$ *and* $i \equiv 0 \pmod 2$.

LEMMA 4. $c_{i,s+j} = c_{s+i,j}$ *for* $-1 \leqq i,\ j \leqq s-2$. *In particular,* $c_{i,s+j} = c_{s+i,j} = 0$ *for* $-1 \leqq i,\ j \leqq s-2$ *and* $j \equiv 1 \pmod 2$.

LEMMA 5. *For* $-1 \leqq i,\ j \leqq r-2$, $c_{i,j} = c_{r+i,r+j}$; $c_{i,s+j} = c_{r+i,3r+j}$; $c_{s+i,j} = c_{3r+i,r+j}$; *and* $c_{s+i,s+j} = c_{3r+i,3r+j}$.

LEMMA 6. *For* $-1 \leqq i \leqq j \leqq r-2$,

(i) $c_{i,r+j} = \begin{cases} 0 \text{ if } i = j \\ p_{i,j}c_{-1,r+j-i-1} \text{ if } i \neq j \end{cases}$

if $c_{i,j} = p_{i,j}c_{-1,j-i-1}$ *where* $p_{i,j}$ *is in* Z_2.

(ii) $c_{i,3r+j} = \begin{cases} 0 \text{ if } i = j \\ q_{i,j}c_{-1,3r+j-i-1} \text{ if } i \neq j \end{cases}$

if $c_{i,s+j} = q_{i,j}c_{-1,s+j-i-1}$ *where* $q_{i,j}$ *is in* Z_2.

(iii) $c_{s+i,3r+j} = \begin{cases} 0 \text{ if } i = j \\ m_{i,j}c_{-1,r+j-i-1} + k_{i,j}c_{-1,r+j-i-2} \text{ if } i \neq j \text{ and } j \equiv 0 \pmod 2 \\ m_{i,j}c_{-1,r+j-i-1} \text{ if } i \neq j \text{ and } j \equiv 1 \pmod 2 \end{cases}$

if

$$c_{s+i,s+j} = \begin{cases} m_{i,j}c_{-1,j-i-1} + k_{i,j}c_{-1,j-i-2} \text{ for } j \equiv 0 \pmod 2 \\ m_{i,j}c_{-1,j-i-1} \text{ for } j \equiv 1 \pmod 2 \end{cases}$$

where $m_{i,j}$ *and* $k_{i,j}$ *are in* Z_2.

To establish the necessity of the condition of the theorem, we proceed by induction on n. The case $n = 2$ is straightforward in view of the preceding lemmas and, for the induction step, we make the following definitions and state a lemma about them:

DEFINITION. If A is an automorphism of $B(2^{n+1})$ with matrix

$(c_{i,j})$, $-1 \leqq i,\ j \leqq 2t-2$, relative to the canonical basis of $B(2^{n+1})$, then the *restriction of A to $B(2^n)$* is the linear transformation A' of $B(2^n)$ onto itself whose matrix $(d_{i,j})$, $-1 \leqq i,\ j \leqq t-2$, relative to the canonical basis of $B(2^n)$ is defined by:

$$d_{i,j} = \begin{cases} c_{i,j} \text{ for } -1 \leqq i,\ j \leqq s-2 \\ c_{s+i,j} \text{ for } s-1 \leqq i \leqq t-2 \text{ and } -1 \leqq j \leqq s-2 \\ c_{i,s+j} \text{ for } -1 \leqq i \leqq s-2 \text{ and } s-1 \leqq j \leqq t-2 \\ c_{s+i,s+j} \text{ for } s-1 \leqq i,\ j \leqq t-2\,. \end{cases}$$

DEFINITION. If A is an automorphism of $B(2^n)$ with matrix $(c_{i,j})$, $-1 \leqq i,\ j \leqq t-2$, relative to the canonical basis of $B(2^n)$, then the *linear transformation of $B(2^{n+1})$ induced by A* is the linear transformation A^* of $B(2^{n+1})$ onto itself whose matrix $(b_{i,j})$, $-1 \leqq i$, $j \leqq 2t-2$, relative to the canonical basis of $B(2^{n+1})$ is defined by:

For $\quad -1 \leqq i,\ j \leqq s-2$,

$$\begin{aligned} &b_{i,j} = b_{s+i,s+j} = c_{i,j}\,, \\ &b_{t+i,t+j} = b_{3s+i,3s+j} = c_{s+i,s+j}\,, \\ &b_{t+i,j} = b_{3s+i,s+j} = c_{s+i,j}\,, \\ &b_{i,t+j} = b_{s+i,3s+j} = c_{i,s+j}, \text{ and} \\ &b_{s+i,j} = b_{i,s+j} = b_{3s+i,t+j} = b_{t+i,3s+j} = b_{3s+i,j} = b_{t+i,s+j} \\ &\qquad = b_{s+i,t+j} = b_{i,3s+j} = 0\,. \end{aligned}$$

LEMMA 7. (i) *If A is an automorphism of $B(2^{n+1})$, then the restriction A' of A to $B(2^n)$ is an automorphism of $B(2^n)$.*

(ii) *If A is an automorphism of $B(2^n)$, then the linear transformation A^* of $B(2^{n+1})$ induced by A is an automorphism of $B(2^{n+1})$.*

The induction step then proceeds as follows: We assume that every automorphism of $B(2^n)$ has matrix of the form A_n relative to the canonical basis of $B(2^n)$ and we let A be an automorphism of $B(2^{n+1})$ with matrix $(d_{i,j})$, $-1 \leqq i,\ j \leqq 2t-2$, relative to the canonical basis of $B(2^{n+1})$. We must show that $(d_{i,j})$ satisfies (i)-(ix).

By the induction hypothesis and Lemmas 5 and 7, we have (i)-(iv). Lemma 1 establishes (v) and (vi)-(ix) follow from the induction hypothesis and Lemmas 4 and 6. Thus we have that $(d_{i,j})$, $-1 \leqq i$, $j \leqq 2t-2$, is of the form A_{n+1}, completing the induction. Therefore every automorphism of $B(2^n)$, $n \geqq 2$, has matrix of the form A_n relative to the canonical basis of $B(2^n)$.

To establish the sufficiency of the condition we first show that $\det A_n = 1$ for $n \geqq 2$. This is accomplished by expanding $\det A_n$ by the cofactors of its first column, expanding the resulting $(t-1) \times (t-1)$

determinant by the cofactors of its first row, expanding the resulting $(t-2) \times (t-2)$ determinant by the cofactors of its first column, and continuing to alternate in this manner. In view of Lemmas 1, 3 and 4, after $t-2$ such steps we have

$$\det A_n = \begin{vmatrix} 1 & c_{-1,0} \\ 0 & 1 \end{vmatrix} = 1 \, .$$

This fact about the determinant says that A_n is the matrix relative to the canonical basis of $B(2^n)$ of a nonsingular linear transformation A of $B(2^n)$ onto itself. Hence it only remains to show that A preserves products of the basis elements of $B(2^n)$ and this may be checked by a straightforward but lengthy calculation. This completes the proof.

3. The automorphism groups. At this point we know that the automorphism group $\mathscr{A}_n$ of $B(2^n)$ over Z_2 is the group formed by all matrices of the form A_n under matrix multiplication. From the definition of the matrices A_n, $n \geqq 2$, it follows that such a matrix has $3q$ elements a_i and $r-1$ elements b_j in its first row. Since any matrix of the form A_n is completely determined by the elements of its first row, this says that the order of $\mathscr{A}_n$ is $2^{3q+r-1} = 2^{5q-1}$.

In order to examine the structure of $\mathscr{A}_n$, we make the following

DEFINITION. For $n \geqq 2$, let

$I_1^n = \{i\colon 0 \leqq i \leqq s-2,\ i \not\equiv 3 \pmod 4\}$,

$I_2^n = \{s + 2i - 2\colon 2 \leqq i \leqq r\}$ and

$I^n = I_1^n \cup I_2^n$.

Then I^n is a subset of $\{i\colon 0 \leqq i \leqq t-2\}$ and consists of $5q-1$ elements.

For i in I^n, define G_i as follows:

(i) If i is in I_1^n, G_i is the matrix of the form A_n with $a_i = 1$, $a_j = 0$ for j in I_1^n and $j \neq i$, and $b_{2k} = 0$ for $2 \leqq k \leqq r$.

(ii) If i is in I_2^n, G_i is the matrix of the form A_n with $b_{i-s+2} = 1$, $b_{j-s+2} = 0$ for j in I_2^n and $j \neq i$, and $a_k = 0$ for k in I_1^n.

Denote by $(i_1, i_2, \cdots, i_m)$, where i_j is in I^n for $1 \leqq j \leqq m$ and $i_1 < i_2 < \cdots < i_m$, the matrix of the form A_n in which for each $j = 1, 2, \cdots, m$, $a_{i_j} = 1$ if i_j is in I_1^n and $b_{i_j - s+2} = 1$ if i_j is in I_2^n while $a_k = 0$ for k in I_1^n and $k \neq i_j$ for any $j = 1, 2, \cdots, m$ and $b_{k-s+2} = 0$ for k in I_2^n and $k \neq i_j$ for any $j = 1, 2, \cdots, m$. Clearly any element of $\mathscr{A}_n$ can be expressed in the form $(i_1, i_2, \cdots, i_m)$ for some $i_1, i_2, \cdots, i_m$.

Using a technique due to Bobo [1], we can show that the set $\{G_i\colon i \text{ in } I^n\}$ generates $\mathscr{A}_n$. Finally, we determine the commutator subgroup of $\mathscr{A}_n$ by using a straightforward induction on n together with the properties of matrix multiplication and matrices of the form G_i. The results are summarized in the following

THEOREM. *The automorphism group* $\mathscr{A}_n$ *of the Jordan algebra* $B(2^n)$, $n \geqq 2$, *can be described abstractly as the group generated by the* $5(2^{n-2}) - 1$ *generators* G_i *where* i *is in* I^n *and where* $G_i^2 = I$ *for all* i *in* I^n, I *being the* $t \times t$ *identity matrix. Moreover, the commuting relations among the generators of* $\mathscr{A}_n$ *may be described inductively as follows*:

In $\mathscr{A}_2$, $G_0G_1 \neq G_1G_0$ *but* $G_0G_1 = G_2G_6G_1G_0$. *For all other* i *and* j *in* I^2, $G_iG_j = G_jG_i$.

If the commuting relations among the generators G'_i, i *in* I^{n-1}, *of* $\mathscr{A}_{n-1}$ *are known, the commuting relations in* $\mathscr{A}_n$ *are given by*:

If i *and* m *are in* I_1^{n-1} *with* $i, m \neq 0$ *and* $i \neq m$ *and if* j *and* k *are in* I_2^{n-1} *with* $j \neq k$, *then*

(a) *If* $i \equiv 1 \pmod 4$,
$G_0G_i = G_{i+1}G_{s+i+1}G_iG_0$,
$G_0G_{i+r} = G_{r+i+1}G_{s+r+i+1}G_{i+r}G_0$,
$G_iG_r = G_{r+i+1}G_rG_i$, *and*
$G_iG_{s+r} = G_{s+r+i+1}G_{s+r}G_i$.

(b) *If* $G'_iG'_m = G'_{i+m+1}G'_mG'_i$,
$G_iG_m = G_{i+m+1}G_mG_i$, *and*
$G_iG_{m+r} = G_{r+i+m+1}G_{m+r}G_i$.

(c) *If* $G'_iG'_j = G'_{i+j+1}G'_jG'_i$,
$G_iG_{j+r} = G_{r+i+j+1}G_{j+r}G_i$;
$G_iG_{j+s} = G_{s+i+j+1}G_{j+s}G_i$, *and*
$G_{i+r}G_{j+r} = G_{s+i+j+1}G_{j+r}G_{i+r}$.

Otherwise, $G_eG_f = G_fG_e$ *for* e *and* f *in* I^n.

The structure of the group $\mathscr{A}_2$ of order 16 becomes clearer if it is recognized as the direct product of two familiar groups. Since $I_1^2 = \{0, 1, 2\}$ and $I_2^2 = \{6\}$, $\mathscr{A}_2$ is generated by G_0, G_1, G_2, and G_6. Let H_1 be the subgroup of $\mathscr{A}_2$ generated by G_0 and G_1. Then, if $a = G_1G_0$ and $b = G_0$, direct computation using the commuting relations in $\mathscr{A}_2$ yields $H_1 = \{I, a, b, ab, a^2, a^3, a^2b, a^3b\} \cong$ the dihedral group D_4. If $H_2 = \{I, G_2\}$ then both H_1 and H_2 are normal in $\mathscr{A}_2$, $H_1 \cap H_2 = \{I\}$, and $H_1H_2 = \mathscr{A}_2$. Hence

$$\mathscr{A}_2 \cong D_4 \times Z_2$$

where D_4 is the dihedral group of order eight and Z_2 is the cyclic group of order two.

Detailed proofs of some of the results summarized in this paper may be found in the author's doctoral thesis written at the University of Virginia under Eugene C. Paige.

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