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TOTALLY INTEGRALLY CLOSED RINGS AND EXTREMAL SPACES

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E. Enochs has defined a ring A (all rings are commutative, with 1) to be *totally integrally closed* (which we abbreviate TIC) if for every integral extension $h: B \to C$ the induced map h^*: Hom $(C, A) \to$ Hom (B, A) is surjective. Our main result here is that A is TIC if and only if A is reduced, each residue class domain of A is normal and has an algebraically closed field of fractions, and Spec A is extremal (disjoint open sets have disjoint clopen neighborhoods). We use this fact to settle negatively the open question, need a localization of a TIC ring be TIC. The proofs depend on the following apparently new characterization of extremal spaces: a topological space X is extremal if and only if there is a Boolean algebra retraction of the family of all subsets of X onto the family of all clopen subsets of X which takes every closed set into a subset of itself.

1. Basic facts. We summarize here some of the facts about TIC rings from [2]. Most results are merely stated informally. The proofs are easy. However, we include simpler proofs of both Lemma 1 (which is, in substance, the "only if" part of Theorem 1 of [2]) and of the existence of "the total integral closure" of a reduced ring than are given in [2], and we also give some useful characterizations of TIC rings which are not stated (though they are implicit) in [2].

We use the term *normal* for a domain integrally closed in its fraction field. When we speak of an integrally closed subring B of a ring C, we mean that B is integrally closed in C, rather than in, say, the fraction field of B.

Trivially, a product of TIC rings is TIC; a retract of a TIC ring is TIC; and an integrally closed subring of a TIC ring is TIC. Proposition 3 of [2] asserts that a domain is TIC if and only if it is normal and has an algebraically closed fraction field. Hence, products of algebraically closed fields are TIC.

Following [2], we define an extension $h: A \to B$ to be *tight* if, equivalently, either (1) for each ideal $I \neq (0)$ of B, $h^{-1}(I) \neq (0)$; or (2) each nonzero element of B has a nonzero multiple in $h(A)$; or (3) if $g: B \to C$ and $g \mid h(A)$ is injective then g is injective.

LEMMA 1. *If A has no proper tight integral extension, then A is reduced.*

Proof. Suppose $c \in A$, $c \neq 0$, but $c^2 = 0$. Let t be an indeterminate over A and let $B = A[t]/(t^2 - c, (\mathrm{Ann}_A c)t)$. It is easy to see that every element of B can be written uniquely in the form $a + \alpha T$, where T is the image of t, $a \in A$, and $\alpha \in A/\mathrm{Ann}_A c$. Since $T^2 - c = 0$, B is integral over A. We can complete the proof by showing that $A \subset B$ is tight, a contradiction. Let $0 \neq b = a + \alpha T \in B$. We want to show $bB \cap A \neq (0)$. If $\alpha = 0$, this is clear. If $ca \neq 0$, then

$$ca = c(a + \alpha T) \in (bB \cap A) - (0) .$$

If $\alpha \neq 0$ but $ca = 0$, then $aT = 0$ and $T(a + \alpha T) = \alpha c \in (bB \cap A) - (0)$.

Notice that if $A \subset B$ and I is an ideal of B maximal with respect to the condition $I \cap A = (0)$, then $A \to B/I$ is a tight extension of A. Hence, if B is integral over A and A has no proper tight integral extension, then A is a retract of B. With this observation we can prove:

PROPOSITION 1. *The following conditions on a ring A are equivalent.*

(1) *A is* TIC.

(2) *A is a retract of every integral extension.*

(3) *A has no proper tight integral extension.*

(4) *A is reduced, and A is a retract of an integrally closed subring of every* TIC *extension.*

(5) *A is a retract of an integrally closed subring of a product of algebraically closed fields.*

(6) *A is a retract of an integrally closed subring of some* TIC *extension.*

Little needs to be said in the way of proof. (1) $\Rightarrow$ (2) trivially, taking $B = A$ and C to be the integral extension in question in the definition of TIC ring. (2) $\Rightarrow$ (3) because the retraction of the tight integral extension will have to be injective, and therefore an isomorphism. (3) $\Rightarrow$ (4) follows from Lemma 1 and the remarks preceding the statement of Proposition 1 (taking B to be the integral closure of A in the TIC extension). (4) $\Rightarrow$ (5) because every reduced ring A can be embedded in a product of algebraically closed fields (for each prime P, let k_P be an algebraic closure of A/P, and embed A in $\prod_P k_P$ in the obvious way). (5) $\Rightarrow$ (6) $\Rightarrow$ (1) follows from our various introductory remarks on TIC rings.

We next observe that every reduced ring A has a tight integral extension ring B which is TIC. (B is called a *total integral closure* of A.) To see this, embed A in any TIC ring C, e.g., a product of algebraically closed fields. By Zorn's lemma, A has a maximal tight integral extension B in C. Now suppose B has a proper tight integral extension B'. Then the map $B \subset C$ extends *injectively* to B' (since $B \subset B'$ is tight), and the image of B' in C under this extension homomorphism gives a proper tight integral extension of B in C. This will be a tight integral extension of A as well, contradicting the maximality of B. Thus, B is TIC.

If B, B' are two total integral closures of A, then there is an A-homomorphism $B \to B'$, because B' is TIC. Any such homomorphism is an isomorphism. (It is surjective because B has no proper tight integral extension.) Thus, the total integral closure of a reduced ring is unique in the same sense that the algebraic closure of a field is.

We conclude this section with a couple of remarks. It is trivial that for a TIC ring A, every monic polynomial $f = f(t)$ of positive degree in $A[t]$, t an indeterminate, has a root in A: Let $B = A$, let $C = A[t]/(f)$, and let T be the image of t in C; then the image of T under any extension of id_A to C is the required root. It follows that every monic polynomial in $A[t]$ factors completely into monic linear factors.

This will also be true for any residue class ring A/I of A, although A/I need not be TIC, even if A is (von Neumann) regular (and I is radical). See § 5. However, if I is prime, A/I will be TIC, for every monic polynomial over a domain has a root in the domain if and only if the domain is normal and has an algebraically closed fraction field. Enochs gives examples in [2] to show that A need not be TIC even if A is reduced and A/P is TIC for every prime P. The missing condition is that Spec A be extremal.

2. Extremal spaces.

We develop here the topological material required in the rest of the paper. Following [5, 326–328], we call a topological space X *extremal* if, equivalently, either (1) any two disjoint open sets have disjoint clopen neighborhoods; or (2) the interior of every closed set is clopen; or (3) the closure of every open set is clopen. The term *extremally disconnected* is used instead if X is Hausdorff. Notice that, trivially, a connected subspace of an extremal space is irreducible (in the sense of algebraic geometry, see [1]). Hence, the maximal irreducible subspaces (i.e., the *irreducible components*) of an extremal space are identical with its connected components. (Notice that the irreducible components are not, in general, disjoint.) Every irreducible space is trivially extremal. It is easy to see that every open and every dense subspace of an extremal space is extremal.

Extremally disconnected spaces have been extensively studied. Let X be a completely regular Hausdorff space. Then X is extremally disconnected if and only if any of the following conditions holds (see [3] for terminology): (1) every open subspace is C^*-embedded; (2) every dense subspace is C^*-embedded; (3) the Stone-Čech compactification βX of X is extremally disconnected; or (4) the lattice $C(X)$ of continuous real-valued functions on X is conditionally complete. A compact space X (*compact* always means compact Hausdorff; we use the term *quasicompact* if the space is not necessarily Hausdorff) is extremally disconnected if and only if $\beta Y = X$ for every dense subspace Y of X. Every infinite compact space has a closed subspace which is not extremally disconnected. If X is a countable infinite discrete space, then βX is extremally disconnected but $\beta X - X$ is a closed subspace which is not. (These results may be found in [3, p. 23, 1H. 6., p. 52, 3N. 6., p. 96, 6M., and p. 98, 6R.]. Condition (4) is treated in detail in [8] and [9].)

Now, if X is any set, let $\mathscr{P}(X)$ be the Boolean algebra of all subsets of X, and if X is a topological space, let $\mathscr{B}(X)$ be the Boolean algebra of all clopen subsets of X. Of course, $\mathscr{B}(X)$ is a subalgebra of $\mathscr{P}(X)$. The following characterization of extremal spaces is the topological crux of the proof of our main result.

PROPOSITION 2. *A space X is extremal if and only if there is a Boolean algebra retraction ρ of $\mathscr{P}(X)$ onto $\mathscr{B}(X)$ such that for every closed set K, $\rho(K) \subset K$. For such a ρ, it is actually true that for each closed set K, $\rho(K) = \operatorname{Int} K$, while for each open set U, $\rho(U) = \operatorname{Cl} U$.*

Proof. $\Leftarrow$ Suppose that there is such a ρ. For each closed K, $\rho(K) \subset K$. Hence, for each open U, $\rho(U) \supset U$, by taking complements. Now, $\rho(K)$ is clopen and $\rho(K) \subset K \Rightarrow \rho(K) \subset \operatorname{Int} K$. But

$$\operatorname{Int} K \subset \rho(\operatorname{Int} K) \subset \rho(K) ,$$

so that also $\operatorname{Int} K \subset \rho(K)$. It follows that $\rho(K) = \operatorname{Int} K$. But then $\operatorname{Int} K$ must be clopen for each closed K, so that X is extremal. The fact that for each open U, $\rho(U) = \operatorname{Cl} U$ follows by taking complements.

$\Rightarrow$ Now suppose X is extremal. We first define ρ on the subalgebra generated by the closed sets; then extend by Zorn's lemma. This subalgebra consists of all sets of the form $Y = \bigcup_{i=1}^{n} (K_i \cap U_i)$, where each K_i is closed and each U_i is open. We want to map Y to $\bigcup_{i=1}^{n} (\operatorname{Int} K_i \cap \operatorname{Cl} U_i)$. To do this we must show that if $\bigcap_{i=1}^{m} (K_i' \cap U_i') = \bigcup_{i=1}^{n} (K_i \cap U_i)$, then $\bigcup_{i=1}^{m} (\operatorname{Int} K_i' \cap \operatorname{Cl} U_i') = \bigcup_{i=1}^{n} (\operatorname{Int} K_i \cap \operatorname{Cl} U_i)$. Clearly, it suffices to show that

$$K \cap U \subset \bigcup_{i=1}^{n} (K_i \cap U_i) \Longrightarrow \operatorname{Int} K \cap \operatorname{Cl} U \subset \bigcup_{i=1}^{n} (\operatorname{Int} K_i \cap \operatorname{Cl} U_i) .$$

If not, let $V = (\operatorname{Int} K \cap \operatorname{Cl} U) - (\bigcup_{i=1}^n (\operatorname{Int} K_i \cap \operatorname{Cl} U_i))$. Then $V \neq \emptyset$ and V is clopen. We then have $V \cap U \subset \bigcup_{i=1}^n (K_i \cap U_i)$ and $V \subset \operatorname{Cl} U$, but V is disjoint from $\bigcup_{i=1}^n (\operatorname{Int} K_i \cap \operatorname{Cl} U_i)$. $V \cap U = \emptyset$ contradicts $V \subset \operatorname{Cl} U$. Thus, if we let $U' = V \cap U$, we have $U' \neq \emptyset$, $U' \subset \bigcup_{i=1}^n (K_i \cap U_i)$, while U' is disjoint from $\bigcup_{i=1}^n (\operatorname{Int} K_i \cap \operatorname{Cl} U_i) \supset \bigcup_{i=1}^n (\operatorname{Int} K_i \cap U_i)$. Then

$$U' \subset \bigcup_{i=1}^n (K_i \cap U_i) - \bigcup_{i=1}^n (\operatorname{Int} K_i \cap U_i) \subset \bigcup_{i=1}^n ((K_i - \operatorname{Int} K_i) \cap U_i) \subset \bigcup_{i=1}^n (K_i - \operatorname{Int} K_i) .$$

But $U' \neq \emptyset$ is open while $\bigcup_{i=1}^n (K_i - \operatorname{Int} K_i)$ is a union of closed sets without interior, and so must also be without interior, a contradiction. This establishes our claim.

Thus, for each $Y = \bigcup_{i=1}^n (K_i \cap U_i)$, we may define $\rho(Y) = \bigcup_{i=1}^n (\operatorname{Int} K_i \cap \operatorname{Cl} U_i)$, and the value of $\rho(Y)$ will not depend on the choice of representation of Y. Clearly, ρ will preserve unions, clopen sets will be mapped to themselves, and for each closed set K, $\rho(K) = \operatorname{Int} K$. We next show that ρ preserves $\cap$. Since $\cap$ distributes over $\cup$, we can reduce to showing that $\rho(Y \cap Y') = \rho(Y) \cap \rho(Y')$ when Y and Y' are both open or both closed. The case where both are closed is trivial. For the case where both are open, let K and K' be their complements. Then, passing to complements, what we need to show is $\operatorname{Int}(K \cup K') = \operatorname{Int} K \cup \operatorname{Int} K'$, i.e., $\rho(K \cup K') = \rho(K) \cup \rho(K')$, which we already know. Finally, knowing that ρ preserves both $\cup$ and $\cap$, we can show that $\rho(Y) = X - \rho(X - Y)$ for every Y by proving it when Y is an open set U or a closed set K. The verification is trivial in each of these cases.

We now apply Zorn's lemma and assume that ρ is a Boolean algebra retraction of a subalgebra $\mathscr{S}$, containing the closed sets, of $\mathscr{P}(X)$ onto $\mathscr{B}(X)$ such that $\rho(K) = \operatorname{Int} K$ for each closed set K, and that ρ is maximal,i.e., cannot be extended to a larger subalgebra. If $\mathscr{S} = \mathscr{P}(X)$ we are done. Otherwise, let $Y \in \mathscr{P}(X) - \mathscr{S}$. The subalgebra $\mathscr{S}'$ of $\mathscr{P}(X)$ generated by $\mathscr{S}$ and Y consists of all sets of the form $(S \cap Y) \cup (T - Y)$, $S, T \in \mathscr{S}$. We wish to define a set $W \in \mathscr{B}(X)$ in such a way that for all S, S', T, and T' in $\mathscr{S}$, $(S \cap Y) \cup (T - Y) = (S' \cap Y) \cup (T' - Y) \Rightarrow (\rho(S) \cap W) \cup (\rho(T) - W) = (\rho(S') \cap W) \cup (\rho(T') - W)$. (This will enable us to extend ρ further, giving the desired contradiction.) The condition on W can be broken up into two parts:

(1) For all S, S' in $\mathscr{S}$, $S \cap Y = S' \cap Y \Rightarrow \rho(S) \cap W = \rho(S') \cap W$; and

(2) For all T, T' in $\mathscr{S}$, $T - Y = T' - Y \Rightarrow \rho(T) - W = \rho(T') - W$.

We can rephrase (1) thus: $Y \subset X - (S \triangle S') \Rightarrow W \subset \rho(X - (S \triangle S'))$, and, similarly, (2) can be rephrased: $T \triangle T' \subset Y \Rightarrow \rho(T \triangle T') \subset W$. Here,

$S \triangle S' = (S - S') \cup (S' - S)$. Since all the sets $T \triangle T'$, $X - (S \triangle S')$ are in $\mathscr{S}$, we merely require that W be a clopen set which contains all the $\rho(T)$ for $T \subset Y$ and is contained in all the $\rho(S)$ for $Y \subset S$. Now, $T \subset Y \subset S \Rightarrow \rho(T) \subset \rho(S)$, so that $U = \bigcup_{T \subset Y} \rho(T) \subset \bigcap_{Y \subset S} \rho(S) K$. Moreover, since each $\rho(T)$ and $\rho(S)$ is clopen, U is open and K is closed. Thus, we need only choose W to be any clopen set between U and K, such as $\mathrm{Cl}\, U$ or $\mathrm{Int}\, K$.

We define ρ' on $\mathscr{S}'$ by

$$\rho'((S \cap Y) \cup (T - Y)) = (\rho(S) \cap W) \cup (\rho(T) - W) .$$

By virtue of the conditions put on W, the value of ρ' does not depend on the choice of S and T in the representation. Moreover, $\rho'(\mathscr{S}') \subset \mathscr{B}(X)$ and ρ' extends ρ. Finally, it is trivial that ρ' preserves unions and complements. Since ρ was supposed to be maximal, we have a contradiction. $\mathscr{S}$ must have equalled $\mathscr{P}(X)$.

We can now tie up the questions of whether a given ring A is TIC and whether $\mathrm{Spec}\, A$ is extremal. For basic facts about $\mathrm{Spec}\, A$ used here, see [1].

PROPOSITION 3. *Let A be a* TIC *ring. Let $X = \mathrm{Spec}\, A$. Then there is a Boolean algebra retraction ρ of $\mathscr{P}(X)$ onto $\mathscr{B}(X)$ such that for each closed set K, $\rho(K) = \mathrm{Int}\, K$. Thus,* Spec A *is extremal.*

Proof. We do not distinguish notationally between points of $\mathrm{Spec}\, A$ and prime ideals of A. Embed A in $\prod_{P \in X} A/P = C$ as usual, and let B be the integral closure of A in C. Since A is TIC, we must have a retraction homomorphism $\phi: B \to A$. The restriction of ϕ to the idempotents of B is a retraction onto the idempotents of A. The idempotents of B are identical with those of C ($c^2 - c = 0 \Rightarrow c$ is integral over A), and these are in one-to-one correspondence with $\mathscr{P}(X)$ via the same map (regard A as a subring of C). Thus, ϕ induces a retraction ρ of $\mathscr{P}(X)$ onto $\mathscr{B}(X)$. Since ϕ preserves products, ρ preserves intersections, and since ϕ preserves sums (in particular, the relation $c_1 + c_2 = 1$), ρ preserves complements. Thus, ρ is a Boolean algebra retraction.

It remains to show that for each closed set K, $\rho(K) = \mathrm{Int}\, K$, and by Proposition 2, it will be enough to show that for each closed set K, $\rho(K) \subset K$. Consider the idempotent $c \in C$ which vanishes precisely on K. For each $a \in A$, $a \in cC$ if and only if a vanishes on K, in which case $a = ca$. But $a = ca \Rightarrow \phi(a) = \phi(c)\phi(a) \Rightarrow a = \phi(c)a$. Since K is closed, for each $P \notin K$ there is an $a \in A$ that vanishes on K but not at P, whence $\phi(c)$ does not vanish at P. Thus, $\phi(c)$ vanishes only on a subset of K.

In the next section we show that this extra topological condition is just what we need to give a reasonably concrete characterization of TIC rings.

3. The main result. The usual (Zariski) topology on Spec A is defined by taking as a basis for the closed sets the sets $K(a) = \{P \in \operatorname{Spec} A: a \in P\}$, $a \in A$. If these sets *and their components* are taken as a subbasis for the *open* sets of a topology on Spec A, Spec A becomes a *Boolean* space, i.e., a totally disconnected compact (Hausdorff) space. We refer to this topology as the *strong* topology on Spec A. See [6, §§ 2 and 8] for details. With this observation, we are ready to prove

THEOREM 1. *A ring A is* TIC *if and only if the following three conditions hold:*

(1) *A is reduced.*

(2) *For each prime P of A, A/P is normal and has an algebraically closed field of fractions.*

(3) *$X =$ Spec A is extremal.*

Proof. We already know that the conditions are necessary. Now assume (1), (2), and (3). Embed A in $C = \prod_{P \in X} A/P$ just as in the proof of Proposition 3. Thus, a is identified with the element of C which has the image of a modulo P as its P-component. Let B be the integral closure of A in C. C is TIC, and hence so is B. To complete the proof, we need only show that A is a retract of B.

For each $Y \subset X$, let $e(Y)$ be the element of C which has P-component $1_{A/P}$ for each $P \in Y$ and $0_{A/P}$ for each $P \notin Y$. (Henceforth, we drop the subscript A/P.) e is the bijection of $\mathscr{P}(X)$ onto the set of idempotents of C (equivalently, of B) utilized in the proof of Proposition 3. We know $e(\mathscr{P}(X)) \subset B$. We shall show that $e(\mathscr{P}(X))$ is a basis for B as an A-module. In fact, it suffices to show that given $b \in B$, there is a finite cover $\{Y_i: 1 \leqq i \leqq n\}$ of X and for each i an element a_i of A such that for each i, $b \mid Y_i = a_i \mid Y_i$. For then $b = \sum_i a_i e(Y_i - \bigcup_{j<i} Y_j)$.

Let b be given, and let $f \in A[t]$ be a monic polynomial in the indeterminate t of degree $d \geqq 1$ such that $f(b) = 0$. Let $P \in X$ be given. By virtue of (2), we can choose d elements $a_P(\nu)$, $1 \leqq \nu \leqq d$, of A such that $g(t) = f(t) - \prod_{\nu=1}^{d} (t - a_P(\nu))$ has all its coefficients in P. Let Y_P be the set of all primes of A which contain all the coefficients of $g(t)$. Then for each P, Y_P is open in the strong topology on Spec A and $P \in Y_P$, so that $\{Y_P: P \in X\}$ is a strongly open cover of X, and has a finite subcover. Thus, it suffices to restrict attention to a single Y_P in looking for the sets Y_i and elements a_i of A. But since $f(b) = 0$ and $g \equiv 0$ modulo Q for each Q in Y_P, we have that for each Q in Y_P the Q-component of b is identical with the Q-component of one of the $a_P(\nu)$, which is exactly what we need.

Thus, every $b \in B$ can be expressed in the form $\sum_{i=1}^n a_i e(Y_i)$, $Y_i \subset X$, for some n. Fix a Boolean algebra retraction ρ of $\mathscr{P}(X)$ onto $\mathscr{B}(X)$ such that $\rho(K) = \operatorname{Int} K$ for each closed set K. This is possible because X is extremal (Proposition 2). Notice that for $Y \subset X$, $e(\rho(Y)) \in A$ (it is a priori an idempotent of B, and turns out to be in A because $\rho(Y)$ is clopen in X). We now define $\phi: B \to A$ by $\phi(\sum_{i=1}^n a_i e(Y_i)) = \sum_{i=1}^n a_i e(\rho(Y_i))$. If we can show that ϕ is a well-defined A-module homomorphism we will be done, because it is obviously then a retraction, and preserves multiplication (this last because, since the $e(Y)$ are a basis for B over A, it suffices to show that for all Y, $Y' \subset X$, $\phi(e(Y)e(Y')) = \phi(e(Y))\phi(e(Y'))$; but this is an immediate consequence of the fact that ρ preserves $\cap$). Thus, to complete the proof we need only check that if $\sum_i a_i e(Y_i) = 0$ then so does $\sum_i a_i e(\rho(Y_i))$, $1 \leqq i \leqq n$. The Boolean subalgebra $\mathscr{Y}$ of $\mathscr{P}(X)$ generated by the Y_i is finite: let $\{W_j : 1 \leqq j \leqq h\}$ be an enumeration of the minimal nonempty sets in it. Then the W's form a finite partition of X, and every set in $\mathscr{Y}$ (in particular, each Y_i) can be expressed as a *disjoint* union of W's. Let $Y_i = \bigcup_{u=1}^{v(i)} W_{J(i,u)}$. Since $\sum_i a_i e(Y_i) = 0$, and since $e(Y_i) = \sum_{u=1}^{v(i)} e(W_{J(i,u)})$(for the union is disjoint), we have $\sum_{j=1}^h a_j' e(W_j) = 0$, where a_j' is the sum of those a_i such that for some u, $1 \leqq u \leqq v(i)$, $J(i, u) = j$. Since the W_j are mutually disjoint, we then have that a_j' vanishes on W for each j. But then a_j' vanishes on $\operatorname{Cl} W_j$, and $\rho(W_j) \subset \rho(\operatorname{Cl} W_j) \subset \operatorname{Cl} W_j$, so that a_j' vanishes on $\rho(W_j)$ for each j, and thus $\sum_{j=1}^h a_j' e(\rho(W_j)) = 0$. Now, for each i, $e(\rho(Y_i)) = e(\bigcup_{u=1}^{v(i)} \rho(W_{J(i,u)})) = \sum_{u=1}^{v(i)} e(\rho(W_{J(i,u)}))$(for the W's mutually disjoint $\Rightarrow$ the $\rho(W)$'s mutually disjoint). Substituting, we find $\sum_i a_i e(\rho(Y_i)) = \sum_{j=1}^h a_j' e(\rho(W_j)) = 0$, as required.

COROLLARY 1. *Let A be a reduced ring such that* Spec A *is extremal. Then the following conditions are equivalent.*

(1) *A is* TIC.

(2) *Every monic $f \in A[t]$ of positive degree factors into monic linear factors.*

(3) *Every residue class domain of A is normal and has an algebraically closed fraction field.*

Proof. (1) $\Rightarrow$ (2), and (2) passes to homomorphic images, so that (2) $\Rightarrow$ (3), while (3) $\Rightarrow$ (1) by Theorem 1.

COROLLARY 2. *Let A be* TIC. *Let B be a reduced homomorphic* *image of A, or a localization of A. Then B is* TIC *if and only if* Spec B *is extremal.*

Proof. B is reduced, and condition (2) of Corollary 1 is inherited by B.

4. **The extremality of Spec A.** In this section, we want to put as concrete as possible an interpretation on the extremality of a topological space X of the form Spec A for some ring A. We regard two cases as well understood: the case where X is Boolean, so that extremal means extremally disconnected, and the case where X is irreducible ($\Rightarrow$ extremal). We want to reduce the general case to a kind of composite of these two cases.

Consider any space X of the form Spec A for some ring A. Let Min X be the set of points of X which are not in the closure of any other point. (These correspond to the minimal primes of A.) Let $\mathscr{C}(X)$ be the set of irreducible components of X (each of which is the set of primes containing a certain minimal one), and let $f\colon \operatorname{Min} X \to \mathscr{C}(X)$ by $f(x) = \operatorname{Cl}\{x\}$. f is a bijection. Let Min X have the relative topology from X, and whenever $\mathscr{C}(X)$ is a partition of X, let $\mathscr{C}(X)$ have the quotient topology. In this case, f is continuous.

PROPOSITION 4. *Suppose $X \approx$ Spec A for some ring A. Then the following conditions are equivalent.*

(1) *X is extremal.*

(2) *Conditions* (a), (b), *and* (c) *below hold.*

(a) *For every open subset of U of* Min X *or of X, the union of the irreducible components of X which meet U is open.*

(b) *$\mathscr{C}(X)$ is the set of connected components of X.*

(c) *f is a homeomorphism, and both* Min X *and* $\mathscr{C}(X)$ *are extremally disconnected Boolean spaces.*

(3) *Conditions* (b′) *and* (c′) *below hold.*

(b′) *If $x, x' \in \operatorname{Min} X$ and $x \neq x'$, then* $\operatorname{Cl}\{x\} \cap \operatorname{Cl}\{x'\} = \varnothing$.

(c′) Min X *is extremal.*

Proof. We show (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Assume (1). To prove (a), we first observe that the set of irreducible components which an open set U of X meets is determined by $U \cap \operatorname{Min} X$: U meets $f(x)$ if and only if $x \in U$. Since the quasicompact open sets form a basis for X, we may assume without loss of generality that U is a quasicompact open subset of X. Then by the first corollary in § 2 of [6], $\operatorname{Cl} U = \bigcup_{x \in U \cap \operatorname{Min} X} f(x)$. Since U is open and X is extremal, $\operatorname{Cl} U$ is open, establishing (a). We already know (b). It thus makes sense to give

$\mathscr{C}(X)$ the quotient topology. We know f is a continuous bijection, and (a) is precisely the condition we need to make f^{-1} continuous. Now $\operatorname{Min} X$ is Hausdorff [4, Corollary 2.4] and $\operatorname{Min} X$ is extremal because it is dense in the extremal space X; on the other hand, $\mathscr{C}(X)$ is quasicompact because it is a quotient of the quasicompact space X, and since f is a homeomorphism, we obtain at once that both $\operatorname{Min} X$ and $\mathscr{C}(X)$ are extremally disconnected Boolean spaces.

(2) $\Rightarrow$ (3) is trivial, since (b) $\Rightarrow$ (b′) and (c) $\Rightarrow$ (c′). Now assume (3). We wish to show that X is extremal. Let U, V be disjoint open sets in X. Then $U \cap \operatorname{Min} X$ is disjoint from $V \cap \operatorname{Min} X$, and by virtue of (c′), these sets have disjoint clopen neighborhoods S and T in $\operatorname{Min} X$. We can assume $T = \operatorname{Min} X - S$. By the first corollary in § 2 of [6], the sets $U' = \bigcup_{s \in S} f(s)$ and $V' = \bigcup_{t \in T} f(t)$ are closed, by (b′) they are disjoint, their union is X, since the union of S and T is $\operatorname{Min} X$, and thus they constitute disjoint clopen neighborhoods of U and V.

This enables us to give a very concrete characterization of TIC rings, which involves only the notion of extremally disconnected Boolean space, rather than the more elusive notion of arbitrary extremal space.

PROPOSITION 5. *The ring A is* TIC *if and only if the following conditions hold.*

(1) *A is reduced.*

(2) *For each prime P of A, A/P is a normal domain whose fraction field is algebraically closed.*

(3) *Distinct minimal primes of A are comaximal.*

(4) *The set of minimal primes of A, in the inherited Zariski topology, is an extremally disconnected Boolean space.*

Using this result, it is not difficult to show that a completely regular Hausdorff space is extremally disconnected if and only if its ring of continuous complex-valued functions is TIC. See [7].

We note that in a TIC ring A, for each $a \in A$, $\operatorname{Ann}_A a$ is generated by an idempotent. To see this, let $D(a) = \{P \in \operatorname{Spec} A : a \notin P\}$. $\operatorname{Cl} D(a)$ is clopen. Thus, it will suffice to show that $\operatorname{Cl} D(a)$ is the set of primes containing $\operatorname{Ann}_A a$. But, by the first corollary in § 2 of [6], $\operatorname{Cl} D(a) = \{P \in \operatorname{Spec} A : P$ contains a prime not containing $a\}$, and P contains a prime not containing a if and only if the image of a in A_P is not zero if and only if $\operatorname{Ann}_A a \subset P$, as required. We will need this in § 5.

Now, let X be any space of the form $\operatorname{Spec} A$ such that X is extremal. Composing the quotient map $X \to \mathscr{C}(X)$ with f^{-1}, we obtain a closed, open, continuous retraction of X onto $\operatorname{Min} X$ which we denote by r. It is quite easy to verify (see Proposition 4 and its proof):

PROPOSITION 6. *$Y \mapsto r(Y)$ is a bijection of $\mathscr{B}(X)$ onto $\mathscr{B}(\mathrm{Min}\, X)$.*

5. Localization and total quotient rings.

PROPOSITION 7. *Let S be a multiplicative system in the* TIC *ring A generated by a family of elements which contains at most finitely many zero divisors. Then $S^{-1}A$ is* TIC. *Also, for each prime P of A, A_P is a* TIC *domain.*

Proof. It suffices, for the first statement, to show that $\mathrm{Spec}\, S^{-1}A$ is extremal. We can break the problem up into two cases: (1) S is finitely generated, and (2) S contains no zero divisors. In the first case, $\mathrm{Spec}\, S^{-1}A$ is open in $\mathrm{Spec}\, A$, hence extremal, while in the second, $A \subset S^{-1}A \Rightarrow \mathrm{Spec}\, S^{-1}A$ dense in $\mathrm{Spec}\, A$, hence, extremal.

To prove the second statement, it will suffice to show that A_P is a domain, for then $\mathrm{Spec}\, A_P$ is irreducible $\Rightarrow$ extremal. Since A is reduced, so is A_P, and the problem therefore reduces to showing that P contains at most one minimal prime of A; but the minimal primes are pairwise comaximal.

We recall that a ring A is (von Neumann) *regular* if for each $a \in A$ there is a $u \in A$ such that $a^2u = a$. A regular $\Rightarrow A$ reduced, and a reduced ring A is regular $\Leftrightarrow$ every prime is maximal $\Leftrightarrow \mathrm{Spec}\, A$ is T_1. In fact, if $\mathrm{Spec}\, A$ is T_1 then it is a Boolean space (totally disconnected compact Hausdorff space). (The properties of regular rings are developed in [1, pp. 172–3, Exercise (15), (16), and (17)], where the term *absolument plat* is used.) Thus, a regular ring A is TIC if and only if each residue class field is algebraically closed and $\mathrm{Spec}\, A$ is extremally disconnected.

For any ring A, let A^* be the total quotient ring of A, i.e., $S^{-1}A$, where S is the set of elements of A which do not divide zero. $A = A^*$ if and only if every noninvertible element of A is a zero divisor, in which case we say that A is a total quotient ring. We always have $A \subset A^*$, and an induced embedding $\mathrm{Spec}\, A^* \subset \mathrm{Spec}\, A$, where the image is the set of primes of A consisting entirely of zero divisors, and always contains $\mathrm{Min}\,\mathrm{Spec}\, A$.

PROPOSITION 8. *If A is a total quotient ring and A is* TIC *then A is regular. An arbitrary ring A is* TIC *if and only if A^* is* TIC *and A is integrally closed in A^*, in which case* $\mathrm{Spec}\, A^*$ *may be identified with* $\mathrm{Min}\,\mathrm{Spec}\, A$ *under the induced inclusion* $\mathrm{Spec}\, A^* \subset \mathrm{Spec}\, A$.

Proof. To prove the first statement, let $a \in A$ be given, and let e be the idempotent generator of $\mathrm{Ann}_A\, a$. $e + a$ is not a zero divisor in A, for $b(e+a)=0 \Rightarrow be=-ba \Rightarrow (be)^2=(be)(-ba)=-b^2ea=0 \Rightarrow be=0 \Rightarrow b=b'e$ for some $b' \Rightarrow b=b'e=b'ee=be=0$, as required. Then $e+a$ has an inverse u, and $a=au(e+a)=a^2u+uea=a^2u$.

We proceed to the second statement. If A^* is TIC and A is integrally closed in A^*, then A is certainly TIC. On the other hand, if A is TIC, A^* is as well, by Proposition 7, and the integral closure of A in A^* will be a tight integral extension of A and hence must equal A. In this situation, the image of Spec A^* in Spec A is a *Hausdorff* space containing Min Spec A; hence it must equal Min Spec A.

We note that if A is TIC and S is any multiplicative system in A then $S^{-1}(A^*)=(S^{-1}A)^*$. For clearly, $S^{-1}(A^*) \subset (S^{-1}A)^*$; but, $S^{-1}(A^*)$ is a localization of a regular ring, hence regular, hence a total quotient ring already. With this observation, we can easily prove:

PROPOSITION 9. *Let A be a* TIC *ring and S a multiplicative system in A. Then the following conditions are equivalent.*

(1) $S^{-1}A$ *is* TIC.

(2) $S^{-1}A^*$ *is* TIC.

(3) *The set of minimal primes of A which fail to meet S (which is always a closed subspace of* Min Spec A) *is extremally disconnected.*

Proof. Let Y be the set of primes described in (3). We have obvious homeomorphisms $Y \approx \text{Min Spec } S^{-1}A \approx \text{Spec } S^{-1}A^*$. Y is closed because it is a quasicompact subspace of a Hausdorff space. Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) is clear, while (2) $\Rightarrow$ (1) because A integrally closed in $A^* \Rightarrow S^{-1}A$ integrally closed in $S^{-1}A^*$. (That (3) $\Rightarrow$ (1) is also easy to see using Proposition 4 and Corollary 2 to Theorem 1.)

We conclude by considering some consequences of this theory of localization. We note, for example, that by Proposition 6 the finite direct product (direct sum) decompositions of a TIC ring are in one-to-one correspondence with those of its total quotient ring. We also note that a TIC ring always has localizations which are not TIC unless it is a finite product of TIC domains. In fact, by virtue of Proposition 9 and the preceding remark, we can pass to the case where A is regular, TIC, and Spec A is an infinite Boolean space. Then $X=\text{Spec } A$ has a closed subspace Y which is not extremally disconnected (cf. the second paragraph of § 2). Let S be the multiplicative system of elements of A not vanishing anywhere on Y, and let I the ideal of elements of A vanishing everywhere on Y. Then it is easy to see that $I=\text{Ker}\,(A \to S^{-1}A)$, and that the induced homomorphism $A/I \to S^{-1}A$ is actually an isomorphism, so that $\text{Spec } S^{-1}A \approx \text{Spec } A/I \approx Y$. Since

Y is not extremally disconnected, $S^{-1}A$ (and A/I) are not TIC. These are the examples mentioned in § 1.

To be completely specific, we first note that if X is any extremally disconnected Boolean space, and Ω is any algebraically closed field, then the ring A of locally constant functions from X to Ω is regular, TIC, and, in fact Spec A may be identified with X. (See the proof of Theorem 6 (c) in [6, § 7].) Let $\underset{\sim}{N}$ be the space of nonnegative integers in the discrete topology, let $X = \beta\underset{\sim}{N}$, let Ω be the field of complex numbers, and let $Y = \beta\underset{\sim}{N} - \underset{\sim}{N}$. (See § 2.)

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