RECURRENCE TIMES FOR THE EHRENFEST MODEL

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1. Introduction and summary. In 1907, P. and T. Ehrenfest [1] used a simple urn scheme as a pedagogic device to elucidate some apparent paradoxes in thermodynamic theory. Their model undergoes fluctuations intuitively related to fluctuations about equilibrium of certain thermodynamic systems. In view of an apparent discord among physicists [6, pp. 139-145] we shall not try to force an analogy with entropy.

The original Ehrenfest scheme was defined as follows. Initially, $2N$ balls are divided in an arbitrary manner between two urns, 1 and 2, the balls being numbered from 1 to $2N$. An integer between 1 and $2N$ is selected at random, each such integer having probability $(2N)^{-1}$, and the ball with the number selected is transferred from one urn to the other. The process is repeated any number of times. If n_1 and n_2 are the numbers of balls in urns 1 and 2 respectively before a transfer, it is clear that the probability is $n_1/(2N)$ that the transfer is from urn 1 to urn 2 and $n_2/(2N)$ that it is in the contrary direction.

Let $x'(n)$ be the number of balls in urn 1 after n transfers, and let $L'_{j,k}$ be be the smallest integer m such that $x'(m) = k$, given that $x'(0) = j$. If $k = j$, we call $L'_{k,k}$ the *recurrence time* for the state k. If $k \neq j$, we call $L'_{j,k}$ the *first-passage time* from j to k. The distribution of $x'(n)$, known classically, was derived by Kac [5] as an example of the use of matrix methods. Kac then found the mean and variance of $L'_{j,k}$, attributing some of his methods to Uhlenbeck. Friedman [4] found the moment-generating function for $x'(n)$ (for the Ehrenfest and more general models) by solving a difference-differential equation.

Instead of the original Ehrenfest model, we shall discuss a modified scheme with a continuous time parameter, which was apparently first suggested by A. J. F. Siegert [9]. In this scheme there are two urns and $2N$ balls initially divided between them arbitrarily. Each ball acts, independently of all the others, as follows: there is a probability of $(1/2)\,dt + o(dt)$ that the ball changes urns between t and $t + dt$, and a probability of $1 - [(1/2)\,dt + o(dt)]$ that the ball remains

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in place between t and $t + dt$. Standard reasoning then shows that the total probability of a change by some ball between t and $t + dt$ is $Ndt + o(dt)$, and that consequently the probability density for the length of time between transfers is $Ne^{-Nt}\, dt$. When a transfer occurs, it is readily seen that the probabilities that it is from urn 1 to urn 2 or from urn 2 to urn 1, respectively, depend on the relative number of balls in the two urns exactly as for the original Ehrenfest model. Thus we see that the present scheme is essentially the original Ehrenfest scheme where the drawings are made at random times. As we shall see, the time-continuous scheme is easier to handle analytically.

Let $x(t)$ be the number of balls in urn 1 at time t; we shall sometimes speak of this number as the *state* of the system. Then $x(t)$ is a random function which can take integer values from 0 to $2N$; $x(t)$ executes a random walk—with a "restoring force"—about the equilibrium value N. It is clear that the random walk is a Markov process.

Let $L_{j,k}$, $j \neq k$, be the *first-passage time* from state j to state k; that is, $L_{j,k}$ is the infimum of t such that $x(t) = k$, given that $x(0) = j$. Let $L_{k,k}$ be the *recurrence time* for the state k; that is, $L_{k,k}$ is the infimum of t such that $x(t) = k$ and $x(\tau) \neq k$ for $0 < \tau < t$, given that $x(0) = k$. We shall discuss the probability distributions of $L_{j,k}$ and $L_{k,k}$.

The probability distribution of $L_{j,k}$ depends, of course, on the size of the model (that is, on the number N). When it is necessary to emphasize this dependence we shall sometimes employ the notation $L_{j,k}^{(N)}$ in place of $L_{j,k}$.

We shall use the notation $P(A)$ for the probability of the event A; $P(A \mid B)$ for the conditional probability of A, given B; $E(X)$ for the mean, or expected value, of the random variable X. By the *distribution* of a random variable X we mean the function (of say u) given by $P(X \leq u)$. The statement that a sequence of distributions converges to a distribution $F(u)$ will mean convergence at all continuity points of $F(u)$.

There are two limiting situations in which the distribution of $L_{j,k}$ is of interest.

(a) Consider a simple thermodynamic system such as an ideal gas in a container. Let us think of the container as consisting of two halves which, however, are not separated by a partition. Suppose that initially the molecules are spread in a rather uniform manner through the two halves of the container. According to classical kinetic theory, if we wait long enough, a time will come, in general, when all the molecules are in one half of the container. Such events, where the fraction of molecules in one half of the container is appreciably different from

(1/2), are evidently enormously rare if the number of molecules is large. Correspondingly, we should like to show that the random variable $L_{N,k}$, where $|k - N|$ is of the order of magnitude of N, is very large with high probability when N is large. Now the mean of $L_{N,k}$ is extremely large when N is large. However, as Kac has observed, the standard deviation is of the same order of magnitude as the mean. Thus we cannot conclude from the values of the first two moments that $L_{N,k}$ is large with high probability. We shall show, however, that *the distribution of* $L_{N,k}/E(L_{N,k})$ *converges to* $1 - e^{-u}$ *as* $N \longrightarrow \infty$ *provided* k/N *remains less than some fixed number* $\lambda_1 < 1$ (Theorem 1).

The situation with respect to $L_{k,k}$, where again $k/N < \lambda_1 < 1$, is somewhat different. If k/N is appreciably different from 0, a very short recurrence time is not improbable. The distribution of $L_{k,k}/E(L_{k,k})$ has for large N a "lump" of probability of magnitude k/N concentrated near 0, the remainder of the distribution being exponential (Theorem 2).

(b) In the theory of the Brownian motion and elsewhere in physics and statistics an important role is played by the stationary Gaussian Markov process $z(t)$ which we scale so that

$$E[z(t)] = 0, \qquad E[z(t)]^2 = 1/2 .$$

This process is defined by the requirement that the joint distribution of $z(t_1), \cdots, z(t_m)$ for any distinct numbers $t_1, \cdots, t_m$ is Gaussian and dependent only on the differences $t_i - t_j$ and that the autocorrelation function is given by

$$E[z(s)\ z(t+s)] = (1/2)\, e^{-|t|} .$$

If N is large, the $z(t)$ process, under the conditional hypothesis that $z(0)$ has an appropriate value, is approximated by the process

$$z_N(t) = \frac{x(t) - N}{N^{1/2}}$$

in a sense described in Section 6. (It should be remembered that $x(t)$ depends on N.) By considering the distribution of $L_{N,k}$ where $(k - N)/N^{1/2} \longrightarrow -\xi_0 < 0$, $N \longrightarrow \infty$, we obtain in Theorem 3 the Laplace transform of the distribution of L, the first time at which $z(t) = -\xi_0$, given $z(0) = 0$. This result is not new, having been obtained by Siegert [10] and by Darling (unpublished). However, the present method of derivation seems instructive.

Results similar to those given under (a) and (b) are obtained for the random variable $L^*_{N,k}$, the first time $|x(t) - N| = N - k$, given $x(0) = N$.

2. The mathematical model. Suppose that there are initially j balls in urn 1 and $2N - j$ in urn 2. Associate with the ith ball a random function $x_i(t)$ defined as follows: $x_i(t)$ is 1 if the ith ball is in urn 1 at time t, and 0 otherwise. From the elementary theory of Markov processes (see, for example, Kolmogorov [7]), we have

$$(1) \qquad P[x_i(t) = x_i(0)] = \frac{1 + e^{-t}}{2},$$

$$P[x_i(t) = 1 - x_i(0)] = \frac{1 - e^{-t}}{2}.$$

We may define the generating function of $x_i(t)$ by

$$P[x_i(t) = 0] + sP[x_i(t) = 1].$$

Then the generating function of $x_i(t)$ is, from (1),

$$(1/2)[1 + e^{-t} + (1 - e^{-t})s],$$

or

$$(1/2)[1 - e^{-t} + (1 + e^{-t})s],$$

according as $x_i(0)$ is 0 or 1. Since the quantities $x_i(t)$, $i = 1, \cdots, 2N$, are independent, the generating function for $x(t) = \Sigma\, x_i(t)$ is

$$(2) \qquad \sum_{k=0}^{2N} P[x(t) = k \,|\, x(0) = j]s^k$$

$$= 2^{-2N}[1 - e^{-t} + (1 + e^{-t})s]^j[1 + e^{-t} + (1 - e^{-t})s]^{2N-j}$$

$$= \sum_{k=0}^{2N} Q_{j,k}(t)s^k,$$

where we have introduced the notation $Q_{j,k}(t)$ for $P[x(t) = k \,|\, x(0) = j]$. Formula (2) was given by Siegert [9 ; 11].

Because of the simple nature of the process under consideration it is easy to show that $L_{j,k}$ and $L_{k,k}$ are (measurable) random variables with absolutely continuous distributions. We omit the proof. We let $P_{j,k}(u)$ be the probability density of $L_{j,k}$,

$$\int_0^u P_{j,k}(y)\,dy = P[L_{j,k} \le u]\,. \tag{3}$$

Define

$$Q_k = \lim_{t\to\infty} Q_{j,k}(t) = \binom{2N}{k}/2^{2N}\,, \tag{4}$$

$$m_k = 1/[(N-k)Q_k]\,, \qquad k < N\,. \tag{5}$$

It is convenient to notice that, as $N \longrightarrow \infty$,

$$m_k = \frac{[\lambda^\lambda(2-\lambda)^{2-\lambda}]^N}{1-\lambda}\left[\frac{\pi\lambda(2-\lambda)}{N}\right]^{1/2}\left[1+\frac{1}{\lambda}O\left(\frac{1}{N}\right)\right], \qquad k \ne 0\,, \tag{6}$$

where we have put $\lambda = k/N$, and $O(1/N)$ is independent of λ.

The quantities $L_{j,k}$, Q_k, and so on, depend on the size of the model; when it is necessary to emphasize this dependence we shall write $L_{j,k}^{(N)}$, $Q_k^{(N)}$, and so on.

3. Distribution of $L_{j,k}$, $j \ne k$. In this section we consider the distribution of the first-passage time from state j to state k for large N, where $|j-k|$ is of the order of magnitude of N. As far as the limiting distributions are concerned, we can restrict ourselves without loss of generality to consideration of $L_{N,k}$, $k < N$. For example, if $j > N > k$ then we can write

$$L_{j,k} = L_{j,N} + L_{N,k}\,.$$

The first-passage time from j to N, representing movement toward equilibrium, is negligible relative to $L_{N,k}$ and does not affect the asymptotic result. On the other hand if $N > j > k$, we have

$$L_{j,k} = L_{N,k} - L_{N,j}$$

and it is not difficult to show that $L_{N,j}$ is negligible compared with $L_{N,k}$.

If the first passage to the state k occurs at time τ, the probability that the state at time t is again k is $Q_{k,k}(t-\tau)$. We have therefore

$$Q_{j,k}(t) = \int_0^t P_{j,k}(\tau)\,Q_{k,k}(t-\tau)\,d\tau\,, \qquad j \ne k\,. \tag{7}$$

Formula (7) is the continuous counterpart of a formula long used for discrete processes and recently exploited by Feller [2]. Taking Laplace transforms of both sides of (7) we have

$$(8)\qquad \int_0^\infty P_{j,k}(t)\, e^{-\sigma t}\, dt = \frac{\int_0^\infty Q_{j,k}(t)\, e^{-\sigma t}\, dt}{\int_0^\infty Q_{k,k}(t)\, e^{-\sigma t}\, dt}, \qquad \Re(\sigma) > 0,$$

Since the quantities $Q_{j,k}(t)$ are polynomials in e^{-t}, as we observe from (2), both the numerator and the denominator in (8) have a simple pole at $\sigma = 0$, and their quotient is therefore analytic in the circle $|\sigma| < 1$.

For simplicity denote $L_{N,k}^{(N)}$ by $L_k^{(N)}$. We have the following result.

THEOREM 1. *The distribution function of* $L_k^{(N)}/m_k^{(N)}$ *converges to* $1 - e^{-u}$, $u \geq 0$, *as* $N \to \infty$, *provided* $k/N \leq \lambda_1 < 1$, *the convergence being uniform in* k *and* u.

The proof will bring out the fact that likewise

$$(9)\qquad E[L_k^{(N)}]/m_k^{(N)} \longrightarrow 1, \quad N \longrightarrow \infty, \quad k/N \leq \lambda_1 .$$

Theorem 1 will follow from this lemma:

LEMMA 1. *For the complex variable* σ, *let*

$$\phi_k^{(N)}(\sigma) = \int_0^\infty P_{N,k}^{(N)}(u) \exp[-\sigma u/m_k^{(N)}]\, du .$$

Further, let $\{k(N)\}$ *be a sequence of nonnegative integers such that* $k(N)/N \longrightarrow \lambda_0 \leq \lambda_1 < 1$ *as* $N \longrightarrow \infty$. *Then* (the convergence being bounded and uniform provided $|\sigma| < \sigma_0 < 1$),

$$(10)\qquad \lim_{N\to\infty} \phi_{k(N)}^{(N)}(\sigma) = \frac{1}{1+\sigma}, \qquad |\sigma| < 1 .$$

Proof of Theorem 1. The function $\phi_k^{(N)}(-\sigma)$ is the moment-generating function" of the quantity $L_k^{(N)}/m_k^{(N)}$. Lemma 1 then implies, as is well known, that

$$(11)\qquad \lim_{N\to\infty} P[L_{k(N)}^{(N)}/m_{k(N)}^{(N)} \leq u] = 1 - e^{-u}$$

uniformly for $u \geq 0$ provided $k(N)/N \longrightarrow \lambda_0$. Lemma 1 also implies, since we have convergence in a complex neighborhood of $\sigma = 0$, that

$$\lim_{N\to\infty} E[L_{k(N)}^{(N)}]/m_{k(N)} = 1 ,$$

so that (11) is still true if we replace $m_{k(N)}^{(N)}$ by $E\,[L_{k(N)}^{(N)}]$.

Now if Theorem 1 were not true then an $\epsilon > 0$ and a sequence $\{h(N)\}, h(N)/N \leq \lambda_1$, would exist such that for infinitely many integers N we would have

$$\sup_{0\leq u<\infty} \left|1 - e^{-u} - P[L_{h(N)}^{(N)}/m_{h(N)}^{(N)} \leq u]\right| > \epsilon .$$

Extracting a convergent subsequence from $\{h(N)/N\}$, we are led to a contradiction of (11).

Proof of Lemma 1. The proof of Lemma 1, which is somewhat indirect, proceeds as follows. We can obtain an expression for $\phi_k^{(N)}(\sigma)$ by substituting $\sigma/m_k^{(N)}$ for σ in (8), obtaining

$$\phi_k^{(N)}(\sigma) = \frac{\int_0^\infty Q_{N,k}(t)\, e^{-\sigma t/m_k}\, dt}{\int_0^\infty Q_{k,k}(t)\, e^{-\sigma t/m_k}\, dt} = \frac{J_1}{J_2} . \tag{12}$$

We can obtain an asymptotic estimate of J_1 as we shall see later. However, a direct estimate of J_2 appears difficult to obtain. We shall therefore resort to another expression for $\phi_k^{(N)}(\sigma)$ which is easier to estimate. Having estimates for $\phi_k^{(N)}(\sigma)$ and for J_1, we can get an estimate of J_2, which will be necessary for Theorem 2.

Since a direct proof of Lemma 1 is easy if all terms in the sequence $\{k(N)\}$ are 0 we can suppose $k > 0$. If $0 < k < N$ we have, from elementary reasoning, the important relation

$$L_{N,0} = L_{N,k} + L_{k,0} . \tag{13}$$

On account of the Markovian nature of the process, $L_{N,k}$ and $L_{k,0}$ are independent random variables and the Laplace transform of the distribution of their sum is the product of the Laplace transforms of their individual distributions. Therefore, using (8) and (13), we have

$$E(e^{-sL_{N,0}}) = E(e^{-sL_{N,k}})\, E(e^{-sL_{k,0}}) , \tag{14}$$

or

$$(15) \qquad \phi_k^{(N)}(\sigma) = \int_0^\infty p_{N,k}(t)\, e^{-\sigma t/m_k}\, dt$$

$$= \frac{\int_0^\infty p_{N,0}(t)\, e^{-\sigma t/m_k}\, dt}{\int_0^\infty p_{k,0}(t)\, e^{-\sigma t/m_k}\, dt} = \frac{\int_0^\infty Q_{N,0}(t)\, e^{-\sigma t/m_k}\, dt}{\int_0^\infty Q_{k,0}(t)\, e^{-\sigma t/m_k}\, dt}$$

$$= \frac{\int_0^\infty (1-e^{-2t})^N\, e^{-\sigma t/m_k}\, dt}{\int_0^\infty (1-e^{-t})^k (1+e^{-t})^{2N-k}\, e^{-\sigma t/m_k}\, dt}$$

The advantage of (15) over (12) is that $Q_{k,0}(t)$ is a simpler function than $Q_{k,k}(t)$.

The numerator of the last fraction in (15) is $(1/2)B\,[N+1,\,(1/2)\sigma/m_k]$. The denominator, with the substitution $e^{-t} = y$, becomes

$$(16) \qquad I = \int_0^1 (1-y)^k (1+y)^{2N-k}\, y^{(\sigma/m_k)-1}\, dy\,.$$

We now have to estimate I as $N \longrightarrow \infty$ under the hypothesis $k/N \longrightarrow \lambda_0 < 1$. [We shall write simply k for $k(N)$.] We shall restrict σ to the circumference of a circle, say $|\sigma| = (1/2)$, since it is clearly sufficient to prove Lemma 1 for such a circle. Write

$$I = \int_0^\epsilon + \int_\epsilon^1 = I_1 + I_2\,, \qquad 0 < \epsilon < 1 - \lambda_0 \leq 1\,.$$

Making use of (6) and the fact that $(1-y)^k\,(1+y)^{2N-k}$ increases to a maximum at $y = 1 - k/N$ and then decreases, $1 - k/N$ being larger than ϵ for sufficiently large N, we have

$$(17) \qquad I_1 = \int_0^{1/N^2} + \int_{1/N^2}^\epsilon$$

$$= \frac{m_k}{\sigma}\,[1 + o(1)] + \int_{1/N^2}^\epsilon (1-y)^k (1+y)^{2N-k}\, y^{(\sigma/m_k)-1}\, dy$$

$$= \frac{m_k}{\sigma}\,[1 + o(1)] + O[(1-\epsilon)^k (1+\epsilon)^{2N-k} \log N]$$

$$= \frac{m_k}{\sigma}\,[1 + o(1)] + o(m_k)\,,$$

where $o(\ )$ is independent of σ for $|\sigma| = 1/2$.

To estimate I_2 we distinguish the cases $\lambda_0 > 0$ and $\lambda_0 = 0$. If $\lambda_0 > 0$, then I_2 can be estimated using the method of Laplace; see [8, p. 77]. We obtain then, setting $k/N = \lambda$, (see (6)),

$$I_2 = [\lambda^\lambda (2-\lambda)^{2-\lambda}]^N \left[\frac{\pi\lambda(2-\lambda)}{N(1-\lambda)}\right]^{1/2} [1 + o(1)] = m_k[1 + o(1)]\,. \tag{18}$$

Going back to (15), we obtain (10) from (17) and (18), since

$$(1/2)B[N+1,\ (1/2)\sigma/m_k] = (m_k/\sigma)[1 + o(1)]\,.$$

This completes the proof of Lemma 1 for the case $\lambda_0 > 0$. If $\lambda_0 = 0$, the integral I_2 can be estimated by making a change of the variable of integration which shows the integral to be asymptotically equivalent to a Beta function. We need not enter into details.

4. Distribution of $L_{k,k}$. We shall establish the following result.

THEOREM 2. *Assume $\lambda = k/N \le \lambda_1 < 1$, and put*

$$F_\lambda(u) = \lambda + (1-\lambda)\,[1 - e^{-(1-\lambda)u}]\,, \qquad u \ge 0\,.$$

Then for every $b > 0$ we have

$$\lim_{N\to\infty} \sup_{b\le u<\infty} |P[L_{k,k}^{(N)}/E(L_{k,k}^{(N)}) \le u] - F_\lambda(u)| = 0$$

uniformly in k.

Proof. As in the case of Theorem 1, it is sufficient to prove that the Laplace transform of the distribution of $NQ_k^{(N)} L_{k,k}^{(N)}$ approaches

$$\lambda_0 + (1-\lambda_0)^2/(1+\sigma-\lambda_0)$$

provided $k/N = \lambda \longrightarrow \lambda_0 < 1$. (We know from the general theory of Markov processes that

$$EL_{k,k}^{(N)} = 1/(NQ_k^{(N)})\,,$$

see Feller, [3, p. 325].)

The relation which replaces (7) when $j = k$ is

$$Q_{k,k}(t) = e^{-Nt} + \int_0^t P_{k,k}(\tau) Q_{k,k}(t-\tau)\, d\tau , \tag{19}$$

the term e^{-Nt} in (19) being the probability that the system remains in state k the entire time from 0 to t. From (19), we have

$$\begin{aligned} \int_0^\infty P_{k,k}(t)\, e^{-\sigma Q_k N t}\, dt \\ = 1 - \frac{1}{N(1+\sigma Q_k) \int_0^\infty Q_{k,k}(t)\, e^{-\sigma Q_k N t}\, dt} , \qquad |\sigma| = \frac{1}{2} . \end{aligned} \tag{20}$$

If we equate the right side of (8), with $j = N$ and with σ replaced by $\sigma N Q_k$, to the right side of (15) with σ/m_k replaced by $\sigma N Q_k$, we obtain

$$\frac{\int_0^\infty Q_{N,k}(t)\, e^{-\sigma N Q_k t}\, dt}{\int_0^\infty Q_{k,k}(t)\, e^{-\sigma N Q_k t}\, dt} = \phi_k^{(N)}(\sigma N Q_k m_k) = \phi_k^{(N)}\left(\frac{\sigma}{1-\lambda}\right),$$

or

$$\begin{aligned} \int_0^\infty Q_{k,k}(t)\, e^{-\sigma N Q_k t}\, dt &= \frac{\int_0^\infty Q_{N,k}(t)\, e^{-\sigma N Q_k t}\, dt}{\phi_k^{(N)} \sigma/(1-\lambda)} \\ &= \frac{I_3}{\phi_k^{(N)} \sigma/(1-\lambda)} . \end{aligned} \tag{21}$$

To estimate I_3, which is the numerator of (12) with σ replaced by $\sigma/(1-\lambda)$, we need two lemmas.

LEMMA 2. *Given $\epsilon > 0$, let*

$$t_N(\epsilon) = \sup\left\{ t : \max_{0 \le r \le 2N} \frac{|Q_{N,r}^{(N)}(t) - Q_r^{(N)}|}{Q_r^{(N)}} \ge \epsilon \right\} .$$

Then

(22) $$t_N(\epsilon) = O(\log N)\,, \qquad N \longrightarrow \infty.$$

Proof. By (2), $Q_{N,r}(t)$ is the coefficient of s^r in

$$(1-e^{-2t})^N(1+zs+s^2)^N/2^{2N}\,,$$

where $z = z(t) = 2(1+e^{-2t})/(1-e^{-2t})$. Since for large N the root of the equation

$$(1-e^{-2t})^N = 1-\delta\,, \qquad 0<\delta<1\,, \qquad \delta \text{ fixed}\,,$$

is approximately $t \sim (1/2)\log N$, it suffices to prove Lemma 2 for the quantities

$$t_N^*(\epsilon) = \sup\left\{t : \max_{0\le r\le 2N} \frac{|c_r^{(N)}(t) - c_r^{(N)}|}{c_r^{(N)}} \ge \epsilon\right\}\,,$$

where we have set

(23) $$(1+zs+s^2)^N = \Sigma c_r^{(N)}(t)\, s^r\,,$$

$$c_r^{(N)} = c_r^{(N)}(\infty) = 2^{2N}Q_r^{(N)}\,.$$

Suppose $\epsilon > 0$ is given. Choose an arbitrary $\alpha > 1$. Let $\epsilon_1 < \epsilon$ be a positive number and define

(24) $$\epsilon_{N+1} = \epsilon_N(1+1/N^{\alpha})\,, \qquad N = 1,2,\cdots.$$

Note that $\{\epsilon_N\}$ is a bounded increasing sequence. We select ϵ_1 small enough so that $\epsilon_N < \epsilon$, for all N. Now define a sequence $\bar{t}_1 \le \bar{t}_2 \le \cdots$ as follows: $\bar{t}_1 = t_1^*(\epsilon_1)$; $\bar{t}_{N+1}$ for $N \ge 1$ is the maximum of $\bar{t}_N$ and the positive root of

(25) $$(1/3)[z(t)-2](1+\epsilon_N)/\epsilon_N = 1/N^{\alpha}.$$

[Note that $z(t)$ is monotone decreasing.] It is then clear from (25) that

(26) $$\bar{t}_N \sim (1/2)\alpha\log N\,, \qquad N \longrightarrow \infty.$$

We now wish to show inductively that

(27) $$\frac{|c_r^{(N)}(t) - c_r^{(N)}|}{c_r^{(N)}} \le \epsilon_N \qquad \text{for} \quad t \ge \bar{t}_N\,, \qquad N = 1,2,\cdots.$$

Clearly (27) holds for $N = 1$, since $\epsilon_1 < \epsilon$ and $\bar{t}_1 = t_1^*$. Suppose that (27) is true for a general N. From (23), we have

$$c_r^{(N+1)}(t) = c_r^{(N)}(t) + zc_{r-1}^{(N)}(t) + c_{r-2}^{(N)}(t)\,, \tag{28}$$
$$c_r^{(N+1)} = c_r^{(N)} + 2c_{r-1}^{(N)} + c_{r-2}^{(N)}\,.$$

Using (27), (28), and the fact that $c_{r-1}^{(N)}/c_r^{(N+1)} \leq 1/3$, we have for $t \geq \bar{t}_N$,

$$\frac{|c_r^{(N+1)}(t) - c_r^{(N+1)}|}{c_r^{(N+1)}} \leq \epsilon_N\left[1 + \frac{1}{3}\,|z(t) - 2|\left(\frac{1+\epsilon_N}{\epsilon_N}\right)\right]. \tag{29}$$

From the definition of ϵ_N it is then clear that (27) holds with N replaced by $N + 1$. Then $t > \bar{t}_N$ implies that the left side of (27) is less than ϵ. Use of (26) now completes the proof of Lemma 2.

LEMMA 3. *Assume $k/N \leq \lambda_1$. Then*
$$\max_{0\leq t\leq\infty} Q_{N,k}^{(N)}(t) < \exp[-3(1-\lambda_1)^2N/5]\,.$$

Proof. Lemma 3 is an immediate consequence of a result of S. Bernstein on sums of independent random variables; see Uspensky [12, p. 205]. To apply Bernstein's result, we consider the $2N$ balls as consisting of N pairs, each pair having initially one ball in urn 1 and one in urn 2, letting Uspensky's random varivariable x_i be the number of balls from the ith pair in urn 1, minus 1, at time t. Now

$$Q_{N,k}^{(N)}(t) = P\left[\sum_{i=1}^{N} x_i = k - N\right] < P\left[\sum_{i=1}^{N} x_i \leq k - N\right],$$

and the applicability of Bernstein's result is obvious.

We now return to the proof of Theorem 2. To estimate the integral I_3 defined in (21), write

$$I_3 = \int_0^\infty (Q_{N,k}(t) - Q_k)\, e^{-\sigma NQ_k t}\, dt + 1/(N\sigma)\,. \tag{30}$$

Write the integral on the right side of (30) as

$$\int_0^{t_N(\epsilon)} + \int_{t_N(\epsilon)}^\infty = I_3' + I_3''$$

for an arbitrary $\epsilon > 0$, where $t_N(\epsilon)$ is defined in Lemma 2. Using Lemmas 2 and 3, we have

$$
(31) \qquad |I_3''| < \epsilon Q_k \int_{t_N(\epsilon)}^{\infty} e^{-\sigma N Q_k t}\, dt < \epsilon/(N\sigma)\,,
$$

$$
|I_3'| = O\{\log N \exp[-3(1-\lambda_1)^2 N/5]\}\,.
$$

Thus $I_3 \sim 1/(N\sigma)$. Putting this estimate in (21) and recalling from Theorem 1 that

$$
\phi_k^{(N)}\left(\frac{\sigma}{1-\lambda}\right) \longrightarrow \frac{1}{1+\sigma/(1-\lambda_0)}\,,
$$

we get the desired result from (20).

5. Intuitive interpretation. Theorem 1 means intuitively that if we take m_k as our time unit, the attainment of the state k is an occurrence of the "chance" type; that is, the probability of attaining k during a given time interval is almost independent of the past history of the process. This interpretation suggests that Theorem 1 should be true for more general types of processes with a central tendency.

Theorem 2 seems to mean that if the initial state is k there is a probability λ of returning to k before leaving its immediate neighborhood; there is a probability $1-\lambda$ of getting completely away from the neighborhood before the first return; in this case the first return has the distribution of first passage times given in Theorem 1.

6. Application to stationary Gaussian Markov processes. In Theorems 1 and 2 we considered rare or microscopic fluctuations of $x(t)$. But if N is large $x(t)$ will for the most part deviate little from its mean value N, and to consider the ordinary fluctuations of $x(t)$ we consider

$$
z_N(t) = [x(t) - N]/N^{1/2}\,.
$$

Let $t_1, \cdots, t_m$ be a fixed set of nonnegative numbers. The joint distribution of $z_N(t_1), \cdots, z_N(t_m)$, given $z_N(0) = 0$, approaches, as $N \longrightarrow \infty$, the joint distribution of $z(t_1), \cdots, z(t_m)$, given $z(0) = 0$, where $z(t)$ is the stationary Gaussian Markov process with

$$
E[z(t)] = 0\,, \qquad E[z(s)\; z(s+t)] = (1/2)\, e^{-|t|}\,.
$$

Define the random variable L to be the smallest value of t for which $z(t) = -\xi_0 < 0$, given $z(0) = 0$. It is intuitively clear that the distribution of L is given by the limiting distribution of $L_{N,k}$ as $N \longrightarrow \infty$ provided we let

$$(k - N)/N^{1/2} \longrightarrow -\xi_0 . \tag{32}$$

A rigorous proof of this statement is not difficult but we omit it.

To find the limiting Laplace transform for the distribution of $L_{N,k}$ under the hypothesis (32), we consider (15) with $\sigma > 0$ in place of σ/m_k, and let $k = N - \xi N^{1/2}$. The substitution $e^{-t} = y/N^{1/2}$ puts the denominator in the form

$$\frac{1}{N^{\sigma/2}} \left[\int_0^{N^\alpha} + \int_{N^\alpha}^{N^{1/2}} \left(1 - \frac{y}{N^{1/2}}\right)^{N-\xi N^{1/2}} \times \left(1 + \frac{y}{N^{1/2}}\right)^{N+\xi N^{1/2}} y^{\sigma-1}\, dy \right], \tag{33}$$

where α is an arbitrary number between 0 and $1/6$. If $0 < y < N^\alpha$, then

$$\left(1 - \frac{y}{N^{1/2}}\right)^{N-\xi N^{1/2}} \left(1 + \frac{y}{N^{1/2}}\right)^{N+\xi N^{1/2}} = e^{-y^2+2\xi y}\,[1 + O(N^{-1/2+3\alpha})] .$$

Hence,

$$\int_0^{N^\alpha} \longrightarrow \int_0^\infty e^{-y^2+2\xi_0 y}\, y^{\sigma-1}\, dy .$$

The second integral inside the bracket in (33) goes to 0 as $N \longrightarrow \infty$.

The numerator of (15), with σ in place of σ/m_k, is

$$(1/2)\, B(N + 1, \sigma/2)\, N^{\sigma/2}.$$

We thus have the following result.

THEOREM 3. *The Laplace transform of the distribution of L is given by*

$$\frac{(1/2)\,\Gamma(\sigma/2)}{\int_0^\infty e^{-y^2+2\xi_0 y}\, y^{\sigma-1}\, dy} . \tag{34}$$

Formula (34) was obtained by Siegert and by Darling through direct consideration of the $z(t)$ process. It is interesting to notice that the present procedure utilizes (13) which has no counterpart for the $z(t)$ process.

7. Two-sided limits. Let $L^{*\prime}_{N,k}$, $N \geq k$, be the first time $|x(t) - N| = N - k$, given $x(0) = N$. Let L^* be the first time $|z(t)| = \xi_0 > 0$, given $z(0) = 0$. Arguments similar to those used for Theorems 1 and 3 give the following two results.

THEOREM 1a. *Under the conditions of Theorem* 1 *the limiting distribution of* $L^*_{N,k}/m_k$ *is* $1 - e^{-2u}$, $u \geq 0$.

THEOREM 3a. *The distribution of* L^* *has the Laplace transform*

$$\frac{(1/2)\Gamma(\sigma/2)}{\int_0^\infty e^{-y^2} y^{\sigma-1} \cosh(2\xi_0 y)\, dy}.$$

8. Added in proof. An argument has been found which rigorizes the remarks of Section 5 and gives a proof of Theorems 1 and 2 for more general processes.

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