TWO THEOREMS ON METRIC SPACES

HSIEN-CHUNG WANG

1. Introduction. Let E be a metric space with distance function d. The space E is called *two-point homogeneous* if given any four points a, a', b, b' with $d(a,a') = d(b,b')$, there exists an isometry of E carrying a, a' to b, b', respectively. In a recent paper [7], the author has determined all the compact and connected two-point homogeneous spaces. It is the aim of the present note to discuss the noncompact case, and prove a conjecture of Busemann which can be regarded also as a sharpening of a theorem of Birkhoff [1]. The results concerning the noncompact two-point homogeneous spaces are not as satisfactory as the results for the compact case; we have to assume certain conditions on the metric.

By a segment in a metric space E, we shall mean an isometric image of a closed interval with the usual metric. A metric space will be said to have the property (L) if given a point p, there exists a neighborhood W of p so that each point x $(\neq p)$ of W can be joined to p by at most one segment in E. The following theorems will be proved:

THEOREM 1. *Let E be a finite-dimensional, finitely compact, convex metric space with property (L). If E is two-point homogeneous, then E is homeomorphic with a manifold.*

THEOREM 2. *Let E be a metric space with all the properties mentioned in Theorem 1. If, moreover,* $\dim E$ *is odd, then E is congruent either to the euclidean space, the hyperbolic space, the elliptic space, or the spherical space.*

Our Theorem 2 justifies the conjecture of Busemann [2, p. 233] that a two-point homogeneous three dimensional S.L. space [2, p. 78] is either elliptic, hyperbolic, or euclidean. It is to be noted that Theorem 2 no longer holds if $\dim E$ is even and greater than two. The complex elliptic spaces [7] and the hyperbolic Hermitian spaces[1] [2, p. 192] serve as counter examples.

Received May 25, 1951.

[1]These spaces were first introduced by H. Poincaré, and then discussed by G. Fubini and E. Study. Following E. Cartan, we call these spaces the hyperbolic Hermitian spaces.

Pacific J. Math. 1(1951), 473-480.

2. Preliminary results. Throughout this note, by a Busemann space [2, p. 11], we shall mean a finitely compact, convex metric space such that at each point p, there exists a neighborhood W with the following property: given any two points x, y of W and any $\epsilon > 0$, we can find a positive number $\delta < \epsilon$ for which a unique point z exists so that

$$d(x,y) + d(y,z) = d(x,z), \quad d(y,z) = \delta.$$

It can easily be verified that the class of all two-point homogeneous, finitely compact, convex metric space with the property (L) coincides with the class of all two-point homogeneous Busemann spaces. In the statements of our Theorems, we use the property (L) instead of Busemann's axioms merely because it is, geometrically, easier to visualize.

Let E be a Busemann space. We shall first see that each d-sphere[1] of sufficiently small radius is locally connected. In fact, let p be a point of E. We choose $\epsilon > 0$ so small that each point x with $0 < d(p,x) \le \epsilon$ can be joined to p by one and only one segment. Let $K(p, \epsilon)$ be the d-sphere with center p and radius ϵ, and R the totality of points y with $0 < d(p,y) < \epsilon$. Then evidently R is an open set of E. Since E is convex, E must be locally connected. It follows then that R is locally connected.

For each point y of $K(p, \epsilon)$, we denote by $P_y(s)$ $(0 \le s \le \epsilon)$ the isometric representation of the segment joining p to y. Let J be the open interval $0 < s < \epsilon$. By our choice of ϵ, the mapping $h: K(p, \epsilon) \times J \longrightarrow R$ defined by $h(y,s) = P_y(s)$ is a one-to-one mapping of the topological product $K(p, \epsilon) \times J$ onto R. Moreover, from Busemann's results [2, I., §3] concerning the convergence of geodesics, we see immediately that h is bicontinuous. This tells us that $K(p, \epsilon) \times J$ and R are homeomorphic. Since R is locally connected, $K(p, \epsilon) \times J$, and hence $K(p, \epsilon)$, is locally connected.

3. Proof of Theorem 1. Let E be a metric space with all the properties mentioned in Theorem 1. From the above discussions, we know that for any point p of E, the d-sphere $K(p, \epsilon)$ with sufficiently small radius ϵ is locally connected. Let Γ be the group of all isometries of E, and Γ_p the totality of all those isometries which leave p invariant. In Γ, we introduce the topology as defined by van Dantzig and van der Waerden [4] (in fact, this is exactly the g-topology of R. Arens).

---

[1] By a d-sphere we mean the totality of points equidistant from a fixed point with respect to the metric d. This should be distinguished from the $(n-1)$-sphere which stands for the $(n-1)$-dimensional topological sphere.

Then Γ_p forms a compact topological group [4]. Evidently, Γ_p is a transformation group of $K(p, \epsilon)$ in the sense of Montgomery and Zippin. From the two-point homogeneity, Γ_p is transitive on $K(p, \epsilon)$. Taking account of the finite dimensionality and local connectedness of $K(p, \epsilon)$ and the compactness of Γ_p, we can conclude [5] that Γ_p is a Lie group, and hence $K(p, \epsilon)$ is locally euclidean (here as well as in what follows, locally euclidean is always used in the topological sense). The set R, being homeomorphic with the topological product of $K(p, \epsilon)$ and the open interval J, must be locally euclidean as well. Hence our space E is locally euclidean at each point of R, and hence locally euclidean at all its points. Moreover, E is obviously separable and connected. It follows then that E is homeomorphic with a manifold.

4. **The structure of d-spheres.** Before proving Theorem 2, we find it convenient to establish some more properties of the d-spheres.

LEMMA. *Let E be a metric space satisfying all the conditions in Theorem* 2. *Then each d-sphere with sufficient small radius is homeomorphic with the* $(n-1)$-*dimensional topological sphere where* $\dim E = n$.

Proof. If $\dim E$ is equal to one, this is trivial. Now we shall assume that $n > 1$. Let p be a point of E, and ϵ so small that each point x with $0 < d(p,x) \leq \epsilon$ can be joined to p by one and only one segment. Set $K(p, \epsilon)$ to be the d-sphere with center p and radius ϵ, and

$$U = \{x \mid d(p, x) < \epsilon\} .$$

We shall show first that U is contractible to a point. Given each point y of $K(p,\epsilon)$, let us denote by $P_y(s)$ the isometric representation of the segment joining p to y. Then the pair (y, s), where $y \in K(p, \epsilon)$ and $0 \leq s < \epsilon$, can be regarded as polar coordinates of points in U. For any real number t with $0 \leq t \leq 1$, we define

$$\phi[t, P_y(s)] = P_y(ts).$$

We see immediately that ϕ is a well-defined mapping of the product $I \times U$, and

$$\phi[1, P_y(s)] = P_y(s), \quad \phi(t, p) = p, \quad \phi[0, P_y(s)] = p,$$

where I denotes the closed interval $\{t \mid 0 \leq t \leq 1\}$. The continuity of ϕ can easily be verified. Thus ϕ gives a contraction of U into the point p, and thus the homotopy group $\pi_i(U)$ vanishes for each i.

Now let us consider the set $R = U - p$. Since U is an n-dimensional open manifold and $n > 1$, the set R is connected and has the same homotopy group π_i as U for all dimensions i less than $n-1$. Thus $\pi_i(R) = 0$, $i = 1, 2, \cdots, n-2$. On the other hand, we have shown in §1 that R is homeomorphic with the topological product $K(p, \epsilon) \times J$, where J denotes an open interval. It follows then that $K(p, \epsilon)$ is connected and

$$(1) \qquad \pi_i[K(p, \epsilon)] = 0, \qquad i = 1, 2, \cdots, n-2.$$

From the proof of Theorem 1, we know that $K(p, \epsilon)$ is a homogeneous space of a compact Lie group. Its connectedness and its simply-connectedness imply that it is an orientable manifold.

Since both $K(p, \epsilon)$ and J are manifolds, we have

$$\dim K(p, \epsilon) + \dim J = \dim R = \dim E = n,$$

and hence $\dim K(p, \epsilon) = n - 1$. It follows immediately from (1) that $K(p, \epsilon)$ is a simply-connected homology sphere of even dimension $n-1$. Therefore [6] $K(p, \epsilon)$ is a topological sphere. The lemma is proved.

5. Proof of Theorem 2. Suppose E to be a metric space with all the properties mentioned in Theorem 2. If E is compact, then our Theorem 2 follows as a direct consequence of [7, Theorem VI]. Thus we can assume from now on that E is not compact. We shall first show that E is an open S. L. space in the sense of Busemann [2, p. 78]. To show this, it suffices [3, p.173] to establish that each geodesic is congruent to a euclidean line; for this, it suffices to demonstrate that given any two distinct points x, y and any $k > 0$, there exists a point z so that

$$d(x, y) + d(y, z) = d(x, z), \qquad d(y, z) = k.$$

In fact, since E is finitely compact and noncompact, E cannot be bounded. There exists then a sequence of points $p_0, p_1, p_2, \cdots$ with $d(p_0, p_i)$ tending to infinity. Thus we can choose i so large that $d(p_0, p_i) \geq d(x, y) + k$. Let τ be a segment joining p_0 to p_i. Evidently there exist three points x', y', z' in τ such that

$$d(x', y') + d(y', z') = d(x', z'), \qquad d(x', y') = d(x, y), \qquad d(y', z') = k.$$

From the two-point homogeneity of E, there is an isometry f of E carrying x', y' to x, y respectively. Then we can see immediately that the point $z = f(z')$ has all the required properties. Thus E is an open S. L. space.

Let $K(p, \epsilon)$ be the d-sphere with center p and radius ϵ, and Γ_p the group of all isometries of E which leave the point p invariant. From the above lemma, we know that $K(p, \epsilon)$ is an $(n - 1)$-sphere and Γ_p a compact and transitive transformation group of $K(p, \epsilon)$. Moreover, it can easily be seen that Γ_p is effective on $K(p, \epsilon)$. In our further discussions, we shall rule out the trivial case where $\dim E = n = 1$. Thus $K(p, \epsilon)$ is connected, and the identity component Γ_p^0 of Γ_p forms a connected, compact, transitive, and effective transformation group of $K(p, \epsilon)$. Since $n - 1$ is even, it follows [6] that Γ_p^0 is either isomorphic with the rotation group R_{n-1} or Cartan's exceptional group G_2. We shall discuss these two cases separately.

Case A. Suppose Γ_p^0 to be isomorphic with the group R_{n-1} of all rotations of the $(n - 1)$-sphere. Let us represent $K(p, \epsilon)$ by the unit sphere in a certain n-dimensional euclidean space, and consider R_{n-1} not only as a topological group but also as a transformation group of $K(p, \epsilon)$ in the usual sense. It is well known that Γ_p^0 and R_{n-1} have the same topological type, that is, there exists a homeomorphism ϕ of $K(p, \epsilon)$ onto itself so that

$$R_{n-1} = \phi \Gamma_p^0 \phi^{-1} = \{\phi f \phi^{-1} \mid f \in \Gamma_p^0\} .$$

Since n is odd, given any point q of $K(p, \epsilon)$, there exists a rotation of period two which leaves fixed *only* q and its diametrically opposite point. It follows then that for each point q of $K(p, \epsilon)$, we can find a transformation f in Γ_p^0 such that (a) f is of period two, (b) f leaves q fixed, and (c) f has only two fixed points on $K(p, \epsilon)$. Now let g be any geodesic through p in E. It intersects $K(p, \epsilon)$ at two points, say q and q'. We consider the transformation f in Γ_p^0 having the above three properties (a), (b), and (c). Since f is an isometry leaving fixed p and q, it leaves the geodesic g pointwise invariant. Moreover, this isometry f cannot have any other fixed point, for otherwise f would have some other fixed points on $K(p, \epsilon)$ besides q and q'. Thus f is a reflection of E about g. Since p is an arbitrary point and g an arbitrary geodesic through p, there exists a reflection of E about each geodesic. From Schur's Theorem [2, p.181], it follows that E is either hyperbolic or euclidean.

Case B. Suppose Γ_p^0 to be isomorphic with the exceptional group G_2. To discuss this case, we have to digress into a few properties of Cayley numbers. Let 1, e_i $(i = 1, 2, \cdots, 7)$ be the units of Cayley algebra. The multiplication rule is given by

$$e_i e_i = -1, \quad e_i e_j = -e_j e_i, \quad e_1 e_2 = e_3, \quad e_1 e_4 = e_5, \quad e_1 e_6 = e_7,$$

$$e_2 e_5 = e_7, \quad e_2 e_4 = -e_6, \quad e_3 e_4 = e_7, \quad e_3 e_5 = e_6,$$

together with the equalities obtained by cyclic permutation of the indices. Let

$$\Theta = \left\{ \sum_{i=1}^{7} x_i e_i \mid x_i = \text{real number}, \quad \sum_{i=1}^{7} (x_i)^2 = 1 \right\}$$

be the totality of all the Cayley numbers with vanishing real part and with norm equal to unity. Evidently, Θ forms a 6-sphere, and each automorphism of the Cayley algebra carries Θ into itself. We can regard therefore the group H of all automorphisms of Cayley algebra as a transformation group of Θ (the topology over H is defined in the usual manner). Now H acts effectively and transitively on Θ. Moreover, it is known that H is isomorphic with the exceptional group G_2.

For each $x = \sum_{i=1}^{7} x_i e_i$ in Θ, we shall denote the Cayley number $x_1 - \sum_{i=2}^{7} x_i e_i$ by x^*, and call it the *symmetric image* of x with respect to e_1. It is evident that

$$(1) \qquad (x^*)^* = x, \quad x^* \begin{cases} = x, & \text{if } x = \pm e_1, \\ \neq x, & \text{otherwise.} \end{cases} \qquad x \in \Theta$$

Moreover, by a direct calculation, we can show that given any two Cayley numbers y, z in Θ, there exists an automorphism f in H such that

$$f(e_1) = e_1, \quad f(y) = y^*, \quad f(z) = z^*.$$

It is to be noted that this f depends on y and z. There is no automorphism of Cayley algebra which carries each x in Θ into its symmetric image x^*.

Now we can proceed to the proof of Theorem 2. Since Γ_p^0 is isomorphic with the exceptional group G_2, $K(p, \epsilon)$ must be six-dimensional [6]. It is known that each transitive transformation group of the 6-sphere which is isomorphic with the exceptional group G_2 has the same topological type as H.[1] Thus we can identify Θ and $K(p, \epsilon)$ in such a manner that Γ_p^0 and H coincide. Let x be a point of $K(p, \epsilon)$. It determines a ray $\overrightarrow{px}$, that is, the totality of points u of E for which either $d(x, u) + d(u, p) = d(x, p)$ or $d(u, x) + d(x, p) = d(u, p)$ [2, p. 76]. For each nonnegative number s, we denote by $P_x(s)$ the point u on the ray $\overrightarrow{px}$ with the property that

[1] This follows as a direct consequence of [6, Lemma 6].

$d(p, u) = s$. Since E is an open S. L. space, each point of E other than p can be represented in a unique way as $P_x(s)$, where $x \in K(p, \epsilon)$ and $s > 0$. Let y, z be any two points of $K(p, \epsilon)$, and let y^*, z^* be, respectively, their symmetric images with respect to e_1 [note that we have identified Θ with $K(p, \epsilon)$]. Then there exists a transformation f in Γ_p^0 such that $f(e_1) = e_1$, $f(y) = y^*$, $f(z) = z^*$. Since f is an isometry of E and leaves p fixed, we have, for any s, $s' \geq 0$, the relations

$$f[P_y(s)] = P_{y^*}(s), \qquad f[P_z(s')] = P_{z^*}(s').$$

This tells us that

$$d[P_y(s),\ P_z(s')] = d[P_{y^*}(s),\ P_{z^*}(s')] \qquad (s,\ s' \geq 0). \tag{2}$$

Now let us consider the mapping $h: E \longrightarrow E$ defined by $h[P_x(s)] = P_{x^*}(s)$, where $x \in K(p, \epsilon)$ and $s \geq 0$. Equality (2) tells us that this mapping h is an isometry of E. Moreover, from (1) we can see that h is of period two and that h has only two fixed points e_1 and $-e_1$ on $K(p, \epsilon)$. It follows then that h is a reflection of E about the geodesic joining p and e_1. However, our space E is two-point homogeneous so that there exists a reflection about every geodesic of E. From Schur's Theorem, we can conclude that E is either hyperbolic or euclidean. Theorem 2 is hereby proved.

6. Remarks. In all the arguments, we use only the weaker two-point homogeneity; that is, there exists a number $\delta > 0$ such that, for any four points x, x', y, y' with $d(x, x') = d(y, y') < \delta$. there exists an isometry of E carrying x, x' to y, y' respectively.

The author wishes to express his thanks to Professor H. Busemann for his helpful suggestions concerning the proof of Theorem 2.

REFERENCES

1. G. Birkhoff, *Metric foundations of geometry I*, Trans. Amer. Math. Soc. 55 (1944), 465-492.

2. H. Busemann, *Metric methods in Finsler spaces and in the foundation of geometry*, Princeton, 1942.

3. ————, *On spaces in which two points determine a geodesic*, Trans. Amer. Math. Soc. 54 (1943), 171-184.

4. D. van Dantzig und B. van der Waerden, *Ueber metrisch homogene Räume*, Abh. Math. Sem. Hamburg 6 (1928), 291-296.

5. D. Montgomery and L. Zippin, *Topological transformation groups I*, Ann. of Math. **41** (1940), 778-791.

6. H. C. Wang, *A new characterisation of spheres of even dimension*, Nederl. Akad. Wetensch. Proc. 52 (1949), 838-845.

7. ———, *Two-point homogeneous spaces*, to appear in Ann. of Math.

LOUISIANA STATE UNIVERSITY