ON A CERTAIN NONLINEAR INTEGRAL EQUATION OF THE VOLTERRA TYPE

J. H. ROBERTS AND W. R. MANN

1. Introduction. In an earlier paper by Mann and Wolf [1], the following problem of heat transfer between a gas at constant unit temperature and the semi-infinite solid was considered:

GAS
Temperature 1°

SOLID
Temperature $U(x,t)$
$U(x, 0) = 0$ for $x \geq 0$

$$U_t(x, t) = U_{x,x}(x, t), \tag{1.1}$$

$$U(x, 0) = 0, \tag{1.2}$$

$$|U(x, t)| < M, \qquad x > 0, \quad t > 0, \tag{1.3}$$

$$U_x(0, t) = \frac{-[1 - U(0, t)]}{K} f[1 - U(0, t)] = -G[U(0, t)]. \tag{1.4}$$

It will be noted that, in boundary condition (1.4), Newton's Law of Cooling has been replaced by the weaker, more realistic hypothesis that the net rate of heat exchange from the gas to the solid, $-KU_x(0,t)$, is some function, $KG[U(0,t)]$, of the surface temperature. In every heat transfer problem of physical significance, the following conditions must be satisfied by $G[U]$:

$$G[U] \text{ is continuous,} \tag{1.5}$$

$$G[1] = 0, \tag{1.6}$$

$$G[U] \text{ is strictly decreasing.} \tag{1.7}$$

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By Duhamel's Principle, the solution $U(x, t)$ of the above boundary value problem is easily constructed once we know the surface temperature, $U(0,t)$, which it can be shown must satisfy the nonlinear integral equation,

$$U(0, t) = \int_0^t \frac{G[U(0,\tau)]}{\pi^{1/2}(t-\tau)^{1/2}}\, d\tau. \tag{1.8}$$

Equation (1.8) was shown in [1] to have at least one solution for all G satisfying (1.5), (1.6), and (1.7). Under the additional *ad hoc* assumption that G satisfy a Lipschitz condition on the unit interval, the solution of (1.8) was proved to be unique and nondecreasing.

It is the purpose of the present paper to show that conditions (1.5), (1.6), and (1.7) alone are sufficient to imply that $U(0,t)$ is not only unique but also strictly increasing. Besides being a stronger result than that previously obtained, it has the advantage of requiring only those conditions imposed upon G by the most elementary physical consideration.

2. The theorems. More general results are obtained without increasing the complexity of the proofs if instead of the function $[\pi(t-\tau)]^{-1/2}$ we write $K(t-\tau)$, or $K(z)$ where $t-\tau = z$, subject to specified conditions, namely:

(2.1) $K(z)$ is positive, continuous, and strictly decreasing for $z > 0$;

(2.2) $\int_0^t K(z)\,dz$ is finite for each $t > 0$;

(2.3) $K(z+\alpha)/K(z)$ is strictly increasing in z for each fixed α greater than zero;

(2.4) $\int_0^t K(z)\,dz \longrightarrow \infty$ as $t \longrightarrow \infty$.

It is easily verified, for example, that $[\pi(t-\tau)]^{-p}$ satisfies the above conditions for $0 < p < 1$.

THEOREM 1. *The equation*

$$y(t) = \int_0^t G[y(\tau)]K(t-\tau)\, d\tau \tag{2.5}$$

can have at most one bounded solution, given that $G[y]$ *satisfies* (1.5), (1.6), *and* (1.7), *and that* $K(z)$ *satisfies* (2.1) *and* (2.2).

THEOREM 2. *In addition to the hypotheses of Theorem* 1, *assume that* K *satisfies* (2.3). *If* $y(t)$ *is a bounded solution of* (2.5), *then* $y(t)$ *is strictly increasing in* t. *If, in addition,* K *satisfies* (2.4), *then* $y(t) \longrightarrow 1$ *as* $t \longrightarrow \infty$.

3. On Theorem 1. In this section we arrive at a proof of Theorem 1.

LEMMA 3.1. *Suppose that* $f(\tau)$ *is continuous for* $a \leq \tau \leq b$, *and that* $\int_a^t f(\tau)\,d\tau$ *is positive for some* t *on* $[a, b]$. *Let* t_1 *be the smallest value of* t *on* $[a, b]$ *for which* $\int_a^t f(\tau)\,d\tau$ *is a maximum. Then either* $f(t_1) = 0$ *or* $t_1 = b$. *Suppose that* $K(\tau)$ *is positive and strictly increasing on* $a \leq \tau < t_1$, *and that* $\int_a^{t_1} K(\tau)\,d\tau$ *exists. Then* $\int_a^{t_1} f(\tau)\,K(\tau)\,d\tau > 0$.

Proof. Set $\int_a^{t_1} f(\tau)\,d\tau = M > 0$. Divide f into its positive and negative parts by writing $f_1(\tau) = \max\,[f(\tau), 0]$ and $f_2(\tau) = -\min\,[f(\tau), 0]$, so that $f(\tau) = f_1(\tau) - f_2(\tau)$. Let $c_0 = a$, and define c_1 to be the smallest number $c\,(c > c_0)$ such that $\int_a^c f_1(\tau)\,d\tau = M$. Then $c_1 \leq t_1$. In general, choose c_{n+1} as the smallest number greater than c_n for which

$$\int_{c_n}^{c_{n+1}} f_1(\tau)\,d\tau = \int_{c_{n-1}}^{c_n} f_2(\tau)\,d\tau. \tag{3.2}$$

Since $\int_{c_1}^{t_1} f_1(\tau)\,d\tau = \int_{c_0}^{t_1} f_2(\tau)\,d\tau$, it follows that for each n we have $c_n \leq t_1$. Let c be the number to which the sequence $c_0, c_1, c_2, \cdots$ converges. Then $c \leq t_1$ and

$$\int_{c_0}^{c} f(\tau)\,d\tau = \int_{c_0}^{c_1} f_1(\tau)\,d\tau + \sum_{n=1}^{\infty} \left[\int_{c_n}^{c_{n+1}} f_1(\tau)\,d\tau - \int_{c_{n-1}}^{c_n} f_2(\tau)\,d\tau\right] = M,$$

since each summand of the infinite series is zero. Thus $c = t_1$.

We have

$$\begin{aligned} (3.3)\quad & \int_{c_0}^{t_1} f(\tau)K(\tau)\,d\tau \\ &= \int_{c_0}^{t_1} [f_1(\tau) - f_2(\tau)]K(\tau)\,d\tau \\ &= \int_{c_0}^{c_1} f_1(\tau)K(\tau)\,d\tau + \sum_{n=1}^{\infty} \left[\int_{c_n}^{c_{n+1}} f_1(\tau)K(\tau)\,d\tau - \int_{c_{n-1}}^{c_n} f_2(\tau)K(\tau)\,d\tau\right]. \end{aligned}$$

Now for $n \geq 1$ we have

$$\int_{c_n}^{c_{n+1}} f_1(\tau)K(\tau)\,d\tau \geq K(c_n) \int_{c_n}^{c_{n+1}} f_1(\tau)\,d\tau,$$

since $K(\tau)$ is strictly increasing; and

$$\int_{c_{n-1}}^{c_n} f_2(\tau)K(\tau)\,d\tau \leq K(c_n) \int_{c_{n-1}}^{c_n} f_2(\tau)\,d\tau.$$

Thus, by (3.2), each summand in the expansion (3.3) is positive or zero, and the first one is positive. Hence $\int_a^{t_1} f(\tau)\,K(\tau)\,d\tau > 0$.

The first assertion of the lemma, namely that either $f(t_1) = 0$ or $t_1 = b$, is obvious.

LEMMA 3.4. *Assume that $f(\tau)$ is continuous on $0 \leq \tau \leq T$ and that $K(z)$ satisfies* (2.1) *and* (2.2). *Suppose furthermore that $F(t)\,f(t) \leq 0$ for $0 \leq t \leq T$, where $F(t) = \int_0^t f(\tau)\,K(t-\tau)\,d\tau$. Then $f(\tau) = 0$ for $0 \leq \tau \leq T$.*[1]

Proof. Assume the lemma to be false. Then for some t we have $\int_0^t f(\tau)\,d\tau \neq 0$. There is no loss of generality in assuming $\int_0^t f(\tau)\,d\tau > 0$, since replacing f by $-f$ results in replacing F by $-F$, so that the inequality $F(t)\,f(t) \leq 0$ persists. Clearly $f(\tau)$ must change signs, so there exists a number b, $0 < b < T$, such that $f(b) = 0$ and, for some $t < b$, $\int_0^t f(\tau)\,d\tau > 0$. Let t_1 be the smallest value of t $(0 < t \leq b)$ for which $\int_0^t f(\tau)\,d\tau$ is a maximum and apply Lemma 3.1 using $K(t_1 - \tau)$ in place of $K(\tau)$. We have

$$F(t_1) = \int_0^{t_1} f(\tau)K(t_1 - \tau)\,d\tau > 0.$$

Then we have $F(t) > 0$ over the segment $(t_1 - \delta, t_1)$ for some $\delta > 0$; and since $\int_{t_1-\delta}^{t_1} f(\tau)\,d\tau > 0$ there is some t between $t_1 - \delta$ and t_1 for which $f(t) > 0$. But for this t we have $F(t)\,f(t) > 0$, violating our hypothesis. Thus $f(t)$ is identically zero on $[0, T]$. This completes the proof of Lemma 3.4 and we are now ready to prove the uniqueness theorem.

Proof of Theorem 1. Suppose $y_1(t)$ and $y_2(t)$ are bounded solutions of (2.5). Obviously both are continuous. Letting $F(t) = y_1(t) - y_2(t)$, and $f(\tau) = G[y_1(\tau)] - G[y_2(\tau)]$, we have $F(t) = \int_0^t f(\tau)\,K(t - \tau)\,d\tau$. If $f(\tau) < 0$ then, since G is

[1] In place of assuming continuity we may assume that $f(\tau)$ has a Lebesgue integral over $[0, T]$ and that the condition $F(t)\,f(t) \leq 0$ holds except for a set of measure zero. Then we may conclude that $f(\tau) = 0$ over $[0, T]$ except at points of a set of measure zero.

strictly decreasing, we have $y_2(\tau) < y_1(\tau)$, whence $F(\tau) > 0$ and $F(\tau)f(\tau) < 0$. Similarly, if $f(\tau) > 0$ it follows that $F(\tau)f(\tau) < 0$. Thus the hypotheses of Lemma 3.4 are satisfied and we can infer that $f(t)$, and hence $F(t)$, is identically zero for $t > 0$. This means that $y_1(t) \equiv y_2(t)$.

4. The function $K(z)$. In preparation for the proof of Theorem 2, we give the following lemma concerning $K(z)$.

LEMMA 4.1. *If $K(z)$ satisfies* (2.1) *and* (2.3), *then*:

(4.1) *For $\alpha > 0$ and $z > 0$, we have*

$$[K(z+\alpha) - K(z+2\alpha)]/[K(z) - K(z+\alpha)] < K(z+\alpha)/K(z)\,;$$

(4.2) *$K(z) - K(z+\alpha)$ is strictly decreasing in z for all fixed $\alpha > 0$;*

(4.3) *$K(z)$ is a convex function;*

(4.4) *For each interval $[0, b]$, there exists a number $R > 0$ such that*

$$K(z) - K(z+\alpha) > R\alpha \text{ for } 0 < z < z + \alpha < b.$$

Proof. By (2.3) we know that $K(z+\alpha)/K(z) < K(z+2\alpha)/K(z+\alpha)$. Subtracting 1 from both sides of this inequality and performing a simple rearrangement of terms, we easily arrive at conclusion (4.1) above.

To prove (4.2) we observe that, by (2.3), $[K(z+\alpha)/K(z)] - 1$ is strictly increasing, so that $[K(z) - K(z+\alpha)]/K(z)$ is strictly decreasing. But by (2.1), both the numerator and the denominator are positive and the denominator is decreasing. Hence, the numerator must also be decreasing.

That $K(z)$ is convex follows readily from (4.2), in view of the hypotheses that $K(z)$ is positive, decreasing, and continuous.

From (4.2) and (4.3) it follows that $K(z)$ has a right-hand derivative at each $z > 0$, and this derivative is negative and strictly increasing. The R of (4.4) can be taken as the negative of this derivative at $z = b$.

5. The function $y(t)$. Sections 5 through 10 are devoted to the proof of Theorem 2. Throughout, $y(t)$ will denote the bounded solution of (2.5), where $K(z)$ satisfies (2.1), (2.2), and (2.3). In §10 we assume in addition that $K(z)$ satisfies (2.4).

LEMMA 5.1. *If $y(t) < 1$ for $0 \leq t < T$, then $y(t)$ is nondecreasing on $[0, T]$.*

Proof. Assume the lemma is false. Then for some subinterval, $[0, b]$, $y(t)$

attains its maximum M at an interior point, a, and we set $y(a) - y(b) = 3\epsilon > 0$. We shall assume that a is the smallest number $(0 < a < b)$ such that $y(a) = M$. Choose $\delta_1 > 0$ so small that

$$\delta_1 \int_a^b K(b - \tau)\, d\tau < \epsilon . \tag{5.2}$$

Set $G[y(a)] = c$ and choose p_1 $(0 < p_1 < a)$ so near to a that (see Fig. 1)

$$G[y(t)] < c + \delta_1 \qquad \text{for } p_1 < t < a . \tag{5.3}$$

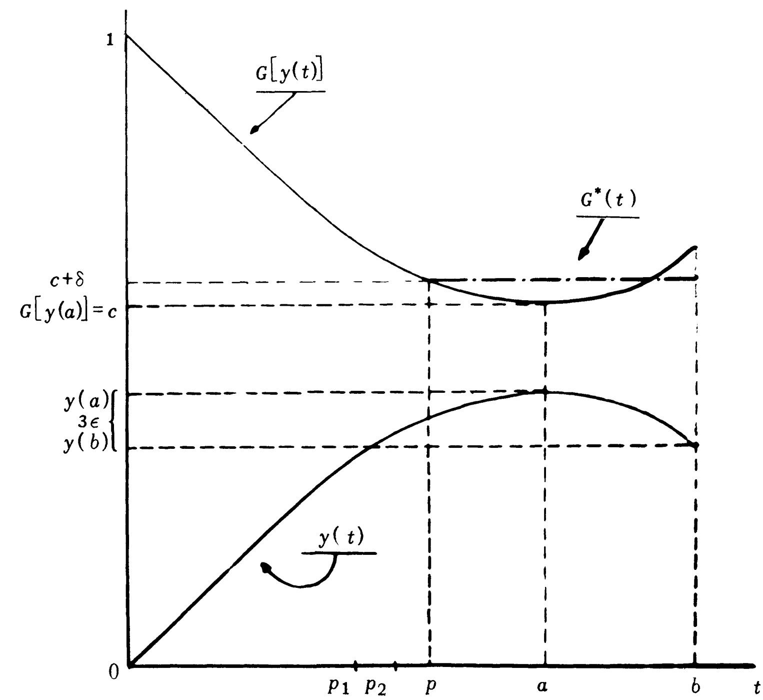

Fig. 1

Next, choose p_2 $(p_1 < p_2 < a)$ so close to a that

$$(c + \delta_1) \int_{p_2}^a K(a - \tau)\, d\tau < \epsilon , \tag{5.4}$$

and

$$(c + \delta_1) \int_{p_2}^{a} K(b - \tau)\, d\tau < \epsilon . \tag{5.5}$$

Define δ so that $0 < \delta < \delta_1$ and $c + \delta \leq G[y(t)]$ for $0 \leq t \leq p_2$. Let p be the largest value of t such that $G[y(t)] \geq c + \delta$ for $t \leq p$. Then $p_2 \leq p < a$. Define

$$G^*[t] = \begin{cases} G[y(t)] & \text{if } t \leq p , \\ c + \delta & \text{if } t > p . \end{cases} \tag{5.6}$$

Now since y attains its maximum on $[0, b]$ at $t = a$, and G is strictly decreasing, we have

$$G - G^* \geq -\delta \qquad \text{for } a \leq t \leq b .$$

We shall show (Lemma 7.1) that $\int_0^t G^*[\tau]\, K(t - \tau) d\tau$ is strictly increasing as t increases from a to b, and therefore $Y(b) > Y(a)$, where we use the following definition:

$$Y(t) = \int_0^t G^*[\tau] K(t - \tau)\, d\tau. \tag{5.7}$$

By (5.4) we have

$$\begin{aligned} |y(a) - Y(a)| &= \left| \int_p^a \{G[y(\tau)] - G^*[\tau]\} K(a - \tau)\, d\tau \right| \\ &\leq \delta_1 \int_p^a K(a - \tau)\, d\tau < \epsilon . \end{aligned} \tag{5.8}$$

Similarly, we obtain

$$\begin{aligned} y(b) - Y(b) &= \int_p^a \{G[y(\tau)] - G^*[\tau]\} K(b - \tau)\, d\tau \\ &\quad + \int_a^b \{G[y(\tau)] - G^*[\tau]\} K(b - \tau)\, d\tau \\ &= \alpha + \beta , \text{ say} . \end{aligned} \tag{5.9}$$

By (5.5), we have $|\alpha| < \epsilon$. As for β, the integrand for any τ is either positive or numerically less than $\delta K(b - \tau)$. Hence, by (5.2), it follows that $\beta > -\epsilon$. From (5.8) and (5.9) we therefore have $y(a) < Y(a) + \epsilon$ and $y(b) > Y(b) - 2\epsilon$.

Subtracting, we get $y(b) - y(a) > Y(b) - Y(a) - 3\epsilon > -3\epsilon$, since $Y(b) - Y(a) > 0$ by Lemma 7.1. This contradicts the definition of ϵ, and thus the proof will be complete when Lemma 7.1 has been established.

6. The function $Y(t)$ for $t \leq a$. We shall establish the following result.

LEMMA 6.1. *With the notation of* §5, *there exist numbers r and s ($p \leq r < s \leq a$) such that $Y(s) > Y(r)$.*

Proof. Define $f(\tau)$ to be $G^*(\tau) - G[y(\tau)]$.

Case 1: for some $q\,(p < q \leq a)$, we have $\int_p^q f(\tau)\,d\tau > 0$. In this case, set $r = p$ and let s be the smallest value of q on $[p, a]$ such that $\int_p^q f(\tau)\,d\tau$ is a maximum. Using $K(s - \tau)$ in place of $K(\tau)$, and p and s, respectively, in place of a and t_1, we see from Lemma 3.1 that $\int_p^s f(\tau)\,K(s - \tau)\,d\tau > 0$. This implies that

$$\int_p^s G^*[\tau]K(s-\tau)\,d\tau > \int_p^s G[y(\tau)]K(s-\tau)\,d\tau. \tag{6.2}$$

Now if $s < a$ then $f(s) = 0$, by Lemma 3.1. That is, $G[y(s)] = c + \delta$, so that $y(s) = y(p)$. If $s = a$, then obviously $y(s) > y(p)$. Since $G^*[\tau] = G[y(\tau)]$ for $\tau \leq p$, we get immediately from (6.2) the result that

$$\begin{aligned}\int_0^s G^*[\tau]K(s-\tau)\,d\tau &> \int_0^s G[y(\tau)]K(s-\tau)\,d\tau \\ &= y(s) \geq y(p) \\ &= \int_0^p G^*[\tau]K(p-\tau)\,d\tau;\end{aligned}$$

that is,

$$Y(s) > Y(p).$$

Case 2: for every q ($p < q \leq a$), we have $\int_p^q f(\tau)\,d\tau \leq 0$. Now $f(\tau)$ is not identically zero on $[p, a]$ since $f(a) = \delta$. Let r be the smallest number q on $[p, a]$ such that $\int_p^q f(\tau)\,d\tau$ is a minimum.

Then $\int_p^r f(\tau)\,d\tau = M < 0$ and $\int_r^t f(\tau)\,d\tau \geq 0$ ($r \leq t \leq a$) by the minimum property for r. Let s be the smallest value of $t\,(r < t \leq a)$ such that $\int_r^t f(\tau)\,d\tau$ is a maximum. We now apply Lemma 3.1 to the interval $[p, r]$, using $K(r - \tau) - K(s - \tau)$ as the function $K(\tau)$ [note that this function is increasing in τ by (4.2)]. We also use $-f(\tau)$ in place of $f(\tau)$. This gives

$$\int_p^r [-f(\tau)][K(r-\tau) - K(s-\tau)]\,d\tau > 0\,. \tag{6.3}$$

Similarly, applying Lemma 3.1 to the interval $[r, s]$ and using $K(s - \tau)$ as the function $K(\tau)$, we get

$$\int_r^s f(\tau)K(s-\tau)\,d\tau > 0\,. \tag{6.4}$$

We are now in a position to show that $Y(s) - Y(r) > 0$. For we have

$$\begin{aligned} Y(s) - Y(r) = &\int_0^p G^*[\tau][K(s-\tau) - K(r-\tau)]\,d\tau \\ &+ \int_p^r G^*[\tau][K(s-\tau) - K(r-\tau)]\,d\tau \\ &+ \int_r^s G^*[\tau]K(s-\tau)\,d\tau\,. \end{aligned}$$

Similarly, we have

$$\begin{aligned} y(s) - y(r) = &\int_0^p G[y(\tau)][K(s-\tau) - K(r-\tau)]\,d\tau \\ &+ \int_p^r G[y(\tau)][K(s-\tau) - K(r-\tau)]\,d\tau \\ &+ \int_r^s G[y(\tau)]K(s-\tau)\,d\tau\,. \end{aligned}$$

We therefore get

$$\begin{aligned} &[Y(s) - Y(r)] - [y(s) - y(r)] \\ &\quad = \int_p^r f(\tau)[K(s-\tau) - K(r-\tau)]\,d\tau + \int_r^s f(\tau)K(s-\tau)\,d\tau \\ &\quad = \int_p^r [-f(\tau)][K(r-\tau) - K(s-\tau)]\,d\tau + \int_r^s f(\tau)K(s-\tau)\,d\tau > 0\,, \end{aligned}$$

by (6.3) and (6.4). But $f(r) = 0$, so that $y(r) = y(p)$. Also either $f(s) = 0$ or $s = a$. In either case we have $y(s) \geq y(p)$. Thus $y(s) - y(r) \geq 0$ and $Y(s) > Y(r)$.

7. The function $Y(t)$ for $t \geq a$. For $t \geq a$ we have the following stronger result.

LEMMA 7.1. *The function $Y(t)$ is strictly increasing for $t \geq a$.*

Proof. Suppose that $e \geq p$, $\alpha > 0$, and $Y(e+\alpha) \geq Y(e)$. We prove first that $Y(e+2\alpha) > Y(e+\alpha)$. Replacing $c + \delta$ by k, we may write

$$Y(e+\alpha) - Y(e) = \int_0^{e+\alpha} G^*(\tau) K(e+\alpha-\tau)\, d\tau - \int_0^{e+\alpha} G^*(\tau) K(e-\tau)\, d\tau$$

$$= \left\{ \int_0^{e+\alpha} kK(e+\alpha-\tau)\, d\tau - \int_0^{e} kK(e-\tau)\, d\tau \right\}$$

$$- \left\{ \int_0^{e} [G^*(\tau) - k][K(e-\tau) - K(e+\alpha-\tau)]\, d\tau \right\}$$

$$= A_1 - B_1, \text{ say}.$$

(We have used the fact that $G^*(\tau) - k = 0$ for $\tau \geq e$.) Similarly, we have

$$Y(e+2\alpha) - Y(e+\alpha) = \left\{ \int_0^{e+2\alpha} kK(e+2\alpha-\tau)\, d\tau - \int_0^{e+\alpha} K(e+\alpha-\tau)\, d\tau \right\}$$

$$- \left\{ \int_0^{e} [G^*(\tau) - k][K(e+\alpha-\tau) - K(e+2\alpha-\tau)]\, d\tau \right\}$$

$$= A_2 - B_2, \text{ say}.$$

Now $A_1 - B_1 \geq 0$ by hypothesis, and we wish to show that $A_2 - B_2 > 0$.

By simple changes of variable we get

$$\int_0^{e} K(e-\tau)\, d\tau = \int_0^{e} K(z)\, dz, \quad \int_0^{e+\alpha} K(e+\alpha-\tau)\, d\tau = \int_0^{e+\alpha} K(z)\, dz,$$

and

$$\int_0^{e+2\alpha} K(e+2\alpha-\tau)\, d\tau = \int_0^{e+2\alpha} K(z)\, dz .$$

Then we have the following:

$$A_1 = k \int_e^{e+\alpha} K(z)\, dz,$$

$$B_1 = \int_0^{e} [G^*(e-z) - k][K(z) - K(z+\alpha)]\, dz,$$

$$A_2 = k \int_{e+\alpha}^{e+2\alpha} K(z)\, dz\,,$$

$$B_2 = \int_0^e [G^*(e-z) - k][K(z+\alpha) - K(z+2\alpha)]\, dz\,.$$

Another change of variable gives

$$A_2 = k \int_e^{e+\alpha} K(z+\alpha)\, dz\,.$$

Now over the interval $e \le z \le e+\alpha$ we have, by (2.3),

$$K(z+\alpha) = K(z)[K(z+\alpha)/K(z)] \ge K(z)[K(e+\alpha)/K(e)]\,.$$

Furthermore, the strict inequality holds except for $z = e$. It follows that $A_2 > [K(e+\alpha)/K(e)]\, A_1$.

To obtain an inequality for B_2/B_1, we note first that $G^*(e-z) - k$ is positive or zero for $0 \le z \le e$. Over this range for z, we have

$$[K(z+\alpha) - K(z+2\alpha)]/[K(z) - K(z+\alpha)]$$
$$< K(z+\alpha)/K(z) \le K(e+\alpha)/K(e)\,,$$

by (4.1) and (2.3). Thus it follows that $B_2 < [K(e+\alpha)/K(e)]\, B_1$. Then

$$A_2 - B_2 > [K(e+\alpha)/K(e)][A_1 - B_1] \ge 0\,.$$

Thus we have seen that if $e \ge p$, $\alpha > 0$, and $Y(e+\alpha) \ge Y(e)$ then $Y(e+2\alpha) > Y(e+\alpha)$. But then it follows that $Y(e+3\alpha) > Y(e+2\alpha)$; $Y(e+4\alpha) > Y(e+3\alpha)$, and so on. Now if $e = r$, and $\alpha = s - r$, we have $Y(e+\alpha) > Y(e)$ by Lemma 6.1. Divide the interval $[r, s]$ into n equal subintervals by the points $x_0 = r, x_1, x_2, \cdots, x_n = s$. It follows that for some i we have $Y(x_{i+1}) > Y(x_i)$. But $x_{i+1} = x_i + \alpha n^{-1}$, so that $Y(x_i + \alpha n^{-1}) > Y(x_i)$. Thus we see that $Y(t)$ is strictly increasing over the points of an arbitrarily fine mesh. Hence, by continuity, it is always increasing for $t \ge s$, therefore *a fortiori* for $t \ge a$. This completes the proof of Lemma 7.1, and thereby establishes Lemma 5.1.

8. A stronger result concerning $y(t)$. We now prove:

LEMMA 8.1. *Under the hypothesis of Lemma* 5.1, *$y(t)$ is strictly increasing on the interval* $[0, T]$.

Proof. If the lemma is false then there exist points p and a $(0 < p < a)$ such that $y(\tau) < y(p)$ if $\tau < p$, and $y(\tau) = y(p)$ if $p \leq \tau \leq a$. Define $G^*(\tau) = G[y(\tau)]$ for $\tau \leq p$, and $G^*(\tau) = G[y(p)]$ for $\tau > p$. Then we have the situation of §7, and $Y(t)$ is strictly increasing for $t \geq p$. But over $[p, a]$, we have $Y(t) = y(t)$.

9. Another result concerning $y(t)$. Our last lemma is the following:

LEMMA 9.1. *For every t $(t \geq 0)$, we have $y(t) < 1$.*

Proof. Assume the lemma is false, and let b be the smallest number such that $y(b) = 1$. Then by (1.5), (1.6), and (1.7) it follows that $G[y(t)]$ strictly decreases from 1 to 0 as t increases from 0 to b. (See Fig. 2.)

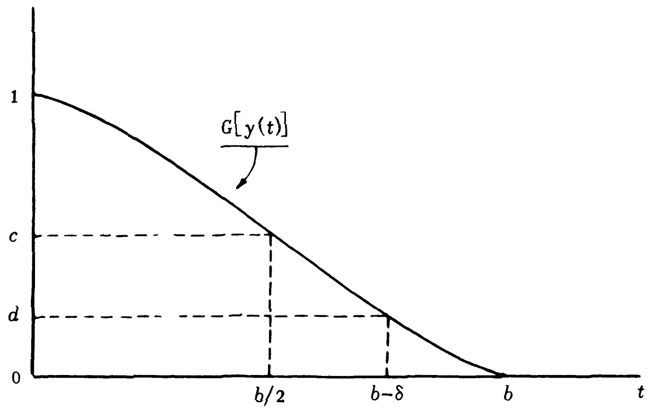

Fig. 2

By (4.4), there exists an $R > 0$ such that for every δ $(0 < \delta < b/2)$ we have

$$K[(b/2) - \delta] - K(b/2) > R\delta \,. \tag{9.1}$$

Set $c = G[y(b/2)]$ and $d = G[y(b - \delta)]$. Then c is fixed and d is a function of δ such that $d \longrightarrow 0$ as $\delta \longrightarrow 0$. Also $K(b) > 0$ and K is continuous. Therefore it is clear that we can fix δ so that

$$(b/2)(c - d)R > 2dK(b) \,, \tag{9.2}$$

$$K(b - \delta) < 2K(b) \,. \tag{9.3}$$

We shall show that for this choice of δ we have $y(b) < y(b - \delta)$, which is a contradiction. Now

$$y(b-\delta) = d\int_0^{b-\delta} K(b-\delta-\tau)\,d\tau + \int_0^{b/2} (G[y(\tau)]-d)K(b-\delta-\tau)\,d\tau$$

$$+ \int_{b/2}^{b-\delta} (G[y(\tau)]-d)K(b-\delta-\tau)\,d\tau$$

$$= \alpha + \beta + \gamma, \text{ say .}$$

Similarly, we have

$$y(b) < d\int_0^{b} K(b-\tau)\,d\tau + \int_0^{b/2} (G[y(\tau)]-d)K(b-\tau)\,d\tau$$

$$+ \int_{b/2}^{b-\delta} (G[y(\tau)]-d)K(b-\tau)\,d\tau = \lambda + \mu + \nu, \text{ say,}$$

where the inequality arises from replacing $G[y(\tau)]$ by the greater quantity d, for $b - \delta < \tau \leq b$. Then

$$y(b) - y(b-\delta) < (\lambda - \alpha) - (\beta - \mu) + (\nu - \gamma) . \tag{9.4}$$

By (2.1) we have

$$\nu - \gamma < 0. \tag{9.5}$$

Furthermore,

$$\lambda - \alpha = d\left[\int_0^{b} K(b-\tau)\,d\tau - \int_0^{b-\delta} K(b-\delta-\tau)\,d\tau\right] = d\int_0^{\delta} K(b-\tau)\,d\tau,$$

since

$$\int_0^{b} K(b-\tau)\,d\tau = \int_0^{\delta} K(b-\tau)\,d\tau + \int_\delta^{b} K(b-\tau)\,d\tau ,$$

and since replacing τ by $z + \delta$ gives $\int_0^{b-\delta} K(b-\delta-z)\,dz$ for the second integral. But by (2.1) it follows that

$$\int_0^{\delta} K(b-\tau)\,d\tau < \delta[K(b-\delta)] ,$$

so that

$$\lambda - \alpha < d\delta[K(b-\delta)] < 2d\delta K(b) . \tag{9.6}$$

Similarly, by (4.2),

$$\beta - \mu = \int_0^{b/2} (G[y(\tau)] - d)[K(b - \delta - \tau) - K(b - \tau)]\, d\tau$$

$$> (c - d) \int_0^{b/2} [K(b - \delta - \tau) - K(b - \tau)]\, d\tau$$

$$> (c - d)[K(b/2 - \delta) - K(b/2)](b/2) .$$

Thus using (9.1) and (9.3) we have

$$\beta - \mu > (c - d) R \delta \,(b/2) . \tag{9.7}$$

In view of (9.2), (9.6), and (9.7), it is clear that $\beta - \mu > \lambda - \alpha$. Hence, from (9.4) and (9.5) we have $y(b) - y(b - \delta) < 0$, a contradiction.

10. Proof of Theorem 2. To complete the proof of Theorem 2, we now assume in addition that $K(z)$ satisfies (2.4).

We know that $y(t)$ is a strictly increasing function of t, $y(0) = 0$, and $y(t) < 1$ for all t. We must show that $y(t) \longrightarrow 1$ as $t \longrightarrow \infty$. Assume on the contrary that $y(t) \longrightarrow k$ as $t \longrightarrow \infty$, where $0 < k < 1$. Then $G[y(t)] > G(k) > 0$ for all t. By (2.5) we have

$$y(t) = \int_0^t G[y(\tau)]K(t - \tau)\, d\tau > \int_0^t G[k]K(t - \tau)\, d\tau$$

$$= G(k) \int_0^t K(t - \tau)\, d\tau = G(k) \int_0^t K(z)\, dz ;$$

but, by (2.4), the last integral increases indefinitely as $t \longrightarrow \infty$, so that we have a contradiction.

11. Conclusion. In conclusion it will be shown that if hypothesis (2.3) on $K(z)$ is replaced by the stipulation that $K(z)$ be convex, then $y(t)$ is not necessarily monotonic increasing.

Let $G(y) = 1 - y$ and $K_1(z) = 1 - z$ $(0 \le z \le 1)$. Then if $y(t)$ denotes the bounded solution of the equation

$$y(t) = \int_0^t G[y(\tau)]K_1(t - \tau)\, d\tau ,$$

it is readily shown that $y(t)$ is actually decreasing over a small segment, $1 - \delta < t < 1$.

To get a similar example where $\int_0^t K(z)dz \longrightarrow \infty$ as $t \longrightarrow \infty$, we select a fixed c, $1 - \delta < c < 1$, and write $K(z) = K_1(z)$ for $z \leq c$, $K(z) = dz^{-1/2}$ for $z > c$, where d is chosen so that the functions $1 - z$ and $dz^{-1/2}$ have the same value at $z = c$; that is, $d = c^{1/2}(1 - c)$.

Reference

1. W. R. Mann and F. Wolf, *Heat transfer between solids and gasses under nonlinear boundary conditions*, to appear soon in Quart. Appl. Math.

Duke University
University of North Carolina