OSCILLATION THEOREMS FOR THE SOLUTIONS OF LINEAR, NONHOMOGENEOUS, SECOND-ORDER DIFFERENTIAL SYSTEMS

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1. **Introduction.** Oscillation theorems for the solutions of the equation

$$\frac{d}{dx}\left[K(x)\frac{dy}{dx}\right] - G(x)y = 0.$$

are classical. It is the purpose of this paper to develop theorems of a similar nature for a class of equations of the type

$$\frac{d}{dx}\left[K(x)\frac{dy}{dx}\right] - G(x)y = A(x).$$

It will be assumed that over an interval X: $a \le x \le b$ $(b > a)$, the functions $K(x)$, $G(x)$, and $A(x)$ are continuous. All quantities used are assumed to be real. Primes will be used to indicate derivatives with respect to x.

Use will be made of the following lemma which gives a modified form of properties of the second-order linear homogeneous equation developed by W. M. Whyburn [3, pp. 633-634].

LEMMA 1. *Let $y(x)$, a solution of $(Ky')' - Gy = 0$ over X, have the m zeros $r_1, \cdots, r_m$ $(m > 2)$ on X. Let the inequalities $K > 0$, $G < 0$ hold, and let GK be a nonincreasing function of x on X. If A is nonvanishing except possibly at a, and for $x > a$ either one of the following is true over X:*

(a) *$A > 0$ and A/G is a strictly decreasing function of x,*

(b) *$A < 0$ and A/G is a strictly increasing function of x,*

then

$$\left|\int_{r_i}^{r_{i+1}} A(t)y(t)\,dt\right| < \left|\int_{r_{i+1}}^{r_{i+2}} A(t)y(t)\,dt\right| \qquad (i = 1, \cdots, m-2).$$

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In order to prove this lemma one needs only to make straightforward modifications in the arguments given by Whyburn.

LEMMA 2. *Under the hypotheses of Lemma* 1, *the zeros of*

$$F(x) = \int_a^x A(t)y(t)\,dt$$

and $y(x)$ *separate each other on* $a < x \leq b$.

This result, which also was given by Whyburn, is an immediate consequence of Lemma 1.

LEMMA 3. *Let* $u(x)$ *be any solution of the system* $(Ky')' - Gy = 0$, $y(b) = 0$. *Under the hypotheses of Lemma* 1, $\int_x^b A(t)u(t)\,dt$ *does not vanish in* $a \leq x < b$.

Proof. If $u(x)$ has no zero except b on X, the conclusion is obvious. Otherwise, by Lemma 1, if q is the last zero of $u(x)$ on X preceding b, then the integral $\int_x^b A(t)u(t)\,dt$ has the sign of $\int_q^b A(t)u(t)\,dt$.

For the sake of brevity we shall henceforth let (H) represent the following set of conditions on X.

(H)
- (1) $K(x) > 0$, $G(x) < 0$.
- (2) $K(x)G(x)$ is a nonincreasing function of x.
- (3) Either one of the following is true:
  - (i) $\beta \leq 0$, $A(x) > 0$ for $x > a$ and $A(x)/G(x)$ is a strictly decreasing function of x.
  - (ii) $\beta \geq 0$, $A(x) < 0$ for $x > a$ and $A(x)/G(x)$ is a strictly increasing function of x.

Let $u_1(x)$ be any solution of $(Ky')' - Gy = 0$ such that $u_1(b) = 0$. Choose another solution $u_2(x)$ such that $K(u_2u_1' - u_2'u_1) \equiv 1$ on X. As a final preliminary result we have the following:

LEMMA 4. *Under the hypotheses* (H) *if* $\beta \neq 0$, *then*

$$\frac{\beta}{u_2(b)} \int_x^b A(t)u_1(t)\,dt > 0$$

over $a \leq x < b$.

Proof. By Lemma 3, $\int_x^b A(t)u_1(t)\,dt$ has the same sign over $a \leq x < b$ as

$\int_{q_f}^{b} A(t)u_1(t)dt$, where q_f is the last zero of $u_1(x)$ on $a \leq x < b$ (or where $q_f = a$ if $u_1(x)$ has no such zero). From $K(u_2u_1' - u_2'u_1) \equiv 1$ we obtain $1/u_2(b) = K(b)u_1'(b)$. Hence

$$\frac{\beta}{u_2(b)} \int_{q_f}^{b} A(t)u_1(t)dt = K(b) \int_{q_f}^{b} [\beta A(t)][u_1'(b)u_1(t)]dt$$

and this latter expression is positive since the integrand is the product of two negative quantities.

Hereafter free use will be made of the facts that any solution of $(Ky')' - Gy = 0$ can have only a finite number of zeros on X and that, under the hypothesis $GK < 0$, the zeros of any two linearly independent solutions separate each other.

2. **Oscillation theorems.** Let $y_1(x)$ be any solution of $(Ky')' - Gy = A$ over X which satisfies the condition $y(b) = \beta$. Then $y_1(x)$ can be expressed in the form

$$y_1(x) = cu_1(x) + \frac{\beta}{u_2(b)} u_2(x) + u_1(x) \int_a^x A(t)u_2(t)dt$$

$$+ u_2(x) \int_x^b A(t)u_1(t)dt,$$

where $u_1(x)$ and $u_2(x)$ are as in Lemma 4, and c is a constant. We shall prove the following result.

THEOREM 1. *Under the hypotheses (H) the zeros of $y_1(x)$ and $u_1(x)$ separate each other on $a \leq x < b$.*

[If $\beta \neq 0$ the restriction that A/G be strictly increasing or decreasing may be modified to the extent of allowing A/G to be a monotone increasing or decreasing function. Under the modified hypotheses it can be shown that

$$\frac{\beta}{u_2(b)} \int_x^b A(t)u_1(t)dt \geq 0,$$

and since $\beta/u_2(b)$ is not zero the proof of the theorem is still valid.]

Proof. The functions $y_1(x)$ and $u_1(x)$ cannot vanish simultaneously on X except at b; for, letting q be a zero of $y_1(x)$ and $u_1(x)$ one obtains

$$y_1(q) = u_2(q) \left[\int_q^b A(t)u_1(t)dt + \beta/u_2(b)\right] = 0.$$

This is impossible since $u_2(q) \neq 0$ and, by Lemmas 3 (if $\beta = 0$) and 4 (if $\beta \neq 0$), the expression in brackets never vanishes.

Suppose now that q and $q' \neq b$, $(q < q')$, are consecutive zeros of $u_1(x)$, and that $y_1(x)$ does not vanish at any point of $q < x < q'$. Then, by Rolle's Theorem, $[u_1(x)/y_1(x)]'$ must vanish at least once in this interval. But

$$\left[\frac{u_1(x)}{y_1(x)}\right]' = \frac{\int_x^b A(t)u_1(t)dt + \beta/[u_2(b)]}{K(x)y_1^2(x)}$$

and, as above, this expression never vanishes.

In a similar manner it can be shown that between two consecutive zeros of $y_1(x)$, $u_1(x)$ must vanish at least once.

COROLLARY 1. *If $\beta \neq 0$, the zeros of $y_1(x)$ and $u_1(x)$ separate each other on $a \leq x \leq b$.*

Proof. If $\beta \neq 0$, the above argument is valid with $q' = b$.

COROLLARY 2. *If $u_1(x)$ has m zeros on X, then $y_1(x)$ has either $m-1$, m, or $m+1$ zeros on X.*

Proof. Let q_0 be the first zero of $u_1(x)$ on X and q_f be the last zero of $u_1(x)$ preceding b; $y_1(x)$ may or may not have a zero in $a \leq x < q_0$. In the interval $q_0 < x < q_f$, $y_1(x)$ has exactly $m-2$ zeros. If $\beta \neq 0$, then $y_1(x)$ has exactly one zero in $q_f < x < b$ by Corollary 1. If $\beta = 0$, $y_1(x)$ may or may not vanish in $q_f < x < b$. (See Theorem 4.)

The next theorem is applicable only if the system $(Ky')' - Gy = 0$, $y(a) = y(b) = 0$ is incompatible. In this case (i) one can select linearly independent solutions $u_1(x)$ and $u_2(x)$ of $(Ky')' - Gy = 0$ such that $u_1(b) = u_2(a) = 0$ and $K(u_2u_1' - u_2'u_1) \equiv 1$ on X and (ii) the nonhomogeneous system $(Ky')' - Gy = A$, $y(a) = 0$, $y(b) = \beta$ has a solution, say $y_2(x)$. We then have the following result.

THEOREM 2. *Let the hypotheses (H) be satisfied. Assume that $u_2(x)$ oscillates on X, and let $a = p_1, p_2, \cdots, p_m$ $(m > 3)$ be its consecutive zeros. Then, for $i \neq 1$, $y_2(p_i) \neq 0$ and either $y_2(x)$ has two zeros in (p_i, p_{i+1}) and none in (p_{i+1}, p_{i+2}) $(2 \leq i \leq m-2)$, or vice versa. In the interval $a < x < p_2$, $y_2(x)$ has either no zero or one zero. In the former case it has two zeros in (p_2, p_3), in the latter case it has no zero in (p_2, p_3). If $y_2(x)$ has two zeros in (p_{m-1}, p_m), it has no zero in $p_m < x < b$.*

Proof. The function $y_2(x)$ can be expressed in the form

$$y_2(x) = \frac{\beta}{u_2(b)}\, u_2(x) + u_1(x) \int_a^x A(t)u_2(t)dt + u_2(x) \int_x^b A(t)u_1(t)\,dt.$$

If $y_2(x)$ has three or more zeros in (p_i, p_{i+1}) $(2 \le i \le m-1)$, Theorem 1 requires that $u_1(x)$ have more than one zero in that interval; this is impossible since the zeros of $u_1(x)$ and $u_2(x)$ separate each other. Also, $y_2(x)$ cannot have a single zero in (p_i, p_{i+1}) $(2 \le i \le m-1)$, for then $y_2(p_i)y_2(p_{i+1}) < 0$ and such a product is always positive. To see this, notice that

$$y_2(p_i) = u_1(p_i) \int_a^{p_i} A(t)u_2(t)dt = u_1(p_i)F(p_i),$$

where $F(x) = \int_a^x A(t)u_2(t)\,dt$ as in Lemma 2. Since the zeros of both $u_1(x)$ and $F(x)$ separate those of $u_2(x)$, the product $u_1(p_i)F(p_i)$ $(2 \le i \le m)$ is consistently positive or negative. Thus $y_2(p_i)y_2(p_{i+1}) > 0$ $(2 \le i \le m-1)$.

The function $u_1(x)$ has a zero in each of (p_i, p_{i+1}) and (p_{i+1}, p_{i+2}) $(1 \le i \le m-2)$. By Theorem 1, $y_2(x)$ must have a zero in (p_i, p_{i+2}). If $y_2(x)$ has no zero in (p_i, p_{i+1}), it must have one, and therefore two, in (p_{i+1}, p_{i+2}). Now assume that $y_2(x)$ has two zeros in (p_i, p_{i+1}). If $y_2(x)$ also has two zeros in (p_{i+1}, p_{i+2}), then $u_1(x)$ must have three zeros in (p_i, p_{i+2}); but this is impossible. Hence $y_2(x)$ has no zero in (p_{i+1}, p_{i+2}).

This same type of argument can be used to prove the part of the theorem pertaining to the interval $a < x < p_2$ and the interval $p_m < x < b$.

REMARK. Theorems 1 and 2 are not true in case $\beta \neq 0$ without the restriction $\beta A(x) < 0$, $x > a$. This is shown by the example

$$\left(\frac{1}{x}\, y'\right)' + xy = -x^3, \quad y(0) = 0, \quad y\left(\sqrt{9\pi}\right) = -9\pi.$$

Here $\beta A(x) = 9\pi x^3 > 0$ on $0 < x < \sqrt{9\pi}$. The solution of the given system is $y(x) = -x^2$, which does not oscillate. However, each of $u_1(x) = -\cos(x^2/2)$, $u_2(x) = \sin(x^2/2)$ has five zeros on $0 \le x \le \sqrt{9\pi}$.

3. **Application to a system involving a parameter.** It will now be supposed that K, G, and A are continuous functions of (x, λ) when $a \le x \le b$, $\Lambda_1 < \lambda < \Lambda_2$. The system

$$[K(x,\lambda)y']' - G(x,\lambda)y = 0, \; y(a,\lambda) = 0, \; y(b,\lambda) = 0,$$

is a system of Sturmian type. Let K and G satisfy conditions sufficient to assure the validity of known oscillation theorems for this system [1, p. 66] to the extent that there exists an infinite set of characteristic numbers λ_i, $\Lambda_1 < \lambda_0 < \cdots < \lambda_m < \cdots < \Lambda_2$, having no limit point except Λ_2, and such that if u_m is the characteristic function corresponding to λ_m then u_m has m zeros in $a < x < b$.

Let $v_2(x, \lambda)$ be the solution of

$$[K(x, \lambda)y']' - G(x, \lambda)y = 0$$

satisfying the initial conditions $v_2(a, \lambda) \equiv 0$, $v_2'(a, \lambda) \equiv \sigma$, where σ is a positive constant. By the fundamental existence theorem [1, p. 7], $v_2(x, \lambda)$ is a continuous function of x and λ. It is well known [2, pp. 229, 232] that as λ increases from Λ_1 a new zero of $v_2(x, \lambda)$ appears at b for $\lambda = \lambda_i$ $(i = 0, 1, \cdots)$, and that each such zero moves continuously towards a as λ increases continuously.

For each λ, let $v_1(x, \lambda)$ be a solution of

$$[K(x, \lambda)y']' - G(x, \lambda)y = 0$$

satisfying the condition $v_1(b, \lambda) = 0$. If $\lambda = \lambda_i$ $(i = 0, 1, \cdots)$, $v_1(x, \lambda)$ is simply a constant multiple of $v_2(x, \lambda)$. For $\lambda \neq \lambda_i$, $v_1(x, \lambda)$ and $v_2(x, \lambda)$ are linearly independent. It follows that on X, for $\lambda < \lambda_0$, $v_1(x, \lambda)$ has a zero only at b; for $\lambda_m \leq \lambda < \lambda_{m+1}$ $(m = 0, 1, \cdots)$, $v_1(x, \lambda)$ has $m + 2$ zeros. Theorem 1 and its corollaries apply to give the following result.

THEOREM 3. *Let the system*

$$[K(x, \lambda)y']' - G(x, \lambda)y = A(x, \lambda), \quad y(b, \lambda) = \beta(\lambda),$$

for each fixed λ *in* (Λ_1, Λ_2), *satisfy the hypotheses* (*H*). *Let* $y_1(x, \lambda)$ *be a solution. Over* X: $a \leq x \leq b$, *if*

$\beta(\lambda) \neq 0$ *and* $\lambda < \lambda_0$, *then* $y_1(x, \lambda)$ *has either no zero or one zero,*

$\lambda = \lambda_m$ $(m \geq 0)$, *then* $y_1(x, \lambda)$ *has* $m + 1$ *zeros,*

$\lambda_m < \lambda < \lambda_{m+1}$ $(m \geq 0)$, *then* $y_1(x, \lambda)$ *has* $m + 1$ *or* $m + 2$ *zeros;*

$\beta(\lambda) = 0$ *and* $\lambda < \lambda_0$, *then* $y_1(x, \lambda)$ *has either one zero or two zeros,*

$\lambda = \lambda_m$ $(m \geq 0)$, *then* $y_1(x, \lambda)$ *has* $m + 1$ *or* $m + 2$ *zeros,*

$\lambda_m < \lambda < \lambda_{m+1}$ $(m \geq 0)$ *then* $y_1(x, \lambda)$ *has* $m + 1$, $m + 2$, *or* $m + 3$ *zeros.*

COROLLARY. *As λ increases in (Λ_1, Λ_2) the number of zeros of the solutions $y_1(x, \lambda)$ increases indefinitely.*

Interesting and more precise results can be obtained in connection with the two-point system

$$(S_\lambda) \qquad \begin{aligned} &[K(x, \lambda)y']' - G(x, \lambda)y = A(x, \lambda), \\ &y(a, \lambda) = 0, \ \ y(b, \lambda) = 0, \end{aligned}$$

where K, G, A conform to the hypotheses (H) for each fixed λ in (Λ_1, Λ_2). If $\lambda = \lambda_i$ $(i = 0, 1, \cdots)$, then (S_λ) is of course incompatible. Otherwise, for each λ one can choose $v_1(x, \lambda)$ such that

$$v_1(b, \lambda) = 0, \ \ v_1'(b, \lambda) = \frac{1}{K(b, \lambda)v_2(b, \lambda)},$$

so that

$$K(x, \lambda)[v_2(x, \lambda)v_1'(x, \lambda) - v_2'(x, \lambda)v_1(x, \lambda)]$$
$$\equiv K(b, \lambda)[v_2(b, \lambda)v_1'(b, \lambda)] \equiv 1$$

on X. It follows that $v_1(a, \lambda) = -1/[\sigma K(a, \lambda)]$ is negative for all λ.

The solution of (S_λ) can be expressed as

$$y_2(x, \lambda) = v_1(x, \lambda) \int_a^x A(t, \lambda)v_2(t, \lambda)\,dt$$
$$+ v_2(x, \lambda) \int_x^b A(t, \lambda)v_1(t, \lambda)\,dt.$$

We now consider an interval L_m: $\lambda_m < \lambda < \lambda_{m+1}$, and let X_0 represent the interval $a < x < b$. For a fixed λ in L_m, each of $v_1(x, \lambda)$ and $v_2(x, \lambda)$ has $m + 1$ zeros on X_0. For the sake of definiteness let $A(x, \lambda)$ be positive over $X_0 L_m$; and let m be odd so that, for λ in L_m, $v_1(x, \lambda)$ and $v_2(x, \lambda)$ each has an even number, $m + 1$, of zeros in X_0. Then $v_1'(b, \lambda) > 0$ and, by virtue of Lemma 3,

$$y_2'(a, \lambda) = \sigma \int_a^b A(t, \lambda)v_1(t, \lambda)\,dt$$

is negative. Let $q_f(\lambda)$ represent the last zero of $v_1(x, \lambda)$ preceding b. By Theorem 1 it follows that $y_2[q_f(\lambda), \lambda]$ is positive over L_m. However, $y_2'(b, \lambda)$ is negative for λ sufficiently close to λ_m, positive for λ sufficiently close to λ_{m+1}, because

$$y_2'(b, \lambda) = v_1'(b, \lambda) \int_a^b A(t, \lambda) v_2(t, \lambda)\, dt,$$

and by Lemma 3

$$\int_a^b A(t, \lambda_m) v_2(t, \lambda_m)\, dt < 0 \quad \text{and} \quad \int_a^b A(t, \lambda_{m+1}) v_2(t, \lambda_{m+1})\, dt > 0.$$

Since $y_2(x, \lambda)$ is a continuous function of (x, λ) over XL_m [1, p. 114], it follows that there exist $\epsilon_m > 0$ and $\epsilon_{m+1} > 0$ such that for $\lambda' - \lambda_m < \epsilon_m$ and $\lambda_{m+1} - \lambda'' < \epsilon_{m+1}$, $y_2(x, \lambda')$ has no zero in $q_f(\lambda') < x < b$, and $y_2(x, \lambda'')$ has one zero in $q_f(\lambda'') < x < b$.

A similar argument can be made in case m is even or in case $A(x, \lambda)$ is negative over X_0. This proves the following result.

THEOREM 4. *Let (S_λ) satisfy the hypotheses (H) for each λ in (Λ_1, Λ_2). On X_0, $y_2(x, \lambda)$ has m zeros for λ sufficiently close to λ_m, $m + 1$ zeros for λ sufficiently close to λ_{m+1} $(m = 0, 1, 2, \cdots)$.*

Letting $p_0(\lambda)$ be the first zero of $v_2(x, \lambda)$ to the right of a, one readily sees that

$$y_2[p_0(\lambda), \lambda] = v_1[p_0(\lambda), \lambda] \int_a^{p_0(\lambda)} A(t, \lambda) v_2(t, \lambda)\, dt$$

is positive or negative according as A is positive or negative. If $A > 0$, then $y_2'(a, \lambda) = \sigma \int_a^b A(t, \lambda) v_1(t, \lambda)\, dt$ is positive or negative over L_m according as m is even or odd. If $A < 0$, $y_2'(a, \lambda)$ is negative or positive over L_m according as m is even or odd. If one uses these relations as well as Theorem 1 and Theorem 2 to sketch graphically several typical cases, he obtains a striking illustration of the effect of the discontinuities of the function $y_2(x, \lambda)$ at the characteristic values of λ. Finally, one may observe that, regardless of the sign of A, for an even value of m the first zero of $v_2(x, \lambda)$ on $a < x < b$ precedes the first zero of $y_2(x, \lambda)$, and for an odd value of m the opposite is the case.

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