FORCES ON THE BOUNDARY OF A DIELECTRIC

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1. Introduction. It has been shown [1, ch. VII] that the component parallel to the axis of x of the resultant force on the matter inside any closed surface S_1 drawn in a medium of specific inductive capacity K is given by

$$X = -\iint_{S_1} (l P_{xx} + m P_{xy} + n P_{xz})\, dS,$$

where (l, m, n) are the direction-cosines of the normal to the surface,

$$P_{xx} = \frac{K}{8\pi} (\overline{X}^2 - \overline{Y}^2 - \overline{Z}^2),$$

$$P_{xy} = \frac{K}{4\pi} \overline{X}\,\overline{Y},$$

$$P_{xz} = \frac{K}{4\pi} \overline{X}\,\overline{Z},$$

and $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are given in terms of the potential by $-\partial\phi/\partial x$, $-\partial\phi/\partial y$, $-\partial\phi/\partial z$, respectively, provided the effect of electrostriction is neglected.

If any other surface S_2 is taken, surrounding S_1, and if

$$\frac{\partial P_{xx}}{\partial x} + \frac{\partial P_{xy}}{\partial y} + \frac{\partial P_{xz}}{\partial z} = 0$$

at all points between the surfaces, that is to say provided $\nabla^2\phi = 0$ at all such points, then, by Green's theorem,

$$X = -\iint_{S_2} (l P_{xx} + m P_{xy} + n P_{xz})\, dS,$$

and similarly for the other components of the resultant force on the matter inside S_1.

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This method can be used to find the resultant forces caused by the refraction of the lines of force at a surface of discontinuity separating one medium of specific inductive capacity K_i from a second of specific inductive capacity K_0.

2. Two-dimensional fields. Instead of applying the foregoing method to two-dimensional fields, we can best obtain the results by using the complex potential. Let

$$\Omega_0 \equiv \phi_0 + i\psi_0$$

be the complex potential of the field in the dielectric K_0. The components of the resultant force on the boundary C are then given by

$$Y_0 + iX_0 = \frac{K_0}{8\pi}\int_C \left(\frac{d\Omega_0}{dz}\right)^2 dz,$$

and the couple Γ_0 is the real part of

$$-\frac{K_0}{8\pi}\int_C \left(\frac{d\Omega_0}{dz}\right)^2 z\,dz.$$

These results follow from the equations of the Introduction with $Z = 0$. The details are omitted since the proof is identical with that of the well-known theorem of Blasius [3, p. 163; 2, p. 91] in fluid flow. The substitution of

$$\Omega_0 = \sum_{n=1}^{p} \frac{c_n - ic_n'}{n} z^n + \sum_{n=1}^{\infty} \frac{d_n + id_n'}{n} z^{-n}$$

and

$$\Omega_i = \sum_{n=1}^{p'} \frac{a_n - ia_n'}{n} z^n + \sum_{n=1}^{\infty} \frac{b_n + ib_n'}{n} z^{-n},$$

and separation into real and imaginary parts, yields the explicit forms

$$\text{(1a)} \quad X = \frac{K_0}{2}\sum_{n=1}^{p-1}(d_n c_{n+1} + d_n' c_{n+1}') - \frac{K_i}{2}\sum_{n=1}^{p'-1}(b_n a_{n+1} + b_n' a_{n+1}'),$$

$$\text{(1b)} \quad Y = \frac{K_0}{2}\sum_{n=1}^{p-1}(d_n c_{n+1}' - d_n' c_{n+1}) - \frac{K_i}{2}\sum_{n=1}^{p'-1}(b_n a_{n+1}' - b_n' a_{n+1}),$$

$$(1c)\qquad \Gamma = \frac{K_0}{2} \sum_{n=1}^{p} (c_n d_n' - c_n' d_n) - \frac{K_i}{2} \sum_{n=1}^{p'} (a_n b_n' - a_n' b_n).$$

CIRCULAR CYLINDER IN A GENERAL FIELD. If a circular cylinder of radius a, filled with homogeneous dielectric of specific inductive capacity K, is placed at the origin of coordinates in a two-dimensional field whose complex potential is $f(z)$ in air, having no singularities inside or on $r = a$, then the complex potentials inside and outside the cylinder are respectively

$$(2)\qquad \begin{cases} \Omega_i = \dfrac{2}{(1+K)} f(z), \\[2ex] \Omega_0 = f(z) + \dfrac{(1-K)}{(1+K)} \bar{f}\left(\dfrac{a^2}{z}\right). \end{cases}$$

It is assumed that there are no other boundaries present, and that the field is caused by isolated singularities (charges, dipoles, etc.). The result can easily be obtained by considering the boundary conditions. Note that by putting $K = 0$ in Ω_0 above we obtain the Circle Theorem [4, p. 84].

If the original real potential is taken to be

$$\phi(r, \theta) = \sum_{n=1}^{p} \left(\frac{E_n r^n \cos n\theta}{n} + \frac{E_n' r^n \sin n\theta}{n} \right),$$

then the potentials inside and outside the dielectric are

$$\phi_i = \frac{2}{(1+K)} \phi(r, \theta)$$

and

$$\phi_0 = \phi(r, \theta) + \frac{(1-K)}{(1+K)} \phi\left(\frac{a^2}{r}, \theta\right).$$

Thus with the above notation we have

$$a_n = \frac{2E_n}{(1+K)},\ a_n' = \frac{2E_n'}{(1+K)},\ b_n = b_n' = 0,$$

$$c_n = E_n,\ c_n' = E_n',\ d_n = \frac{(1-K)}{(1+K)} a^{2n} E_n,\ d_n' = \frac{(1-K)}{(1+K)} a^{2n} E_n'.$$

Hence the resultant forces on the boundary are given by

$$(3)\quad \begin{cases} X = \dfrac{(1-K)}{2(1+K)} \displaystyle\sum_{n=1}^{p-1} a^{2n} (E_n E_{n+1} + E_n' E_{n+1}'), \\ Y = \dfrac{(1-K)}{2(1+K)} \displaystyle\sum_{n=1}^{p-1} a^{2n} (E_n E_{n+1}' - E_n' E_{n+1}), \\ \Gamma = 0. \end{cases}$$

Equations (2) can be extended in the form of infinite series to the case where there is also present a conducting surface $r = b$ ($b < a$). Infinite series are also obtained when $r = b$ is a line of flow. These two cases can then be used to obtain results for a dielectric elliptic cylinder.

3. Three-dimensional fields. In spherical polar coordinates (r, θ, ω), the components of force are

$$(4)\quad \begin{cases} Z = \iint (F \cos\theta - G \sin\theta)\, dS, \\ Y = \iint [(F \sin\theta + G \cos\theta) \sin\omega + H \cos\omega]\, dS, \\ X = \iint [(F \sin\theta + G \cos\theta) \cos\omega - H \sin\omega]\, dS, \end{cases}$$

where

$$F = F_i - F_0,\; G = G_i - G_0,\; H = H_i - H_0,$$

$$F_0 = \frac{K_0}{8\pi} \left\{ \left(\frac{\partial\phi_0}{\partial r}\right)^2 - \left(\frac{\partial\phi_0}{r\partial\theta}\right)^2 - \left(\frac{\partial\phi_0}{r \sin\theta\, \partial\omega}\right)^2 \right\},$$

$$G_0 = \frac{K_0}{4\pi} \left(\frac{\partial\phi_0}{\partial r}\right) \left(\frac{\partial\phi_0}{r\partial\theta}\right),$$

$$H_0 = \frac{K_0}{4\pi} \left(\frac{\partial\phi_0}{\partial r}\right) \left(\frac{\partial\phi_0}{r \sin\theta\, \partial\omega}\right),$$

with similar expressions for F_i, G_i, H_i. As before, ϕ_0 is the potential inside the dielectric K_0. The integration is performed over the sphere of radius r.

The couple components are

$$(5)\quad \begin{cases} N = \iint H r \sin\theta\, dS, \\ M = \iint [G r \cos\omega - H r \sin\omega \cos\theta]\, dS, \\ L = \iint [H r \cos\omega \cos\theta - G r \sin\omega]\, dS, \end{cases}$$

Considering components due to F_0, G_0, H_0 only, and making the change of variable

$$\mu = -\cos\theta,\ dS = r^2\, d\omega\, d\mu,$$

we obtain

$$(6)\qquad \frac{8\pi Z_0}{K_0}$$

$$\int_0^{2\pi}\int_{-1}^{1}\left\{(1-\mu^2)\left(\frac{\partial\phi_0}{\partial\mu}\right)^2 + \frac{1}{(1-\mu^2)}\left(\frac{\partial\phi_0}{\partial\omega}\right)^2 - r^2\left(\frac{\partial\phi_0}{\partial r}\right)^2\right\}\mu d\omega\, d\mu$$

$$- 2\int_0^{2\pi}\int_{-1}^{1}\left(\frac{\partial\phi_0}{\partial r}\right)\left(\frac{\partial\phi_0}{\partial\mu}\right)(1-\mu^2)\, r\, d\omega\, d\mu,$$

and proceed similarly for X_0, Y_0, N_0, M_0, L_0.

These integrals can be evaluated if the potential ϕ_0 is expanded with the usual notation [1, ch. VII, p. 239, and elsewhere], in the form

$$(7)\qquad \phi_0 = \sum_{n=1}^{p} r^n S_n + \sum_{n=1}^{\infty} \frac{W_n}{r^{n+1}},$$

where

$$(8)\qquad S_n = A_{0,n} P_n + \sum_{s=1}^{n} (A_{s,n} \cos s\omega + B_{s,n} \sin s\omega)\, P_n^s,$$

$$(9)\qquad W_n = a_{0,n} P_n + \sum_{s=1}^{n} (a_{s,n} \cos s\omega + b_{s,n} \sin s\omega)\, P_n^s,$$

and P_n^s satisfies the differential equation

$$\frac{d}{d\mu}\left\{(1-\mu^2)\frac{dP_n^s}{d\mu}\right\} = -\left\{n(n+1) - \frac{s^2}{(1-\mu^2)}\right\} P_n^s.$$

With the usual notation for associated Legendre functions of the first kind, we have

$$P_n^s = (1-\mu^2)^{s/2} \frac{d^s P_n}{d\mu^s} = \frac{1}{2^n\, n!} (1-\mu^2)^{s/2} \frac{d^{n+s}}{d\mu^{n+s}} (\mu^2-1)^n .$$

The potential ϕ_i has a similar expansion.

The recurrence and integral formulae used are

(a) $$\sqrt{(1-\mu^2)}\; P_n^s = \frac{1}{(2n+1)} (P_{n+1}^{s+1} - P_{n-1}^{s+1}),$$

(b) $$(2n+3)\,\mu P_{n+1}^s = (n+s+1) P_n^s + (n-s+2) P_{n+2}^s,$$

(c) $$2s\mu\, P_{n+1}^s = \sqrt{(1-\mu^2)}\; P_{n+1}^{s+1} + (n+s+1)(n-s+2)\sqrt{(1-\mu^2)}\; P_{n+1}^{s-1},$$

(d) $$(n+s+1) P_n^s = (n-s+1)\,\mu P_{n+1}^s + \sqrt{(1-\mu^2)}\; P_{n+1}^{s+1},$$

(e) $$(1-\mu^2)\frac{dP_{n+1}^s}{d\mu} = (n+2)\,\mu P_{n+1}^s - (n-s+2) P_{n+2}^s$$

$$= (n+s+1) P_n^s - \mu(n+1) P_{n+1}^s \cdot$$

$$= \sqrt{(1-\mu^2)}\; P_{n+1}^{s+1} - s\mu\, P_{n+1}^s,$$

(f) $$\int_{-1}^{1} P_n^s P_{n'}^s \, d\mu = \begin{cases} 0 \text{ if } n \neq n', \\ \dfrac{(n+s)!}{(n-s)!} \cdot \dfrac{2}{(2n+1)} \text{ if } n = n', \end{cases}$$

(g) $$\int_{-1}^{1} P_n^{s+1} P_{n'}^{s-1} \, d\mu = \begin{cases} 0 \text{ if } n' > n \text{ or } n-n' \text{ odd}, \\ -\dfrac{(n+s-1)!}{(n-s-1)!} \cdot \dfrac{2}{(2n+1)} \text{ if } n = n', \\ \dfrac{4s\,(n+s-3)!}{(n-s-1)!} \text{ if } n = n'+2, \end{cases}$$

(h) $$\int_{-1}^{1} \mu\, P_n^s P_{n'}^s \, d\mu = \begin{cases} 0 \text{ if } n' \neq n \pm 1, \\ \dfrac{(n+s+1)!}{(n-s)!} \cdot \dfrac{2}{(2n+1)(2n+3)} \text{ if } n' = n+1, \end{cases}$$

(i) $$\int_{-1}^{1} \frac{P_n^s P_{n+1}^{s+1}}{\sqrt{1-\mu^2}} \, d\mu = \frac{2(n+s)!}{(n-s)!},$$

(j) $$\int_{-1}^{1} \frac{P_n^s P_{n+1}^{s-1}}{\sqrt{1-\mu^2}} \, d\mu = 0,$$

(k) $$\int_{-1}^{1} \sqrt{1-\mu^2}\, P_n^s P_{n'}^{s+1} \, d\mu = \begin{cases} 0 \text{ if } n' \neq n \pm 1, \\ \dfrac{2}{(2n+1)(2n+3)} \cdot \dfrac{(n+s+2)!}{(n-s)!} \text{ if } n' = n+1, \end{cases}$$

(l) $$\int_{-1}^{1} \frac{P_{n+1}^s \; P_{n'+1}^s}{(1-\mu^2)} \, d\mu = \begin{cases} \dfrac{(n+s-1)!}{s(n-s-1)!} \text{ if } n = n'+2, \\ \dfrac{(n+s+1)!}{s(n-s+1)!} \text{ if } n = n'. \end{cases}$$

Some of these formulae may be found in textbooks [1].

The Z force is given by

$$Z = Z_i - Z_0,$$

where Z_0 is given by (6) above with a similar expression for Z_i. Consider, first, the integral

$$\begin{aligned} I_1 &= \int_0^{2\pi} \int_{-1}^{1} \mu(1-\mu^2) \left(\frac{\partial \phi_0}{\partial \mu}\right)^2 d\omega \, d\mu \\ &= \int_0^{2\pi} \int_{-1}^{1} \mu(1-\mu^2) \left[\sum_{n=1}^{p} r^n S_n' + \sum_{n=1}^{\infty} \frac{W_n'}{r^{n+1}}\right]^2 d\omega \, d\mu, \end{aligned}$$

where, from (8) and (9),

$$S_n' = A_{0,n} \frac{\partial P_n}{\partial \mu} + \sum_{s=1}^{n} (A_{s,n} \cos s\omega + B_{s,n} \sin s\omega) \frac{\partial P_n^s}{\partial \mu}$$

and

$$\mathbb{W}'_n = a_{0,n} \frac{\partial P_n}{\partial \mu} + \sum_{s=1}^{n} (a_{s,n} \cos s\omega + b_{s,n} \sin s\omega) \frac{\partial P_n^s}{\partial \mu} .$$

Now the integral has to be independent of r, so that

$$I_1 = \int_0^{2\pi} \int_{-1}^{1} 2\mu(1-\mu^2) \sum_{n=1}^{p-1} S'_{n+1} \mathbb{W}'_n \, d\omega \, d\mu$$

$$= 2\pi \sum_{n=1}^{p-1} \int_{-1}^{1} \left[2A_{0,n+1} a_{0,n+1} \frac{\partial P_{n+1}}{\partial \mu} \frac{\partial P_n}{\partial \mu} \right.$$

$$\left. + \sum_{s=1}^{n} (A_{s,n+1} a_{s,n} + B_{s,n+1} b_{s,n}) \frac{\partial P_{n+1}^s}{\partial \mu} \frac{\partial P_n^s}{\partial \mu} \right] \mu(1-\mu^2) \, d\mu .$$

But

$$\int_{-1}^{1} \mu(1-\mu^2) \frac{\partial P_{n+1}^s}{\partial \mu} \frac{\partial P_n^s}{\partial \mu} d\mu \tag{10}$$

$$= \int_{-1}^{1} (1-\mu^2) \frac{\partial P_n^s}{\partial \mu} \left[\left(\frac{n+s+1}{2n+3} \right) \frac{\partial P_n^s}{\partial \mu} + \left(\frac{n-s+2}{2n+3} \right) \frac{\partial P_{n+2}^s}{\partial \mu} \right] d\mu$$

$$+ (n-s+1) \int_{-1}^{1} P_{n+1}^s P_{n+1}^s \, d\mu - (n+1) \int_{-1}^{1} \mu P_n^s P_{n+1}^s \, d\mu \text{ (by } (b),(e))$$

$$= \left(\frac{n+s+1}{2n+3} \right) \int_{-1}^{1} P_n^s \left[n(n+1) - \frac{s^2}{(1-\mu^2)} \right] P_n^s \, d\mu$$

$$+ \left(\frac{n-s+2}{2n+3} \right) \int_{-1}^{1} P_n^s \left[(n+2)(n+3) - \frac{s^2}{(1-\mu^2)} \right] P_{n+2}^s \, d\mu$$

$$+ (n-s+1) \frac{(n+s+1)!}{(n-s+1)!} \cdot \frac{2}{(2n+3)}$$

$$- (n+1) \frac{(n+s+1)!}{(n-s)!} \cdot \frac{2}{(2n+1)(2n+3)} \qquad \text{(by } (f), (h))$$

$$= \frac{(n+s+1)\, n(n+1)}{(2n+3)} \int_{-1}^{1} P_n^s P_n^s \, d\mu - s^2 \int_{-1}^{1} \frac{\mu P_n^s P_{n+1}^s}{(1-\mu^2)} d\mu$$

$$+ \frac{(n+s+1)!}{(n-s)!} \frac{2n}{(2n+3)(2n+1)} ,$$

which gives an expression for I_1.

In a similar manner,

$$(11)\quad I_2 = \int_0^{2\pi}\int_{-1}^{1} r^2 \left(\frac{\partial \phi_0}{\partial r}\right)^2 \mu\, d\omega\, d\mu$$

$$= -2\pi \sum_{n=1}^{p-1} (n+1)^2 \int_{-1}^{1} \Bigg[2A_{0,n+1}\, a_{0,n}\, P_{n+1}\, P_n + \sum_{s=1}^{n} (A_{s,n+1}\, a_{s,n} + B_{s,n+1}\, b_{s,n})\, P_{n+1}^s\, P_n^s \Bigg] \mu\, d\mu,$$

$$(12)\quad I_3 = \int_0^{2\pi}\int_{-1}^{1} \left(\frac{\partial \phi_0}{\partial \omega}\right)^2 \frac{\mu}{(1-\mu^2)}\, d\omega\, d\mu$$

$$= 2\pi \sum_{n=1}^{p-1} \int_{-1}^{1} \sum_{s=1}^{n} s^2 (A_{s,n+1}\, a_{s,n} + B_{s,n+1}\, b_{s,n})\, P_{n+1}^s\, P_n^s \frac{\mu}{(1-\mu^2)}\, d\mu,$$

$$(13)\quad I_4 = \int_0^{2\pi}\int_{-1}^{1} r \left(\frac{\partial \phi_0}{\partial \mu}\right)\left(\frac{\partial \phi_0}{\partial r}\right)(1-\mu^2)\, d\omega\, d\mu$$

$$= \pi \sum_{n=1}^{p-1} (n+1) \int_{-1}^{1} \Bigg[2A_{0,n+1}\, a_{0,n} \frac{\partial P_n}{\partial \mu} P_{n+1} + \sum_{s=1}^{n} (A_{s,n+1}\, a_{s,n} + B_{s,n+1}\, b_{s,n}) \frac{\partial P_n^s}{\partial \mu} P_{n+1}^s \Bigg] (1-\mu^2)\, d\mu$$

$$- \pi \sum_{n=1}^{p-1} (n+1) \int_{-1}^{1} \Bigg[2A_{0,n+1}\, a_{0,n} \frac{\partial P_{n+1}}{\partial \mu} P_n + \sum_{s=1}^{n} (A_{s,n+1}\, a_{s,n} + B_{s,n+1}\, b_{s,n}) \frac{\partial P_{n+1}^s}{\partial \mu} P_n^s \Bigg] (1-\mu^2)\, d\mu.$$

We see that from (6),

$$\frac{8\pi Z_0}{K_0} = I_1 - I_2 + I_3 - 2I_4,$$

from which, by using (10), (11), (12), (13), we get

$$(14)\qquad Z_0 = \frac{K_0}{2}\sum_{n=1}^{p-1}\left[2(n+1)\,A_{0,n+1}\,a_{0,n} + \sum_{s=1}^{n}(A_{s,n+1}\,a_{s,n} + B_{s,n+1}\,b_{s,n})\,\frac{(n+s+1)!}{(n-s)!}\right].$$

This determines Z, for the component Z_i can be written down in terms of the coefficients of the expansion of ϕ_i. In the same way, when the components Y_0, X_0, N_0, M_0, L_0 are determined then the components Y, X, N, M, L may be written down.

By using the same method as for Z_0 above, the following results are obtained:

$$(15)\qquad Y_0 = \frac{K_0}{4}\sum_{n=1}^{p-1}\left[2(n+1)(n+2)\,B_{1,n+1}\,a_{0,n} - 2n(n+1)\,A_{0,n+1}\,b_{1,n} - \sum_{s=1}^{n}\frac{(n+s+2)!}{(n-s)!}(A_{s+1,n+1}\,b_{s,n} - B_{s+1,n+1}\,a_{s,n}) - \sum_{s=1}^{n-1}\frac{(n+s+1)!}{(n-s-1)!}(A_{s,n+1}\,b_{s+1,n} - B_{s,n+1}\,a_{s+1,n})\right],$$

$$(16)\qquad X_0 = \frac{K_0}{4}\sum_{n=1}^{p-1}\left[2(n+1)(n+2)\,A_{1,n+1}\,a_{0,n} - 2n(n+1)\,A_{0,n+1}\,a_{1,n} + \sum_{s=1}^{n}\frac{(n+s+2)!}{(n-s)!}(A_{s+1,n+1}\,a_{s,n} + B_{s+1,n+1}\,b_{s,n}) - \sum_{s=1}^{n-1}\frac{(n+s+1)!}{(n-s-1)!}(A_{s,n+1}\,a_{s+1,n} + B_{s,n+1}\,b_{s+1,n})\right],$$

$$(17)\qquad N_0 = \frac{K_0}{2}\sum_{n=1}^{p}\sum_{s=1}^{n}\frac{s(n+s)!}{(n-s)!}(B_{s,n}\,a_{s,n} - A_{s,n}\,b_{s,n}),$$

$$(18)\qquad M_0 = \frac{K_0}{4}\sum_{n=1}^{p}\Bigl[2n(n+1)\,(A_{1,n}\,a_{0,n} - A_{0,n}\,a_{1,n})$$

$$+ \sum_{s=1}^{n-1} \frac{(n+s+1)!}{(n-s-1)!} (A_{s+1,n}\, a_{s,n} + B_{s+1,n}\, b_{s,n} - A_{s,n}\, a_{s+1,n} - B_{s,n}\, b_{s+1,n})\Bigg].$$

$$(19) \qquad L_0 = \frac{K_0}{4} \sum_{n=1}^{p} \Bigg[2n(n+1)(A_{0,n}\, b_{1,n} - B_{1,n}\, a_{0,n})$$

$$+ \sum_{s=1}^{n-1} \frac{(n+s+1)!}{(n-s-1)!} (A_{s+1,n}\, b_{s,n} + A_{s,n}\, b_{s+1,n} - B_{s+1,n}\, a_{s,n} - B_{s,n}\, a_{s+1,n})\Bigg].$$

It can be shown that the field becomes two-dimensional if

$$(20) \begin{cases} s \text{ even}, \quad b_{sn} = B_{s,n} = 0, \quad a_{s,n} = \dfrac{2(n-s)!}{n!}\, a_{0,n}, \quad A_{s,n} = \dfrac{2\,n!}{(n+s)!}\, A_{0,n}, \\[2ex] s \text{ odd}, \quad a_{sn} = A_{s,n} = 0, \quad b_{s,n} = \dfrac{(n-s)!}{(n-1)!}\, b_{1,n}, \quad B_{s,n} = \dfrac{(n+1)!}{(n+s)!}\, B_{1,n}. \end{cases}$$

SPHERE IN A GENERAL FIELD. If a sphere of radius a, filled with homogeneous dielectric of specific inductive capacity K, is placed in air, with its center at the origin of coordinates in any electrostatic field whose potential function is $\phi(x, y, z)$ having no singularities inside or on $r = a$, then the potential inside and outside the sphere are respectively

$$\phi_i = \frac{2}{(1+K)}\, \phi(x, y, z) + \frac{(K-1)}{(K+1)^2} \int_0^1 t^{-K/(K+1)}\, \phi(xt, yt, zt)\, dt$$

and

$$\phi_0 = \phi(x, y, z) - \frac{(K-1)}{(K+1)} \frac{a}{r}\, \phi(x_1, y_1, z_1)$$

$$+ \frac{(K-1)}{(K+1)^2} \frac{a}{r} \int_0^1 t^{-K/(K+1)}\, \phi(x_1 t, y_1 t, z_1 t)\, dt,$$

where

$$x_1 = \frac{a^2 x}{r^2}, \quad y_1 = \frac{a^2 y}{r^2}, \quad z_1 = \frac{a^2 z}{r^2} \quad \text{and} \quad r^2 = x^2 + y^2 + z^2.$$

It is assumed that there are no other boundaries present.

These results can be obtained either by the method of P. Weiss [6], or by expanding $\phi(x, y, z)$ in harmonics and using the boundary conditions

$$\phi_i = \phi_0, \quad K \frac{\partial \phi_i}{\partial r} = \frac{\partial \phi_0}{\partial r} \qquad \text{on } r = a.$$

If the original potential is given by

$$\phi(x, y, z) = \sum_{n=1}^{p} r^n S_n,$$

where

$$S_n = A_{0,n} P_n + \sum_{s=1}^{n} (A_{s,n} \cos s\omega + B_{s,n} \sin s\omega) P_n^s,$$

and if we assume the interference potential to be given by

$$\phi_1 = \sum_{n=1}^{\infty} \frac{W_n}{r^{n+1}},$$

where

$$W_n = a_{0,n} P_n + \sum_{s=1}^{n} (a_{s,n} \cos s\omega + b_{s,n} \sin s\omega) P_n^s,$$

then, by using the above result, we get

$$a_{0,n} = \begin{cases} \dfrac{n(1-K)}{n(1+K)+1} a^{2n+1} A_{0,n} & \text{if } n \leq p, \\ 0 & \text{if } n > p, \end{cases}$$

with similar expressions for $a_{s,n}$, $b_{s,n}$. The forces are thus

$$Z = \frac{1}{2} \sum_{n=1}^{p-1} \frac{n(K-1) a^{2n+1}}{n(K+1)+1} \Bigg[2(n+1) A_{0,n} A_{0,n+1}$$

$$+ \sum_{s=1}^{n} (A_{s,n} A_{s,n+1} + B_{s,n} B_{s,n+1}) \frac{(n+s+1)!}{(n-s)!} \Bigg],$$

$$X = \frac{1}{4}\sum_{n=1}^{p-1} \frac{n(K-1)\,a^{2n+1}}{n(K+1)+1}\Bigg[2(n+1)(n+2)\,A_{0,n}\,A_{1,n+1} - 2n(n+1)\,A_{1,n}\,A_{0,n+1}$$

$$+ \sum_{s=1}^{n} (A_{s,n}\,A_{s+1,n+1} + B_{s,n}\,B_{s+1,n+1})\,\frac{(n+s+2)!}{(n-s)!}$$

$$- \sum_{s=1}^{n-1} (A_{s+1,n}\,A_{s,n+1} + B_{s+1,n}\,B_{s,n+1})\,\frac{(n+s+1)!}{(n-s-1)!}\Bigg],$$

$$Y = \frac{1}{4}\sum_{n=1}^{p-1} \frac{n(K-1)\,a^{2n+1}}{n(K+1)+1}\Bigg[2(n+1)(n+2)\,A_{0,n}\,B_{1,n+1} - 2n(n+1)\,A_{0,n+1}B_{1,n}$$

$$- \sum_{s=1}^{n} (B_{s,n}\,A_{s+1,n+1} - A_{s,n}\,B_{s+1,n+1})\,\frac{(n+s+2)!}{(n-s)!}$$

$$- \sum_{s=1}^{n-1} (B_{s+1,n}\,A_{s,n+1} - B_{s,n+1}\,A_{s+1,n})\,\frac{(n+s+1)!}{(n-s-1)!}\Bigg],$$

$$N = M = L = 0.$$

The potential inside, being of the form

$$\phi_i = \sum_{n=1}^{p} r^n\,S_n,$$

contributes nothing to the forces.

The forces on bodies with surfaces $r = a + \epsilon P_n$ (ϵ small) can also be easily evaluated.

As an example [1, p. 290, ex. 31], take a positive point-charge e at the point $(0, 0, c)$, $c > a$. We have

$$A_{0,n} = \frac{e}{c^{n+1}}, \quad A_{s,n} = 0,$$

and so

$$Z = e^2 \sum_{n=1}^{\infty} \frac{n(n+1)(K-1)}{n(K+1)+1} \cdot \frac{a^{2n+1}}{c^{2n+3}}, \quad Y = 0, \quad X = 0.$$

The resultant attraction between the sphere and point-charge may thus be written

$$\frac{e^2a^3\alpha}{2c^3}\left\{\frac{2c^2}{(c^2-a^2)^2}+\frac{(1+\alpha)}{(c^2-a^2)}-\frac{c(1-\alpha^2)}{a^3}\left(\frac{a}{c}\right)^{\alpha}\int_0^{a/c}\frac{x^{2-\alpha}}{(1-x^2)}\,dx\right\},$$

where

$$\alpha=\frac{K-1}{K+1}.$$

Equations (14) - (19) can be used for the forces on a body in a liquid moving irrontationally and extending to infinity, simply by putting $K_0 = 4\pi\rho$. Elementary cases have been considered [5].

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