IDENTIFICATIONS IN SINGULAR HOMOLOGY THEORY

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INTRODUCTION

0.1. Given a Mayer complex M, a subcomplex M' is termed an *unessential identifier* for M if the natural projections from M onto the factor complex M/M' induce isomorphisms-onto on the homology level (see [1, § 1.2]). The present paper is a continuation and improvement of certain results obtained by Radó and Reichelderfer (see [1] and [3]) concerning unessential identifiers for the singular complex R of Radó (see [1, § 0.1]). We shall make use of the results, terminology, and notation in [1] and [3] with one exception. Because of a conflict in notation in [1] and [3], we shall use the notation η_p for the homomorphisms

$$\eta_p : C_p^S \longrightarrow C_p^R,$$

defined as the trivial homomorphism for $p < 0$, and for $p \geq 0$ as follows:

$$\eta_p(d_0, \cdots, d_p, T)^S = (d_0, \cdots, d_p, T)^R$$

(see [1, § 0.3]).

0.2. The principal results of the present paper may be described as follows. Let $N(\sigma_p \beta_p^R)$ denote the nucleus of the product homomorphism

$$\sigma_p \beta_p^R : C_p^R \longrightarrow C_p^S.$$

THEOREM. *The system* $\{N(\sigma_p \beta_p^R)\}$ *is an unessential identifier for* R.

Furthermore, for each p we have

$$N(\sigma_p \beta_p^R) \supset \hat{\Delta}_p^R \supset \hat{\Gamma}_p^R,$$

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where $\{\hat{\Delta}_p^R\}$ and $\{\hat{\Gamma}_p^R\}$ are the largest unessential identifiers for R obtained by Reichelderfer [3, §3.6] and Radó [1, §4.7], respectively. Thus $\{N(\sigma_p \beta_p^R)\}$ is the largest unessential identifier presently known for R and imposes all the classical identifications in R.

Let $N(\beta_p^S)$ denote the nucleus of the barycentric homomorphism

$$\beta_p^S : C_p^S \longrightarrow C_p^S.$$

THEOREM. *The system* $\{N(\beta_p^S)\}$ *is an unessential identifier for* S.

It is interesting to note that the foregoing theorem gives for the Eilenberg complex S the result corresponding to that of Reichelderfer for the Radó complex R (see [3, §3.2]).

I. PRELIMINARIES

1.1. Let $v_0, \cdots, v_p$ denote $p+1$ points in Hilbert space E_∞. The barycenter $b = b(v_0, \cdots, v_p)$ of these points is given by

$$b = (v_0 + \cdots + v_p)/(p+1).$$

The following lemmas are easily verified.

1.2. LEMMA. *Let* v_j $(j = 0, \cdots, p)$ *denote* $p+1$ *points in* E_∞, *and*

$$x = \sum_{j=0}^{p} \mu_j\, b(v_0, \cdots, v_j),$$

where μ_j *is real for* $j = 0, \cdots, p$. *Then*

$$x = \sum_{j=0}^{p} \sum_{l=j}^{p} \frac{\mu_l}{l+1} v_j, \quad \text{with} \quad \sum_{j=0}^{p} \sum_{l=j}^{p} \frac{\mu_l}{l+1} = \sum_{j=0}^{p} \mu_j.$$

1.3. LEMMA. *Let* v_j $(j=0, \cdots, p)$ *denote* $p+1$ *points in* E_∞, *and*

$$x = \sum_{j=0}^{p} \mu_j v_j,$$

with μ_j $(j = 0, \cdots, p)$ *real and satisfying*

$$\mu_0 \geq \mu_1 \geq \cdots \geq \mu_p \geq 0.$$

Then

$$x = \sum_{j=0}^{p} \lambda_j b(v_0 \cdots v_j),$$

with

$$\lambda_j = (j+1)(\mu_j - \mu_{j+1}) \text{ for } j = 0, \cdots, p-1 \text{ (provided } p-1 \geq 0),$$

$$\lambda_p = (p+1)\mu_p,$$

and

$$\sum_{j=0}^{p} \lambda_j = \sum_{j=0}^{p} \mu_j .$$

1.4. As in [1], let $d_0, d_1, d_2, \cdots$ denote the sequence of points $(1, 0, 0, 0, \cdots)$, $(0, 1, 0, 0, \cdots)$, $(0, 0, 1, 0, \cdots)$, $\cdots$ in E_∞. For integers p, q such that $p \geq 0$, $0 \leq q \leq p+1$, the homomorphism

$$q_{*p} : C_p \longrightarrow C_{p+1}$$

in the formal complex K of E_∞ is defined by the relation

$$q_{*p}(v_0, \cdots, v_p) = \begin{cases} (d_{p+1}, v_0, \cdots, v_p) & \text{for } q = 0, \\ (-1)^q (v_0, \cdots, v_{q-1}, d_{p+1}, v_q, \cdots, v_p) & \text{for } 1 \leq q \leq p, \\ (-1)^{p+1}(v_0, \cdots, v_p, d_{p+1}) & \text{for } q = p+1. \end{cases}$$

1.5. For $p \geq 0$, let τ_p denote an element of $T_{p\,0}$ (see [3, §1.9]), and let $(i_0, \cdots, i_p)$ denote the permutation of $0, \cdots, p$ which gives rise to τ_p. Then we let sgn τ_p denote the sign of the permutation $(i_0, \cdots, i_p)$: i.e., sgn τ_p is $+1$ or -1 according as an even or odd number of transpositions is required to obtain $(i_0, \cdots, i_p)$.

The following lemmas are then obvious.

1.6. LEMMA. *For $p \geq 0$ and $\tau_{p+1} \in T_{p+1\,0}$, there exists a unique $\pi_p \in T_{p0}$,* *and a unique q,* $0 \leq q \leq p+1$, *such that*

$$\tau_{p+1}(d_0, \cdots, d_{p+1}) = q_{*p}\, \pi_p\, (p+1)_{p+1}(d_0, \cdots, d_{p+1}).$$

1.7. LEMMA. *For* $p \geq 0$, *let* E_{p+1} *denote the set of ordered pairs* (q, π_p), $0 \leq q \leq p+1$, $\pi_p \in T_{p0}$. *There exists a biunique correspondence*

$$\xi : T_{p+1\,0} \longrightarrow E_{p+1}$$

with

$$\xi \tau_{p+1} = (q, \pi_p),$$

such that

$$\tau_{p+1}(d_0, \cdots, d_{p+1}) = q_{*p}\pi_p(p+1)_{p+1}(d_0, \cdots, d_{p+1})$$

and

$$\operatorname{sgn} \tau_{p+1} = (-1)^{p+q+1} \operatorname{sgn} \pi_p.$$

1.8. Let

$$h_p : C_p \longrightarrow C_q$$

denote a homomorphism in K such that

$$h_p(d_0 \cdots d_p) = \pm (w_0, \cdots, w_q).$$

Then $[h_p]$ will denote the usual affine mapping from the convex hull $|d_0, \cdots, d_q|$ of the points $d_0, \cdots, d_q$ onto the convex hull $|w_0, \cdots, w_q|$ of the points $w_0, \cdots, w_q$ such that $[h_p](d_i) = w_i$ for $i = 0, \cdots, q$.

1.9. Let β_p^R denote the barycentric homomorphism in R, and ρ_{*p}^R the barycentric homotopy operator in R of Reichelderfer (see [**3**, § 2.1]). The barycentric homomorphism

$$\beta_p^S : C_p^S \longrightarrow C_p^S$$

in S may be given by

$$\beta_p^S = \sigma_p\, \beta_p^R\, \eta_p \qquad \text{(see [2, § 3.7]).}$$

The corresponding homotopy operator

$$\rho_{*p}^{S} : C_p^S \longrightarrow C_{p+1}^S$$

is given by

$$\rho_{*p}^{S} = \sigma_{p+1}\, \rho_{*p}^{R}\, \eta_p ,$$

1.10. Employing the structure theorems for β_p^R, ρ_{*p}^R (see [3, § 2.2]) we obtain the following:

LEMMA. *For* $p \geq 0$,

$$\beta_p^S(d_0, \cdots, d_p, T)^S = \sum_{\tau_p \in T_{p0}} \operatorname{sgn} \tau_p (d_0, \cdots, d_p, T[0_{p+1}\, b_{p0}\, \tau_p])^S,$$

$$\rho_{*p}^S(d_0, \cdots, d_p, T)^S = \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} (-1)^k \operatorname{sgn} \tau_p (d_0, \cdots, d_{p+1}, T[b_{pk}\, \tau_p])^S.$$

Proof. We have

$$\beta_p^S(d_0, \cdots, d_p, T)^S = \sigma_p\, \beta_p^R(d_0, \cdots, d_p, T)^R$$

$$= \sigma_p \sum_{\tau_p \in T_{p0}} (0_{p+1}\, b_{p0}\, \tau_p(d_0, \cdots, d_p), T)^R$$

$$= \sum_{\tau_p \in T_{p0}} \operatorname{sgn} \tau_p (d_0, \cdots, d_p, T[0_{p+1}\, b_{p0}\, \tau_p])^S,$$

and

$$\rho_{*p}^S(d_0, \cdots, d_p, T)^S = \sigma_{p+1}\, \rho_{*p}^R(d_0, \cdots, d_p, T)^R$$

$$= \sigma_{p+1} \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} (b_{pk}\, \tau_p(d_0, \cdots, d_p), T)^R$$

$$= \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} (-1)^k \operatorname{sgn} \tau_p (d_0, \cdots, d_{p+1}, T[b_{pk}\, \tau_p])^S.$$

1.11. In [2], Radó makes use of the following identities which we state in terms of ρ_{*p}^R :

(1) $\sigma_{p+1}\, \rho_{*p}^R\, \eta_p\, \sigma_p = \sigma_{p+1}\, \rho_{*p}^R, \qquad -\infty < p < \infty,$

(2) $\sigma_p\, \beta_p^R\, \eta_p\, \sigma_p = \sigma_p\, \beta_p^R, \qquad -\infty < p < \infty.$

The proof of (1) may be modeled after the proof for the corresponding identity stated in terms of the classical homotopy operator ρ_p^R (see [2, § 3.5]). From identities (1) and (2), we have

(3) $\beta_p^S\, \sigma_p = \sigma_p\, \beta_p^R$,

(4) $\rho_{*p}^S\, \sigma_p = \sigma_{p+1}\, \rho_{*p}^R$,

(5) $\beta_{p+1}^S\, \rho_{*p}^S\, \sigma_p = \sigma_{p+1}\, \beta_{p+1}^R\, \rho_{*p}^R$

for all integers p.

1.12. Let P_1 and P_2 denote the following propositions:

P_1. Let c_p^S denote a p-chain of S such that

$$\beta_p^S\, c_p^S = 0.$$

Then

$$\beta_{p+1}^S\, \rho_{*p}^S\, c_p^S = 0.$$

P_2. Let c_p^R denote a p-chain of R such that

$$\sigma_p\, \beta_p^R\, c_p^R = 0.$$

Then

$$\sigma_{p+1}\, \beta_{p+1}^R\, \rho_{*p}^R\, c_p^R = 0.$$

THEOREM. *$P_1 \equiv P_2$; i.e., P_1 is true if and only if P_2 is true.*

Proof. Assume P_1, and let c_p^R denote a p-chain of R such that

$$\sigma_p\ \beta_p^R\ c_p^R = 0.$$

Then via identity (3) we have

$$\beta_p^S\ \sigma_p\ c_p^R = 0.$$

Therefore

$$\beta_{p+1}^S\ \rho_{*p}^S\ \sigma_p\ c_p^R = 0.$$

But via identity (5), we have

$$\sigma_{p+1}\ \beta_{p+1}^R\ \rho_{*p}^R\ c_p^R = 0,$$

and P_2 follows.

Now assume P_2, and let c_p^S denote a p-chain of S such that

$$\beta_p^S\ c_p^S = 0.$$

Then since

$$\beta_p^S = \sigma_p\ \beta_p^R\ \eta_p,$$

we have

$$\sigma_p\ \beta_p^R\ \eta_p\ c_p^S = 0.$$

Therefore, via P_2, we have

$$\sigma_{p+1}\ \beta_{p+1}^R\ \rho_{*p}^R\ \eta_p\ c_p^S = 0.$$

But via (5) and the fact that $\sigma_p\ \eta_p = 1$, we have

$$\sigma_{p+1}\ \beta_{p+1}^R\ \rho_{*p}^R\ \eta_p\ c_p^S = \beta_{p+1}^S\ \rho_{*p}^S\ \sigma_p\ \eta_p\ c_p^S = \beta_{p+1}^S\ \rho_{*p}^S\ c_p^S = 0,$$

and P_1 follows.

II. The proof of P_1

2.1. We shall use throughout this section the notation T for the p-cell

$(d_0, \cdots, d_p, T)^S$ when there is little chance for ambiguity. Under this convention a chain c_p^S having the representation

$$c_p^S = \sum_{j=1}^{n} \lambda_j (d_0, \cdots, d_p, T_j)^S$$

may be written $\sum_{j=1}^{n} \lambda_j T_j$. Thus T represents both a transformation from the convex hull $|d_0, \cdots, d_p|$ into the topological space X and the p-cell $(d_0, \cdots, d_p, T)^S$.

2.2. For $p < 0$, the proposition P_1 is trivial. For $p = 0$, P_1 is also trivial. For since $\beta_0^R = 1$ and $\sigma_0 \ \eta_0 = 1$, we have

$$\beta_0^S \ c_0^S = 0$$

implying

$$\sigma_0 \ \beta_0^R \ \eta_0 \ c_0^S = \sigma_0 \ \eta_0 \ c_0^S = c_0^S = 0,$$

whence clearly

$$\beta_1^S \ \rho_{*0}^S \ c_0^S = 0.$$

Now, take a fixed $p \geq 1$. Let

$$c_p^S = \sum_{j=1}^{n} \lambda_j \ T_j \qquad (\lambda_j \neq 0)$$

denote a p-chain of S such that

$$\beta_p^S \ c_p^S = 0.$$

Via §1.10,

$$(1) \qquad \beta_p^S \ c_p^S = \sum_{j=1}^{n} \sum_{\tau_p \in T_{p0}} \lambda_j \operatorname{sgn} \tau_p \ T_j [0_{p+1} \ b_{p0} \ \tau_p].$$

Let E denote the set of ordered pairs (j, τ_p), $1 \leq j \leq n$, $\tau_p \in T_{p0}$. Then

$$(2) \qquad \beta_p^S c_p^S = \sum_{(j,\tau_p)\in E} \lambda_j \operatorname{sgn} \tau_p \, T_j[0_{p+1}\, b_{p0}\, \tau_p].$$

We now define a binary relation "$\equiv$" on E as follows:

$$(j, \tau_p) \equiv (j', \tau_p')$$

if and only if $T_j[0_{p+1}\, b_{p0}\, \tau_p]$, $T_{j'}[0_{p+1}\, b_{p0}\, \tau_p']$ are identical p-cells. Then "$\equiv$" as defined is obviously a true equivalence relation and induces a partitioning of E into nonempty, mutually disjoint sets E_s $(s = 1, \cdots, t)$ with

$$E = \bigcup_{s=1}^{t} E_s.$$

Therefore, via (2), we have

$$(3) \qquad \beta_p^S c_p^S = \sum_{s=1}^{t} \sum_{(j,\tau_p)\in E_s} \lambda_j \operatorname{sgn} \tau_p \, T_j[0_{p+1}\, b_{p0}\, \tau_p].$$

Take $1 \le s < s' \le t$. Then for $(j, T_p) \in E_s$, $(j', T_p') \in E_{s'}$, the p-cells $T_j[0_{p+1}\, b_{p0}\, \tau_p]$, $T_{j'}[0_{p+1}\, b_{p0}\, \tau_p']$ are distinct. Therefore, since

$$\beta_p^S c_p^S = 0,$$

we must have for each s, $1 \le s \le t$,

$$(4) \qquad \sum_{(j,\tau_p)\in E_s} \lambda_j \operatorname{sgn} \tau_p \, T_j[0_{p+1}\, b_{p0}\, \tau_p] = 0,$$

and hence

$$(5) \qquad \sum_{(j,\tau_p)\in E_s} \lambda_j \operatorname{sgn} \tau_p = 0,$$

since all p-cells occuring in (4) are identical.

2.3. Again via §1.10,

$$(6) \qquad \beta^S_{p+1}\,\rho^S_{*p}\,c^S_p = \sum_{j=1}^{n}\sum_{k=0}^{p}\sum_{\tau_p\in T_{pk}}\sum_{\tau_{p+1}\in T_{p+1\,0}} (-1)^k \operatorname{sgn}\tau_p \operatorname{sgn}\tau_{p+1}\,\lambda_j\,T_j[b_{pk}\,\tau_p]\,[0_{p+2}\,b_{p+1\,0}\,\tau_{p+1}].$$

Applying the lemma of § 1.7, we obtain

$$(7) \qquad \beta^S_{p+1}\,\rho^S_{*p}\,c^S_p = \sum_{k=0}^{p}\sum_{q=0}^{p+1}(-1)^{p+q+k+1}\left\{\sum_{j=1}^{n}\sum_{\tau_p\in T_{pk}}\sum_{\pi_p\in T_{p0}}\lambda_j \operatorname{sgn}\tau_p \operatorname{sgn}\pi_p\,T_j[b_{pk}\,\tau_p]\,[0_{p+2}\,b_{p+1\,0}\,q_{*p}\,\pi_p(p+1)_{p+1}]\right\}.$$

Thus, to prove that

$$\beta^S_{p+1}\,\rho^S_{*p}\,c^S_p = 0,$$

we are led to consider for a fixed k and q, $0 \le k \le p$, $0 \le q \le p+1$, the expression

$$(8) \qquad Y_{kq} = \sum_{j=1}^{n}\sum_{\tau_p\in T_{pk}}\sum_{\pi_p\in T_{p0}}\lambda_j \operatorname{sgn}\tau_p \operatorname{sgn}\pi_p\,T_j[b_{pk}\,\tau_p]\,[0_{p+2}\,b_{p+1\,0}\,q_{*p}\,\pi_p(p+1)_{p+1}].$$

Now to prove P_1 we need only show that $Y_{kq} = 0$. Therefore k and q will remain fixed throughout the remainder of this section; and even though subsequent definitions will depend upon k and q, they will not be displayed in the notation.

2.4. For

$$\tau_p = \tau_p(i_0, \cdots, i_p) \in T_{p0}$$

(see [3, § 1.9]) there exists a unique permutation $(n_0, \cdots, n_k)$ of $0, \cdots, k$ such that $i_{n_0} < \cdots < i_{n_k}$. Let

$$\bar{\tau}_p = \bar{\tau}_p(j_0, \cdots, j_p),$$

where $j_l = i_{n_l}$ for $l = 0, \cdots, k$, and $j_l = i_l$ for $k+1 \le l \le p$. Then there exists a unique permutation $(m_0, \cdots, m_k)$ of $0, \cdots, k$, namely $(n_0, \cdots, n_k)^{-1}$, such that

$$\tau_p = \tau_p(j_{m_0}, \cdots, j_{m_k}, j_{k+1}, \cdots, j_p).$$

Furthermore, let $A(\tau_p)$ denote the set of $\pi_p \in T_{p0}$ defined as follows. For

$$\pi_p = \pi_p(u_0, \cdots, u_p) \in T_{p0}$$

we have a unique set of integers $l_0, \cdots, l_k$, $0 \leq l_0 < \cdots < l_k \leq p$ such that $(u_{l_0}, \cdots, u_{l_k})$ is a permutation of $0, \cdots, k$. Set $\pi_p \in A(\tau_p)$ if and only if $m_0 = u_{l_0}, \cdots, m_k = u_{l_k}$.

2.5. Let B denote the set of ordered pairs (τ_p, π_p), $\tau_p \in T_{p0}$, $\pi_p \in A(\tau_p)$, and B' the set of ordered pairs (τ_p', π_p'), $\tau_p' \in T_{pk}$, $\pi_p' \in T_{p0}$. We define a mapping

$$\gamma : B \longrightarrow B'$$

as follows:

$$\gamma(\tau_p, \pi_p) = (\tau_p', \pi_p')$$

where $\tau_p' = \overline{\tau}_p$ and $\pi_p' = \pi_p$. One shows with little difficulty that γ is biunique. Therefore

$$(9) \qquad Y_{kq} = \sum_{j=1}^{n} \sum_{\tau_p \in T_{p0}} \sum_{\pi_p \in A(\tau_p)} \lambda_j \operatorname{sgn} \overline{\tau}_p \operatorname{sgn} \pi_p \, T_j [b_{pk} \overline{\tau}_p]$$

$$[0_{p+2} \, b_{p+1\,0} \, q_{*p} \, \pi_p (p+1)_{p+1}].$$

2.6. Let $A = A(\tau_p(0, \cdots, p))$. For $\tau_p \in T_{p0}$ we define

$$f_{\tau_p} : A \longrightarrow A(\tau_p)$$

as follows. For $\pi_p(u_0, \cdots, u_p) \in A$, there exist integers $l_0, \cdots, l_k$, $0 \leq l_0 < \cdots < l_k \leq p$, such that $u_{l_0} = 0, \cdots, u_{l_k} = k$. Define

$$f_{\tau_p} \pi_p = \pi_p'(u_0', \cdots, u_p')$$

as follows. Let

$$\bar{\tau}_p = \bar{\tau}_p(j_0, \cdots, j_p) \quad \text{and} \quad \tau_p = \tau_p(j_{m_0}, \cdots, j_{m_k}, j_{k+1}, \cdots, j_p),$$

where $(m_0, \cdots, m_k)$ is a permutation of $0, \cdots, k$. Set $u'_{l_0} = m_0, \cdots, u'_{l_k} = m_k$, and $u'_r = u_r$ for $r \neq l_0, \cdots, l_k$. Here again it is easy to show that f_{τ_p} is biunique. We have then

$$(10) \qquad Y_{kq} = \sum_{j=1}^{n} \sum_{\tau_p \in T_{p0}} \sum_{\pi_p \in A} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p T_j [b_{pk} \bar{\tau}_p]$$

$$[0_{p+2}\, b_{p+1\,0}\, q_{*p}\, f_{\tau_p}\, \pi_p (p+1)_{p+1}],$$

and hence

$$(11) \qquad Y_{kq} = \sum_{s=1}^{t} \sum_{\pi_p \in A} \sum_{(j,\, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p T_j [b_{pk} \bar{\tau}_p]$$

$$[0_{p+2}\, b_{p+1\,0}\, q_{*p}\, f_{\tau_p}\, \pi_p (p+1)_{p+1}]$$

(see § 2.2).

2.7. LEMMA. *Take $\pi_p(u_0, \cdots, u_p) \in T_{p0}$ and let*

$$\alpha = [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p (p+1)_{p+1}].$$

Let

$$x = \sum_{j=0}^{p+1} \mu_j d_j,$$

with

$$\mu_j \geq 0, \ j = 0, \cdots, p+1, \ \text{and} \ \sum_{j=0}^{p+1} \mu_j = 1,$$

denote a point of $|d_0, \cdots, d_{p+1}|$. Then

$$\alpha(x) = \sum_{j=0}^{p+1} a_j d_j,$$

where

(i) $a_j \geq 0,\ j = 0, \cdots, p+1;$

(ii) $\sum_{j=0}^{p+1} a_j = 1;$

(iii) $a_{u_0} \geq a_{u_1} \geq \cdots \geq a_{u_p};$

(iv) $a_{u_0}, \cdots, a_{u_p}, a_{p+1}$ *are independent of* π_p; *i.e., if* $\pi_p' = \pi_p'(u_0', \cdots, u_p') \in T_{p0}$ *and*

$$\alpha' = [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p'(p+1)_{p+1}],$$

then

$$\alpha'(x) = \sum_{j=0}^{p+1} a_j' d_j$$

with

$$a_{u_0} = a_{u_0'}', \cdots, a_{u_p} = a_{u_p'}', \quad a_{p+1} = a_{p+1}' .$$

Proof. We consider only the case $1 \leq q \leq p$ since the fringe cases $q = 0$, $p+1$ follow in a completely analogous manner. In case $1 \leq q \leq p$ we have

$$\alpha = [b(w_0)b(w_0, w_1) \cdots b(w_0, \cdots, w_{p+1})],$$

where

$$w_l = d_{u_l},\ l = 0, \cdots, q-1,\ w_q = d_{p+1},\ w_l = d_{u_{l-1}},\ l = q+1, \cdots, p+1.$$

Therefore,

$$\alpha(x) = \sum_{j=0}^{p+1} \mu_j\, b(w_0, \cdots, w_j) = \sum_{j=0}^{p+1} \left(\sum_{l=j}^{p+1} \frac{\mu_l}{l+1} \right) w_j$$

(see §1.2). Let

$$a_{p+1} = \sum_{l=q}^{p+1} \frac{\mu_l}{l+1}, \quad a_{u_r} = \sum_{l=r}^{p+1} \frac{\mu_l}{l+1} \quad \text{for } r = 0, \cdots, q-1$$

and

$$a_{u_r} = \sum_{l=r+1}^{p+1} \frac{\mu_l}{l+1} \quad \text{for } r = q, \cdots, p.$$

Clearly, $a_{u_0}, \cdots, a_{u_p}, a_{p+1}$ are independent of π_p in the sense of (iv), and $a_{u_0} \geq \cdots \geq a_{u_p}$. Furthermore, $a_j \geq 0$ $(j = 0, \cdots, p+1)$, and

$$\sum_{j=0}^{p+1} a_j = \sum_{j=0}^{p+1} \mu_j = 1.$$

Also,

$$\alpha(x) = \sum_{j=0}^{q-1} a_{u_j} d_{u_j} + a_{p+1} d_{p+1} + \sum_{j=q}^{p} a_{u_j} d_{u_j} = \sum_{j=0}^{p+1} a_j d_j,$$

and the lemma follows.

2.8. LEMMA. *Take* (j, τ_p) *and* $(j', \tau_p') \in E_s$ *(see §2.2),* $1 \leq s \leq t$, *and* $\pi_p^* \in A$. *Then*

$$T_j [b_{pk} \bar{\tau}_p] [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, f_{\tau_p}\, \pi_p^* (p+1)_{p+1}]$$

$$= T_{j'} [b_{pk} \bar{\tau}_p'] [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, f_{\tau_p'}\, \pi_p^* (p+1)_{p+1}].$$

Proof. Since (j, τ_p), (j', τ_p') lie in E_s, we have

$$T_j [0_{p+1}\, b_{p0}\, \tau_p] = T_{j'} [0_{p+1}\, b_{p0}\, \tau_p'],$$

Let

$$\pi_p = f_{\tau_p} \pi_p^* = \pi_p (u_0, \cdots, u_p), \quad \pi_p' = f_{\tau_p'} \pi_p^* = \pi_p' (u_0', \cdots, u_p'),$$

$$\alpha = [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p (p+1)_{p+1}], \quad \alpha' = [0_{p+2}\, b_{p+1\,0}\, q_{*p}\, \pi_p' (p+1)_{p+1}],$$

$$\gamma = [b_{pk} \bar{\tau}_p], \text{ and } \gamma' = [b_{pk} \bar{\tau}_p'].$$

Furthermore, let

$$\tau_p = \tau_p(i_0, \cdots, i_p),\ \bar{\tau}_p = \bar{\tau}_p(j_0, \cdots, j_p),$$

$$\tau_p' = \tau_p'(i_0', \cdots, i_p'),\ \bar{\tau}_p' = \bar{\tau}_p'(j_0', \cdots, j_p').$$

We have permutations $(m_0, \cdots, m_k)$, $(n_0, \cdots, n_k)$ of $0, \cdots, k$ such that

$$\tau_p = \tau_p(j_{m_0}, \cdots, j_{m_k},\ j_{k+1}, \cdots, j_p),$$

$$\tau_p' = \tau_p'(j'_{n_0}, \cdots, j'_{n_k},\ j'_{k+1}, \cdots, j'_p)$$

Take an arbitrary point of $|d_0, \cdots, d_{p+1}|$, say

$$x = \sum_{j=0}^{p+1} \mu_j d_j \qquad\qquad \mu_j \geq 0,\ \sum_{j=0}^{p+1} \mu_j = 1.$$

Then via the lemma of § 2.7 we have

$$\alpha(x) = \sum_{j=0}^{p+1} a_j d_j \text{ with } a_j \geq 0,\ \sum_{j=0}^{p+1} a_j = 1,\ a_{u_0} \geq \cdots \geq a_{u_p},$$

and

$$\alpha'(x) = \sum_{j=0}^{p+1} a_j' d_j \text{ with } a_j' \geq 0,\ \sum_{j=0}^{p+1} a_j' = 1,\ a'_{u_0'} \geq \cdots \geq a'_{u_p'},$$

with

$$a_{u_0} = a'_{u_0'}, \cdots, a_{u_p} = a'_{u_p'} \text{ and } a_{p+1} = a'_{p+1}.$$

Now

$$\gamma = [d_{j_0}, \cdots, d_{j_k},\ b(d_{j_0}, \cdots, d_{j_k}), \cdots, b(d_{j_0}, \cdots, d_{j_p})].$$

Hence

$$\gamma\alpha(x) = a_0 d_{j_0} + \cdots + a_k d_{j_k} + a_{k+1} b(d_{j_0}, \cdots, d_{j_p}) + \cdots + a_{p+1} b(d_{j_0}, \cdots, d_{j_p})$$

$$= a_{m_0} d_{j_{m_0}} + \cdots + a_{m_k} d_{j_{m_k}} + a_{k+1} b(d_{j_0}, \cdots, d_{j_k}) + \cdots + a_{p+1} b(d_{j_0}, \cdots, d_{j_p})$$

$$= a_{m_0} d_{j_{m_0}} + \cdots + a_{m_k} d_{j_{m_k}} + a_{k+1} b(d_{j_{m_0}}, \cdots, d_{j_{m_k}}) + \cdots + a_{p+1} b(d_{j_{m_0}}, \cdots, d_{j_{m_k}}, d_{j_{k+1}}, \cdots, d_{j_p})$$

$$= a_{m_0} d_{i_0} + \cdots + a_{m_k} d_{i_k} + a_{k+1} b(d_{i_0}, \cdots, d_{i_k}) + \cdots + a_{p+1} b(d_{i_0}, \cdots, d_{i_p}).$$

Now take integers $l_0, \cdots, l_k$, $0 \leq l_0 < \cdots < l_k \leq p$, such that $(u_{l_0}, \cdots, u_{l_k})$ is a permutation of $0, \cdots, k$. Since $\pi_p \in A(\tau_p)$, we have $m_0 = u_{l_0}, \cdots, m_k = u_{l_k}$. Hence $a_{m_0} \geq \cdots \geq a_{m_k}$.

In a similar fashion we obtain

$$\gamma'\alpha'(x) = a'_{n_0} d_{i'_0} + \cdots + a'_{n_k} d_{i'_k} + a'_{k+1} b(d_{i'_0}, \cdots, d_{i'_k}) + \cdots + a'_{p+1} b(d_{i'_0}, \cdots, d_{i'_p}),$$

with $a'_{n_0} \geq \cdots \geq a'_{n_k}$; and if $l'_0, \cdots, l'_k$, $0 \leq l'_0 < \cdots < l'_k \leq p$, are integers such that $(u'_{l'_0}, \cdots, u'_{l'_k})$ is a permutation of $0, \cdots, k$, we have

$$n_0 = u'_{l'_0}, \cdots, n_k = u'_{l'_k}.$$

Applying § 1.3, we get

$$a_{m_0} d_{i_0} + \cdots + a_{m_k} d_{i_k} = \sum_{l=0}^{k} \gamma_l b(d_{i_0}, \cdots, d_{i_l})$$

with

$$\gamma_l = (l+1)(a_{m_l} - a_{m_{l+1}}) \text{ for } l = 0, \cdots, k-1,$$

$$\gamma_k = (k+1) a_{m_k} ,$$

and

$$\sum_{l=0}^{k} \gamma_l = \sum_{l=0}^{k} a_{m_l} .$$

Similarly,

$$a'_{n_0} d_{i'_0} + \cdots + a'_{n_k} d_{i'_k} = \sum_{l=0}^{k} \gamma'_l b(d_{i'_0}, \cdots, d_{i'_l})$$

with

$$\gamma'_l = (l+1)(a'_{n_l} - a'_{n_{l+1}}) \text{ for } l = 0, \cdots, k-1,$$

$$\gamma'_k = (k+1) a'_{n_k}$$

and

$$\sum_{l=0}^{k} \gamma'_l = \sum_{l=0}^{k} a'_{n_l} .$$

However, since

$$\pi_p = f_{\tau_p} \pi^*_p, \ \pi'_p = f_{\tau'_p} \pi^*_p,$$

we have

$$l_0 = l'_0, \cdots, l_k = l'_k \text{ and } u_r = u'_r \text{ for } r \neq l_0, \cdots, l_k .$$

Therefore, $a_{u_{l_0}} = a'_{u'_{l'_0}}, \cdots, a_{u_{l_k}} = a'_{u'_{l'_k}}$, and hence

$$a_{m_0} = a'_{n_0}, \cdots, a_{m_k} = a'_{n_k} .$$

Thus

$$\gamma_r = \gamma'_r \text{ for } r = 0, \cdots, k.$$

Furthermore,

$$a_{u_r} = a'_{u'_r} \text{ for } r \neq l_0, \cdots, l_k, \text{ and } a_{p+1} = a'_{p+1}.$$

Therefore,

$$\gamma\alpha(x) = \sum_{l=0}^{k} \gamma_l\, b(d_{i_0}, \cdots, d_{i_l}) + \sum_{l=k}^{p} a_{l+1}\, b(d_{i_0}, \cdots, d_{i_l}),$$

$$\gamma'\alpha'(x) = \sum_{l=0}^{k} \gamma_l\, b(d_{i'_0}, \cdots, d_{i'_l}) + \sum_{l=k}^{p} a_{l+1}\, b(d_{i'_0}, \cdots, d_{i'_l}),$$

with

$$\sum_{l=0}^{k} \gamma_l + \sum_{l=k}^{p} a_{l+1} = \sum_{l=0}^{p+1} a_l = 1.$$

Let

$$y = \sum_{j=0}^{p} h_j\, d_j$$

with

$$h_j = \gamma_j \text{ for } j = 0, \cdots, k-1,$$

$$h_k = \gamma_k + a_{k+1},$$

$$h_j = a_{j+1} \text{ for } j = k+1, \cdots, p.$$

Clearly,

$$h_j \geq 0 \quad (j = 0, \cdots, p), \text{ and } \sum_{j=0}^{p} h_j = 1.$$

Then

$$\gamma\alpha(x) = \sum_{l=0}^{p} h_l\, b(d_{i_0}, \cdots, d_{i_l}) = [0_{p+1}\, b_{p0}\, \tau_p](y)$$

and

$$\gamma' \alpha'(x) = \sum_{l=0}^{p} h_l \, b(d_{i_0'}, \cdots, d_{i_l'}) = [0_{p+1} \, b_{p0} \, \tau_p'](y).$$

Therefore, since

$$T_j [0_{p+1} \, b_{p0} \, \tau_p](y) = T_j' [0_{p+1} \, b_{p0} \, \tau_p'](y),$$

we have

$$T_j \, \gamma \, \alpha(x) = T_j' \, \gamma' \, \alpha'(x).$$

Since x is arbitrary in $|d_0, \cdots, d_{p+1}|$, our lemma follows.

2.9. LEMMA. *For any* s, $1 \leq s \leq t$, *and* $\pi_p^* \in A$,

$$\sum_{(j, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p^* = 0.$$

Proof. Since

$$\operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p^* = \operatorname{sgn} \tau_p \operatorname{sgn} \pi_p^*,$$

we have

$$\sum_{(j, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \bar{\tau}_p \operatorname{sgn} f_{\tau_p} \pi_p^* = \operatorname{sgn} \pi_p^* \sum_{(j, \tau_p) \in E_s} \lambda_j \operatorname{sgn} \tau_p = 0$$

via (5) of § 2.2.

2.10. Employing §§ 2.8, 2.9, and (11) of § 2.6, we see that $Y_{kq} = 0$, and hence P_1 follows. Let us note also that since $P_1 \equiv P_2$, P_2 also is valid.

III. RESULTS

3.1. In [1, § 4.2], Radó has established a lemma, which we state here for the barycentric homotopy operator ρ_{*p}^R.

LEMMA. *Let* $\{G_p\}$ *be an identifier for* R, *such that the following conditions hold*:

(i) $G_p \supset A_p^R$ (*see* [1, § 3.4]),

(ii) $c_p^R \in G_p$ *implies that* $\sigma_p \beta_p^R c_p^R = 0$,

(iii) $c_p^R \in G_p$ *implies that* $\rho_{*p}^R c_p^R \in G_{p+1}$.

Then $\{G_p\}$ *is an unessential identifier for* R.

The proof of this lemma is identical with the proof of the corresponding lemma as given by Radó with ρ_p^R (classical homotopy operator) replacing ρ_{*p}^R.

Since

$$\sigma_p \beta_p^R : C_p^R \longrightarrow C_p^S$$

is a chain mapping, the system $\{N(\sigma_p \beta_p^R)\}$ of nuclei of the homomorphisms $\sigma_p \beta_p^R$ is an identifier for R (see [**1**, § 1.2]). Furthermore,

$$N(\sigma_p \beta_p^R) \supset A_p^R \text{ since } \sigma_p \beta_p^R = \beta_p^S \sigma_p$$

(see § 1.11). Applying P_2 directly, we see that $N(\sigma_p \beta_p^R)$ satisfies (iii) of the foregoing lemma. Therefore, since $N(\sigma_p \beta_p^R)$ is the largest identifier, satisfying (ii), we have the following maximum result yielded by the same lemma:

THEOREM. *The system* $\{N(\sigma_p \beta_p^R)\}$ *is an unessential identifier for* R.

3.2. In order to compare our results with those of Radó [**1**] and Reichelderfer [**3**] let us first note that

$$\hat{N}(\sigma_p \beta_p^R) = N(\sigma_p \beta_p^R),$$

where $\hat{N}(\sigma_p \beta_p^R)$ is the division hull of $N(\sigma_p \beta_p^R)$, since C_p^R is a free Abelian group. Then since

$$N(\sigma_p \beta_p^R) \supset \Delta_p^R = N(\beta_p^R) + A_p^R$$

(see [**3**, § 3.6]) we have

$$N(\sigma_p \beta_p^R) \supset \hat{\Delta}_p^R \supset \hat{\Gamma}_p^R$$

(see [**1**, § 4.7]).

The writer has been unable to determine as yet whether $N(\sigma_p \beta_p^R)$ is effectively larger than either $\hat{\Delta}_p^R$ or $\hat{\Gamma}_p^R$.

3.3. The following lemma (see [**1**, §4.1]) is immediate from the fact that ρ_{*p}^S satisfies the well-known "homotopy identity,"

$$\partial_{p+1}^S \rho_{*p}^S + \rho_{*p-1}^S \partial_p^S = \beta_p^S - 1.$$

LEMMA. *Let* $\{G_p\}$ *be an identifier for* S *such that the following conditions hold*:

(i) $c_p^S \in G_p$ *implies that* $\beta_p^S c_p^S = 0$,

(ii) $c_p^S \in G_p$ *implies that* $\rho_{*p}^S c_p^S \in G_{p+1}$.

Then $\{G_p\}$ *is an unessential identifier for* S.

The system of nuclei $\{N(\beta_p^S)\}$ clearly is an identifier for S since β_p^S is a chain mapping. Therefore, applying P_1 we obtain the maximum result of the foregoing lemma.

THEOREM. *The system* $\{N(\beta_p^S)\}$ *is an unessential identifier for* S.

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