A NOTE ON THE HÖLDER MEAN

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1. Introduction. Of the two better-known generalizations of the simple arithmetic mean, the Hölder mean and the Cesàro mean, the latter has been the more extensively studied. This is primarily due to the equivalence of the two when used to define summability methods and to the following formulas. If we define C_n^k, the k^{th} order Cesàro mean of the terms $S_0, S_1, \cdots, S_n$, by the relation

$$C_n^k = \binom{n+k}{k}^{-1} S_n^k,$$

where

$$S_n^0 = S_n \quad \text{and} \quad S_n^k = \sum_{v=0}^{n} S_v^{k-1} \quad \text{for } n \geq 0, \quad k = 1, 2, \cdots,$$

then it follows [1, p. 96] that

$$S_n^{k+m} = \sum_{v=0}^{n} \binom{n-v+m-1}{m-1} S_v^k \tag{1.1}$$

and

$$S_n^k = \sum_{v=0}^{m} (-1)^v \binom{m}{v} S_{n-v}^{k+m} \qquad (m = 1, 2, \cdots). \tag{1.2}$$

The only known analogues to these formulas for the Hölder mean that this writer has been able to find are as follows. Denoting the k^{th} order Hölder mean of the terms $S_0, S_1, \cdots, S_n$ by H_n^k, and recalling the definition that

$$H_n^0 = S_n \quad \text{and} \quad H_n^k = \frac{1}{n+1} \sum_{v=0}^{n} H_v^{k-1} \qquad \text{for } n \geq 0, \quad k = 1, 2, \cdots,$$

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it can be proved [**1**, p. 250] that

$$(1.3)\qquad H_n^{k+m} = \sum_{v=0}^{n} (-1)^v \binom{n}{n-v} [\Delta^v (n+1-v)^{-m}]\, H_{n-v}^k$$

and

$$(1.4)\qquad H_n^k = \sum_{v=0}^{m} (-1)^v \binom{n}{v} [\Delta^v (n+1-v)^m]\, H_{n-v}^{k+m} \qquad (m = 1, 2, \cdots),$$

where $\Delta u(n) = u(n+1) - u(n)$. These formulas follow from a more general expression for the coefficients in any Hausdorff transformation. It is easily seen that the coefficients involved in (1.3) and (1.4) in many respects are not as convenient to work with as those of (1.1) and (1.2).

In § 2 below, the coefficients of (1.4) are obtained in different form, being expressed in terms of a particular set of polynomials. A few of the properties of these polynomials are considered in § 3, while applications with respect to Hölder summability are dealt with in § 4.

2. A set of polynomials. It follows from the definition of the Hölder mean that

$$(n+1)\, H_n^{k+1} - nH_{n-1}^{k+1} = H_n^k$$

for integers $k \geq 0$ and $n \geq 0$. By iteration, it follows that there exist coefficients $A_j^m(n)$ such that

$$(2.1)\qquad H_n^k = \sum_{j=0}^{m} (-1)^j A_j^m(n)\, H_{n-j}^{k+m} \qquad (m = 0, 1, 2, \cdots)$$

if

$$(2.2)\qquad A_j^{m+1}(n) = (n-j+1)\,[A_j^m(n) + A_{j-1}^m(n)]$$

for $0 \leq j \leq m$, where

$$(2.3)\qquad A_0^0(n) = 1 \text{ and } A_j^m(n) = 0$$

for $j < 0$ or $j > m$. By virtue of the identity

$$\Delta^j (n+1-j)^{m+1} = (n+1-j)\, \Delta^j (n+1-j)^m + j\Delta^{j-1}(n+2-j)^m,$$

it follows that the coefficient of (1.4),

$$A_j^m(n) = \binom{n}{j} \Delta^j (n+1-j)^m ,$$

is a solution of (2.2) satisfying the boundary condition (2.3).

Another form of this solution is obtained when we consider the following set of polynomials. For arbitrary nonnegative integers m and j, $0 \le j \le m$, let

$$(2.4) \qquad \begin{aligned} F_0^m(x) &= x^{m+1}, \\ F_1^m(x) &= \sum_{m+1} x^i (x-1)^j, \\ \cdots \quad &\cdots \quad \cdots \\ F_j^m(x) &= \sum_{m+1} x^p (x-1)^q \cdots (x-j)^s, \\ \cdots \quad &\cdots \quad \cdots \\ F_m^m(x) &= x(x-1) \cdots (x-m), \end{aligned}$$

the symbol

$$\sum_{m+1} x^p (x-1)^q \cdots (x-j)^s$$

denoting the sum of all possible but different such products where $p, q, \cdots, s$ are positive integers such that $p + q + \cdots + s = m + 1$. If we further let

$$(2.5) \qquad F_j^m(x) = 0$$

whenever $j < 0$ or $j > m$, it follows that

$$(2.6) \qquad F_j^{m+1}(x) = (x-j) [F_{j-1}^m(x) + F_j^m(x)]$$

for integers j and $m \ge 0$. To prove the latter relation, apply (2.4) to get

$$(2.7) \quad (x-j)[F_{j-1}^m(x) + F_j^m(x)] = \sum_{m+1} x^p (x-1)^q \cdots (x-j+1)^r (x-j)$$

$$+ \sum_{m+1} x^p (x-1)^q \cdots (x-j+1)^r (x-j)^{s+1}$$

for $0 < j \leq m$. In the first sum on the right, the exponents $p, q, \cdots, r$ take on all possible positive integral values such that $(p+q+\cdots+r)+1 = m+2$. In the second sum, the integers $p, q, \cdots, r, s$ take on all possible integral values such that $(p+q+\cdots+r)+(s+1) = m+2$. It follows that if we consider both sums on the right of (2.7) together, then their sum is $F_j^{m+1}(x)$, thus completing the proof of (2.6) when $0 < j \leq m$. Its truth for $j \leq 0$ or $j > m$ follows when we further consider (2.5) as well as (2.4).

Reconsidering equations (2.4), we note that each of the polynomials defined there has x as a factor. Consequently there exists a unique polynomial $G_j^m(x)$ such that

$$(2.8) \qquad F_j^m(x) = xG_j^m(x)$$

for integral $m \geq 0$ and j. Substituting into (2.5) and (2.6), and noting that $G_0^0(x) = 1$ for all x, we see that $G_j^m(n+1)$ is a solution for (2.2) satisfying the boundary conditions (2.3). Consequently, we assert that

$$(2.9) \qquad H_n^k = \sum_{j=0}^{m} (-1)^j \, G_j^m(n+1) \, H_{n-j}^{k+m}$$

for integers $k \geq 0$ and $m \geq 0$.[1]

3. Properties of the polynomials $G_j^m(x)$. In the work that follows, it will be more convenient to consider the polynomials $G_j^m(x)$ defined by (2.8). As might be expected, we find a considerable number of recurrence relations and other formulas involving these polynomials and their coefficients. Before proceeding to the particular applications in view, we shall list a few such relations. For integral $m \geq 0$ and j,

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[1]The author is indebted to the referee for suggesting the above derivation of (2.9) which is somewhat simpler than the proof originally presented. The referee also proposed the following alternative derivation. We write

$$H^k(x) = \sum_{n=0}^{\infty} H_n^k x^n,$$

and then with $D = d/dx$,

$$(1-x)\, D\{xH^{k+1}(x)\} = H^k(x),$$

and symbolically,

$$[(1-x)\, Dx]^m \, H^{k+m}(x) = H^k(x).$$

Interpretation of the operator leads to the same results. This derivation is worth noting, for it is analogous to the classical development of equations (1.1) and (1.2).

$$G_j^{m+1}(x) = (x-j)\,[G_{j-1}^m(x) + G_j^m(x)]; \tag{3.1}$$

for integral $m \geq 1$ and j,

$$G_j^{m+1}(x) = (x-1)\,G_{j-1}^m(x-1) + xG_j^m(x); \tag{3.2}$$

and for integral $m \geq 0$ and j,

$$(j/2+x)\,G_j^m(j/2+x) = (-1)^{m+1}\,(j/2-x)\,G_j^m(j/2-x). \tag{3.3}$$

Equation (3.1) is obtained by substituting from (2.8) into (2.6). The proof of (3.2) is carried out by first deriving the relation

$$F_j^{m+1}(x) = x[F_{j-1}^m(x-1) + F_j^m(x)]$$

in the same manner as we derived (2.6), then substituting from (2.8). Equation (3.3) follows from the defining equation of $F_j^m(x)$ when (-1) is factored from each of the factors of the defining sum giving

$$F_j^m(x) = (-1)^{m+1}\,F_j^m(j-x)$$

for $0 \leq j \leq m$. Replacing x by $(j/2)+x$ and substituting from (2.8) yields the desired result. This relation displays the symmetric nature of the polynomials $F_j^m(x) = xG_j^m(x)$ in that they are symmetric with respect to the line $x = j/2$ when m is odd, and symmetric with respect to the point $(j/2, 0)$ when m is even.

Determine coefficients ${}_jA_{m,i}$ such that

$$G_j^m(x) = {}_jA_{m,0}\,x^m + {}_jA_{m,1}\,x^{m-1} + \cdots + {}_jA_{m,m-1}\,x + {}_jA_{m,m} \tag{3.4}$$

for $m > 0$. It follows from the definition that

$${}_jA_{m,i} = 0 \tag{3.5}$$

for either $i < 0$, $i > m > 0$, $j < 0$, or $j > m > 0$, and in particular ${}_0A_{m,0} = 1$ while ${}_0A_{m,i} = 0$ for $i > 0$. The following is a table of the polynomials $G_j^m(x)$ when $m = 1, 2, 3,$ and 4:

$k = 1$	$k = 2$
$G_0^1(x) = x$	$G_0^2(x) = x^2$
$G_1^1(x) = x - 1$	$G_1^2(x) = 2x^2 - 3x + 1$
	$G_2^2(x) = x^2 - 3x + 2$

$k = 3$	$k = 4$
$G_0^3(x) = x^3$	$G_0^4(x) = x^4$
$G_1^3(x) = 3x^3 - 6x^2 + 4x - 1$	$G_1^4(x) = 4x^4 - 10x^3 + 10x^2 - 5x + 1$
$G_2^3(x) = 3x^3 - 12x^2 + 15x - 6$	$G_2^4(x) = 6x^4 - 30x^3 + 55x^2 - 45x + 14$
$G_3^3(x) = x^3 - 6x^2 + 11x - 6$	$G_3^4(x) = 4x^4 - 30x^3 + 80x^2 - 90x + 36$
	$G_4^4(x) = x^4 - 10x^3 + 35x^2 - 50x + 24$

Substituting from (3.4) into (3.1), collecting like terms with respect to x, replacing m by $m - 1$, and equating coefficients, yields the recurrence relation

$$(3.6) \qquad {}_jA_{m,i} = ({}_jA_{m-1,i} + {}_{j-1}A_{m-1,i}) - j({}_jA_{m-1,i-1} + {}_{j-1}A_{m-1,i-1})$$

for integral $m \geq 1$ and j. Summing the latter expression with respect to j results in the relation

$$(3.7) \qquad \sum_{v=0}^{j} (-1)^v \, {}_vA_{m,i} = (-1)^j \, {}_jA_{m-1,i} - j(-1)^j \, {}_jA_{m-1,i-1} + \sum_{v=0}^{j-1} (-1)^v \, {}_vA_{m-1,i-1}$$

for $0 \leq i \leq m$. An interesting particular case of the latter formula is obtained by letting $j = m$ and considering (3.5). It follows that

$$\sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} = \sum_{v=0}^{m-1} (-1)^v \, {}_vA_{m-1,i-1}.$$

From repeated substitution, we conclude that

$$\sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} = {}_0A_{0,i-m},$$

whence

$$(3.8) \qquad \sum_{v=0}^{m} (-1)^v \, {}_vA_{m,i} = \begin{cases} 0 \text{ for } i < m \\ 1 \text{ for } i = m \end{cases}$$

when $m \geq 1$.

Recalling the factorial notation $x^{(m+1)} = x(x-1) \cdots (x-m)$, $m \geq 0$, we obtain

$$x G_m^m(x) = x^{(m+1)}.$$

But by definition, the numbers $s_{m,v}$ such that

$$x^{(m)} = s_{m,m} x^m + s_{m,m-1} x^{m-1} + \cdots + s_{m,1} x$$

are the Stirling numbers of the first kind [2, p. 143].[2] It now follows, since

$$G_m^m(x) = s_{m+1,m+1} x^m + s_{m+1,m} x^{m-1} + \cdots + s_{m+1,1},$$

that

$$(3.9) \qquad {}_mA_{m,i} = s_{m+1,m-i+1}.$$

In turn, letting $i = 0$ in (3.6), we find that

$${}_jA_{m,0} = {}_jA_{m-1,0} + {}_{j-1}A_{m-1,0}.$$

As a consequence of the initial conditions that ${}_0A_{m,0} = 1$ and ${}_jA_{m,0} = 0$ for $j > 0$, it follows [2, p. 615] that the solution of this partial difference equation is

$$(3.10) \qquad {}_jA_{m,0} = \binom{m}{j}.$$

When considering the polynomials $G_j^m(x)$ as displayed in the table, we see that, for any m, the coefficients considered by rows in light of (3.9) and (3.6) give a possible extension of the Stirling numbers. On the other hand, when the coefficients are considered by columns in light of (3.10), they present a possible extension of the binomial coefficients. This latter property is better displayed when we consider the known formula [2, p. 169]

$$\sum_{v=1}^{m} (-1)^v \binom{m}{v} v^j = (-1)^m\, m!\, S_{j,m} \qquad (j \geq 1),$$

where $S_{j,m}$ is the Stirling number of the second kind and thus $S_{j,m} = 0$ for $0 < j < m$. Make the definitions

$$P^m(i, j) = \sum_{v=1}^{m} (-1)^v\, {}_vA_{m,i}\, v^j \quad \text{and} \quad Q^m(i,j) = \sum_{v=0}^{m} (-1)^v\, {}_vA_{m,i}\, (v+1)^j,$$

where $m \geq 1$. It follows from a straightforward induction proof that

$$(3.11) \qquad P^m(0, 0) = -1 \quad \text{and} \quad P^m(i, j) = 0$$

[2] The notation used here for the Stirling numbers of the first and second kind is not the same as that used by Jordan in [2].

whenever $0 \leq i < m$, $0 \leq j < m - i$, and $i + j \neq 0$. The induction can be carried out by using the identity

$$P^{m+1}(i, j) = [P^m(i, j) - Q^m(i, j)] - [P^m(i-1, j+1) - Q^m(i-1, j+1)]$$

and the fact that the truth of (3.11) implies that both

$$Q^m(i, j) = 0 \tag{3.12}$$

for $0 \leq i < m$, $0 \leq j < m - i$, and

$$Q^m(i, m-i) = P^m(i, m-i)$$

for $0 \leq i \leq m$.

It is of interest that

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + in) = \sum_{i=0}^{m-1} n^{m-i} \, P^m(i, m-i) + 1 \tag{3.13}$$

for $m \geq 1$, $n = 0, \pm 1, \pm 2, \cdots$, and all x. That is, the sum

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + in)$$

is a function of n and m alone, independent of x. This follows from (3.8), (3.11), (3.12), and the identity

$$\begin{aligned}\sum_{i=0}^{m} (-1)^i \, G_i^m(x + in) &= \{ {}_0A_{m,0} + P^m(0, 0) \} \, x^m \\ &+ \sum_{j=1}^{m-1} \left[\sum_{v=0}^{j} \binom{m-v}{j-v} n^{j-v} \, P^m(v, j-v) \right] x^{m-j} \\ &+ \sum_{i=0}^{m-1} n^{m-i} \, P^m(i, m-i) + P^m(m, 0),\end{aligned}$$

where $m \geq 1$. Since the sum on the left of (3.13) is independent of x, we can write

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + in) = \sum_{i=0}^{m} (-1)^i \, G_i^m(in)$$

for $m \geq 1$, $n = 0, \pm 1, \pm 2, \cdots$, and all x. Letting $n = 1$, recalling that $G_i^m(x)$ has $(x - i)$ as a factor for $i > 0$ and that $G_0^m(x) = x^m$, we see that

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + i) = 0$$

for $m \geq 1$ and all x. If we let $n = 0$ in (3.13), then

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x) = 1 \tag{3.14}$$

for $m \geq 1$ and all x. It turns out that $n = 0, 1$ are the only two cases where the sum

$$\sum_{i=0}^{m} (-1)^i \, G_i^m(x + in)$$

is independent of m as well as x.

Consideration of (1.4) with (2.9) yields

$$G_j^m(n) = \binom{n-1}{j} \Delta^j \, (n - j)^m. \tag{3.15}$$

As might be expected, more is found concerning the nature of the coefficients of the polynomial $G_j^m(x)$ by studying the expression on the right of (3.15). Substituting into (3.15) from the identity

$$\Delta^j \, x^m = \sum_{v=1}^{m+1} v^{(j)} \, S_{m,v} \, x^{(v-j)},$$

where $S_{m,v}$ denotes the Stirling number of the second kind [2, p. 181], and simplifying, we obtain the relation

$$G_j^m(n) = \frac{(n-j)}{n} \sum_{v=j}^{m} \binom{v}{j} S_{m,v} \, n^{(v)}.$$

Substituting from the defining relation for the Stirling numbers of the first kind,

$$x^{(v)} = \sum_{i=1}^{v} s_{v,i} \, x^i,$$

collecting like terms with respect to n^v, $v = 0, 1, \cdots, m$, and equating coefficients, yields the relation

$$ {}_jA_{m,i} = \sum_{v=0}^{m-j} \binom{j+v}{j} S_{m,j+v}(s_{j+v,m-i} - j s_{j+v,m-i+1}) $$

for integral $m \geq 0$, i, and j.

4. Application to Hölder summability. For the remainder of this paper $\{S_n\}$ denotes the sequence of partial sums of the arbitrary infinite series $\sum a_n$, and H_n^k denotes the k^{th} order Holder mean of the terms $S_0, S_1, \cdots, S_n$. If

$$ \lim_{n \to \infty} H_n^k = S, $$

then $\sum a_n$ is said to be *summable Hölder of order k to S*, and this fact is denoted by

$$ \sum a_n = S(H, k). $$

In the same manner, the sequence $\{C_n^k\}$ defines Cesàro summability of order k. Likewise, Cesàro summability of order k is denoted by

$$ \sum a_n = S(C, k). $$

The Hölder and Cesàro summability methods are equivalent in that

$$ \sum a_n = S(H, k) $$

if and only if

$$ \sum a_n = S(C, k). $$

At times it will be convenient to use the operator form of denoting the Hölder mean. That is, the k^{th} order Hölder mean of the terms $p_0, p_1, \cdots, p_n$ is denoted by $H^k(p_n)$. If $p_n = S_{n-k}$, $k > 0$, and $S_m = 0$ for $m < 0$, then we have

$$ H^1(S_{n-k}) = \frac{1}{n+1} \sum_{v=0}^{n-k} S_v, \qquad H^k(S_{n-k}) = H^1(H^{k-1}(S_{n-k})) $$

for $k > 1$, and

$$H^0(S_{n-k}) = S_{n-k}.$$

It follows that

$$H^m(H^k(p_n)) = H^{m+k}(p_n) \tag{4.1}$$

and

$$H^m(p_n + q_n) = H^m(p_n) + H^m(q_n), \tag{4.2}$$

where m and k are nonnegative integers.

Letting $k = -m$ in (2.9), $m \geq 0$, we have the following definition for Hölder means of negative integral order.

DEFINITION 1. *For* $m \geq 0$,

$$H_n^{-m} = \sum_{i=0}^{m} (-1)^i \, G_i^m(n+1) \, S_{n-i}. \tag{4.3}$$

Referring to the defining equation for the Cesàro mean,

$$C_n^m = \binom{n+m}{m}^{-1} S_n^m,$$

we see that the first factor on the right is undefined for negative m when n is sufficiently large.

From Definition 1, it follows that (2.9) can be extended to all integral values of k. The Hölder method of summation is said to be *regular* since

$$\sum a_n = S$$

implies

$$\sum a_n = S(H, m)$$

for $m > 0$. With respect to negative order summation, the following extended sense of regularity is immediate.

(i) If $\sum a_n$ is divergent, then it is not summable $(H, -m)$ for any $m \geq 0$.

(ii) If

$$\sum a_n = S(H, -m)$$

for $m \geq 0$, then

$$\sum a_n = S(H, p)$$

for all $p \geq -m$.

Also, the right side of (4.3) can be used to define the operator H^{-m}. From this definition, it follows that properties (4.1) and (4.2) are true for all integral m and k.

Applying summation by parts to (4.3), considering (3.14), and using the operator notation, we find that

$$H^{-m}(S_n) = \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \; G_j^m(n+1) \right) a_{n-i} + S_{n-m} \tag{4.4}$$

for $m \geq 0$. Applying the operator H^{q+m}, we see that

$$H^q(S_n) = H^{q+m} \left[\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \; G_j^m(n+1) \right) a_{n-i} \right] + H^{q+m}(S_{n-m}) \tag{4.5}$$

for integers $m \geq 0$ and q. Since

$$\lim_{n \to \infty} H^q(S_n) = S$$

implies

$$\lim_{n \to \infty} H^{q+m}(S_{n-m}) = S$$

for $m \geq 0$, we have the following theorem as a formal statement of our results.

THEOREM 1. *If*

$$\sum a_n = S(H, q+m), \quad m \geq 0,$$

then

$$\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \; G_j^m(n+1) \right) a_{n-i} = 0(H, q+m)$$

is a necessary and sufficient condition that

$$\sum a_n = S(H, q).$$

Letting $q = 0$ in Theorem 1 yields a Tauberian theorem, that is, a theorem in which ordinary convergence is deduced from the fact that the series is summable and satisfies some further condition (which will vary with the method of summation).

Letting $q = -m$ in Theorem 1, we have the following corollary with respect to negative order summation.

COROLLARY 1. *If*

$$\sum a_n = S,$$

then

$$\lim_{n \to \infty} \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m (n+1) \right) a_{n-i} = 0$$

is a necessary and sufficient condition that

$$\sum a_n = S(H, -m), \quad m \geq 0.$$

Noting that

$$\sum_{j=0}^{i} (-1)^j G_j^m (n+1)$$

is a polynomial at least of degree m, it follows that

$$\lim_{n \to \infty} n^m a_n = 0$$

implies

$$\lim_{n \to \infty} \left[\sum_{j=0}^{i} (-1)^j G_j^m (n+1) \right] a_{n-i} = 0,$$

and consequently we assert:

COROLLARY 2. *If*

$$\sum a_n = S,$$

then

$$\lim_{n\to\infty} n^m a_n = 0, \; m > 0,$$

is sufficient for

$$\sum a_n = S(H, -m).$$

Letting $m = 1$ in (4.5) we have

$$H^q(S_n) = H^{q+1}((n+1)\,a_n) + H^{q+1}(S_{n-1}),$$

or, applying the distributive property of this operator,

$$H^q(S_n) = H^{q+1}(na_n) + H^{q+1}(S_n). \tag{4.6}$$

This relation is equivalent to a well-known analogue to Kronecker's theorem [3, p. 485] which states that if $\sum a_n$ is summable (C, q), then

$$H^1(na_n) = 0\,(C, q).$$

Conversely, it follows from (4.6) that if $\sum a_n$ is summable $(H, q+1)$, then a necessary and sufficient condition that it be summable (H, q) is that

$$na = 0\,(H, q+1).$$

For integral $q \geq 0$ this is analogous to Theorem 65 of [1]. However, in the foregoing case, the statement is true for all integral q. As a further extension of the analogue to Kronecker's theorem, we have the following.

COROLLARY 3. *If*

$$\sum a_n = S(H, q),$$

then

$$\sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j \, G_j^m(n+1) \right) a_{n-i} = 0\,(H, q+m)$$

for integral $m > 0$.

For a special case where the condition of Corollary 2 is necessary as well as sufficient, we shall prove the following.

THEOREM 2. *If* $\sum a_n$ *is a convergent alternating series, then*

$$\lim_{n\to\infty} n^m a_n = 0, \ m \geq 0,$$

is a necessary and sufficient condition for $\sum a_n$ *to be summable* $(H, -m)$.

Proof. Letting $i = 0$ in (3.7), we conclude that there exist constants ${}_k a_{m,j}$, $j = 1, 2, \cdots, m$, such that

$$(4.7) \qquad \sum_{j=0}^{k} (-1)^j G_j^m(n) = (-1)^k\, {}_kA_{m-1,0}\, n^m + {}_k a_{m,1}\, n^{m-1} + {}_k a_{m,2}\, n^{m-2} \cdots + {}_k a_{m,m}$$

for $0 \leq k < m$. We recall from the definition of $G_k^m(x)$ that ${}_kA_{m-1,0} > 0$ for $0 \leq k < m$. Consequently, for a given m, it follows that there exists an n_0 such that for all even k,

$$\sum_{i=0}^{k} (-1)^i G_i^m(n) > 0;$$

and for all odd k,

$$\sum_{i=0}^{k} (-1)^i G_i^m(n) < 0$$

whenever $n \geq n_0$. But by hypothesis, a_{n-k} is alternating in sign with respect to m, whence

$$(4.8) \qquad \left| \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m(n) \right) a_{n-i-1} \right| = \sum_{i=0}^{m-1} \left| \sum_{j=0}^{i} (-1)^j G_j^m(n) \right| |a_{n-i-1}|$$

for $n \geq n_0$. Also, it follows from (4.7) that

$$\lim_{n\to\infty} n^{-m} \left| \sum_{j=0}^{i} (-1)^j G_j^m(n) \right| = {}_iA_{m-1,0},$$

consequently there exist positive constants $n_1 > n_0$, $M(m)$, and $N(m)$ such that

$$n^m M(m) \le \left| \sum_{j=0}^{i} (-1)^j G_j^m(n) \right| \le n^m N(m)$$

for $0 \le i < m$ and $n \ge n_1$. Considering this with (4.8) yields

$$M(m) \sum_{i=0}^{m-1} \left(\frac{n}{n-i-1}\right)^m (n-i-1)^m |a_{n-i-1}| \le \left| \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m(n) \right) a_{n-i-1} \right|$$

$$\le N(m) \sum_{i=1}^{m-1} \left(\frac{n}{n-i-1}\right)^m (n-i-1)^m |a_{n-i-1}|$$

for $n \ge n_1$. We conclude that

$$\lim_{n\to\infty} \sum_{i=0}^{m-1} \left(\sum_{j=0}^{i} (-1)^j G_j^m(n) \right) a_{n-i-1} = 0$$

if and only if

$$\lim_{n\to\infty} n^m a_n = 0.$$

The theorem now follows from Corollary 1.

Letting $q = -1$ in (4.6), we see that any convergent series for which

$$\lim_{n\to\infty} na_n \ne 0$$

is not summable Hölder for any negative order. On the other hand, $\sum 1/(n+1)^2$ is convergent and

$$\lim_{n\to\infty} n^2 a_n \ne 0,$$

yet it follows from direct application of Corollary 1 that this series is summable $(H, -2)$.

References

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