GENERALIZED WALSH TRANSFORM

R. G SELFRIDGE

Introduction. The Walsh functions were first defined by Walsh [6] as a completion of the Rademacher functions in the interval $(0,1)$. As originally defined $\psi_n(x)$ took on the values ± 1. The generalization of Chrestenson [1] permits $\psi_n(x)$ to have the values $e^{2\pi n i/\alpha}$ for some integer α, and also leads to a complete orthonormal system over $[0,1]$. Fine [2] considers the original Walsh function, but with arbitrary subscript, attained by consideration of certain dyadic groups. This paper combines these two generalizations by starting with the Walsh functions as defined by Chrestenson and then using a subsidiary result of Fine to define a Walsh function $\psi_y(x)$ for arbitrary subscript.

With $\psi_y(x)$ one can define a Walsh-Fourier transform for functions in $L_p(0, \infty)$, $1 \le p \le 2$. Many of the ordinary Fourier transform theorems carry over, with certain modifications. For $1 < p \le 2$ the transform is defined as a limit in the appropriate mean, with a Plancherel theorem holding for $p = 2$. Since the proofs carry over from ordinary transforms, or from the L_1 theory only a few theorems are stated for these cases with only brief proofs. The case of L_1 requires considerably more preparation.

Section one is devoted to definitions and obtaining certain varied but very necessary results, such as the evaluation of definite integrals of $\psi_y(x)$ over specified intervals, which are used constantly throughout the paper. Walsh-Fourier series are introduced, and some of the sufficient conditions for convergence of such a series to the generating function are listed but not proved, since the proofs are available in Chrestenson's paper [1].

Section two covers certain basic results for L_1 transforms and associated kernals. A Riemann-Lebesgue theorem follows simply from results of section one, as do sufficient conditions for the convergence to $f(x)$ of the inverse transform of the transform of $f(x)$. The function $P_\beta(x \ominus y)$ is defined as

Received May 26, 1953. This paper is part of a thesis, done under the direction of Professor Paul Civin, done towards the requirements for a Ph.D. at the University of Oregon.

Pacific J. Math. 5 (1955), 451-480

$$\int_0^\beta \overline{\psi_y(t)}\, \psi_x(t)\, dt,$$

and is considered in some detail. With this function it is possible to show that

$$f(y) = \lim_{\beta \to \infty} \int_0^\infty P_\beta(x \ominus y) f(x)\, dx,$$

under fairly general conditions.

Section three is devoted almost entirely to $C, 1$ summability of transforms. If $f(x) \in L_1$ and has a transform $F(y)$ then one has

$$f(x) = \lim_{\beta \to \infty} \int_0^\beta (1 - [y]/\beta)\, F(y)\, \overline{\psi_x(y)}\, dy$$

provided $f(y)$ is locally essentially bounded at x and

$$\int_0^h |f(x+t) - f(x)|\, dt = o(h).$$

It is not possible to remove completely the requirement of local essential boundedness, so that it cannot be said that the inverse transform is $C, 1$ summable almost everywhere to $f(x)$. Again one has that if x has a finite expansion in powers of α then the conditions need only be right-hand conditions.

Finally section four considers transforms of functions in L_p, $1 < p \leq 2$. For $p = 2$ one has a Plancherel theorem, and for all p, $1 < p \leq 2$ one has, if $f(x) \in L_p$, then the transform $F(y) \in L_{p/p-1}$ and

$$F(x) = \frac{d}{dx} \int_0^\infty f(y) P_x(y)\, dy, \quad f(x) = \frac{d}{dx} \int_0^\infty F(y) \overline{P}_x(y)\, dy.$$

One also has $C, 1$ summability of the inverse transform yielding $f(x)$ for almost every x at which $f(y)$ is locally essentially bounded.

1. Definitions and lemmas. Throughout this paper α is taken to be a fixed, but arbitrary integer greater than one. For such an α each $x \geq 0$ has an expansion $x = \sum_{i=N}^\infty x_i \alpha^{-i}$, $0 \leq x_i < \alpha$, that is unique if $x_N \neq 0$ and, in case of choice, a finite expansion. Under these conditions one has:

DEFINITION 1. Degree of $x = D(x) = -N$, where $x = \sum_{i=N}^\infty x_i \alpha^{-i}$, $x_N \neq 0$.

For convenience in what follows it will be supposed that, by addition of dummy coefficients if necessary, $x = \sum_{i=M}^{\infty} x_i \alpha^{-i}$, $M \leq 0$, so that $D(x)$ is not necessarily $-M$.

In all that follows the use of a subscript on a variable will mean the corresponding coefficient of the α-expansion. All intervals will be closed on the left and open on the right unless otherwise stated, and any number that has a finite α-expansion will be called an α-adic rational.

Now if $\omega = e^{2\pi i/\alpha}$ one defines in sucession:

DEFINITION 2. (a) $\phi_0(x) = \omega^{x_1}$

(b) $\phi_n(x) = \phi_0(\alpha^n x)$ $n \geq 0$

(c) If $n = \sum_{i=-N}^{0} n_i \alpha^{-i}$, $\psi_n(x) = \prod_{i=0}^{N} (\phi_i(x))^{n_{-i}}$

(d) $\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y)$, $[x]$ = greatest integer in x.

Since the addition of an integer will not change x_1 it is easy to see that $\phi_0(x)$ has period one, and hence $\phi_n(x)$ has period α^{-n}. Further one has that $\phi_n(x) = \omega^{x_{n+1}}$ and $\psi_n(x) = \omega^z$, $z = \sum_{i=0}^{N} n_{-i} x_{i+1}$, so that $\psi_n(x)$ is of period one.

$\phi_0(x)$ compares with the original Rademacher function, and for the case $\alpha = 2$ the $\psi_n(x)$ reduce to the ordinary Walsh functions. For $\alpha \geq 2$ the completeness and orthonormality of $\psi_n(x)$ over $[0, 1]$ have been shown by Chrestenson [1].

Since $k_i = 0$, $i > 0$, if k is an integer, it is clear that $\psi_n(k) = 1$. Thus Definition 2d) extending the $\psi_n(x)$ to arbitrary subscript is consistent with Definition 2c). Further, by symmetry, one has $\psi_y(x) = \psi_x(y)$.

Before defining transforms for functions there are a number of preliminary steps and additional terminology that can be used to advantage.

LEMMA 1. *If*

$$x = \sum_{i=-N}^{\infty} x_i \alpha^{-i}, \quad y = \sum_{i=-M}^{\infty} y_i \alpha^{-i},$$

then

$$\psi_y(x) = \psi_x(y) = \omega^z, \quad z = \sum_{i=-N}^{M+1} x_i y_{1-i}.$$

Proof. From Definitions (2a, b, c) it is easy to see that $\psi_{[y]}(x) = \omega^{\mu}$ where $\mu = \sum_{i=0}^{M} y_{-i}x_{i+1}$, and similarly for $\psi_{[x]}(y)$. Combining the two yields the result.

DEFINITION 3. *If*

$$x = \sum_{i=-N}^{\infty} x_i \alpha^{-i}, \quad y = \sum_{i=-M}^{\infty} y_i \alpha^{-i},$$

then $z = x \oplus y$ is defined as

$$z = \sum_{i=-T}^{\infty} z_i \alpha^{-i}, \quad \text{where } z_i \equiv x_i + y_i \ (\mathrm{mod}\ \alpha),$$

provided that $z_i \neq \alpha - 1$ for $i \geq K$, and $T = \max(M, N)$.

Similarly $z = x \ominus y$ is defined as

$$z = \sum_{i=-T}^{\infty} z_i \alpha^{-i} \ \text{ where } z_i \equiv x_i - y_i \ (\mathrm{mod}\ \alpha).$$

Clearly for each x, $z = x \oplus y$ or $z = x \ominus y$ is defined for almost every y. The following lemmas then follow quite simply from Lemma 1 and Definition 3.

LEMMA 2. (a) $\psi_y(\alpha^n x) = \psi_x(\alpha^n y) \quad -\infty < n < \infty$

(b) $\psi_y(x)\, \psi_y(z) = \psi_y(x \oplus z)$ each x, a.e. z.

(c) $\psi_y(x)\, \overline{\psi_y(z)} = \psi_y(x \ominus z) = \overline{\psi_y(z \ominus x)}$, each x, a.e. z.

The last identity of Lemma 2 can be written in a slightly different form, which is useful in case of integration with respect to y.

LEMMA 3. *If* $0 \leq y < \beta$ *then there is a* q *such that* $\psi_y(x)\, \overline{\psi_y(z)} = \psi_y(q)$.

Proof. It is only necessary to define q in case $x \ominus z$ is undefined. Take n such that $\beta \leq \alpha^n$, and define $q_i \equiv x_i - z_i$ (mod α) for $i < n$, and then set $q \doteq \sum_{i=-\infty}^{n} q_i \alpha^{-i}$. Clearly this definition will suffice.

DEFINITION 4. (a) $D_n(t) = \sum_{i=0}^{n-1} \psi_i(t)$

(b) $C_b(t) = 1,\ 0 \leq t < b,\ = 0\ \ b \leq t.$

LEMMA 4. (a) $D_{\alpha^N}(t) = \alpha^N C_{\alpha^{-N}}(\{t\})$, where $\{t\} = t - [t]$; $D_n(t) = D_n(\{t\})$

(b) $D_n(t) = D_p(\alpha^N t) D_{\alpha^N}(t) + \psi_{p\alpha^N}(t) D_q(t)$

where $n = p\alpha^N + q$, $0 \leq q < \alpha^N$.

(c) $|D_n(t)| \leq \alpha/2\{t\}$.

Proof. (a) Let t_n be the first non-zero coefficient in the expansion of t. If $n > N$ then by Lemma 1 $\psi_k(t) = 1$ and $D_{\alpha^N}(t) = \alpha^N$. If $n \leq N$, consider the range $p\alpha^n \leq k < (p+1)\alpha^n$. In this range one has

$$\psi_k(t) = \omega^z, \quad z = \sum_{i=n-1}^{N} k_{-i} t_{i+1} \quad \text{or} \quad \psi_k(t) = A\omega^{k_{i-n} t_n}, \quad \text{with } t_n \neq 0.$$

Thus one has

$$\sum_{k=p\alpha^n}^{(p+1)\alpha^n - 1} \psi_k(t) = \sum_{k=p\alpha^n}^{(p+1)\alpha^n - 1} A\omega^{k_{1-n} t_n} = A\alpha^{n-1} \sum_{k=0}^{\alpha-1} \omega^{t_n k} = 0.$$

Now summing on p yields the desired result. The second part is immediate since $\psi_k(t) = \psi_k(\{t\})$.

(b) Since $q < \alpha^N$ one has $n = p\alpha^N \oplus q$ and

$$\sum_{k=0}^{n-1} \psi_k(t) = \sum_{k=0}^{p\alpha^N - 1} \psi_k(t) + \sum_{k=p\alpha^N}^{n-1} \psi_k(t). = \sum_{k=0}^{p-1} \sum_{i=0}^{\alpha^N - 1} \psi_{i+k\alpha^N}(t)$$

$$+ \sum_{i=0}^{q-1} \psi_{p\alpha^N + i}(t) = \sum_{k=0}^{p-1} \psi_{k\alpha^N}(t) \sum_{i=0}^{\alpha^N - 1} \psi_i(t) + \psi_{p\alpha^N}(t) \sum_{i=0}^{q-1} \psi_i(t)$$

$$= \sum_{k=0}^{p-1} \psi_k(\alpha^N t) D_{\alpha^N}(t) + \psi_{p\alpha^N}(t) D_q(t) = D_p(\alpha^N t) D_{\alpha^N}(t) + \psi_{p\alpha^N}(t) D_q(t).$$

(c) Let $D(\{t\}) = N$. Then

$$D_n(t) = D_p(\alpha^N t) D_{\alpha^N}(t) + \psi_{p\alpha^N}(t) D_q(t) = \psi_{p\alpha^N}(t) D_q(t).$$

Now $|D_q(t)| \leq \alpha^N/2 \leq \alpha/2\{t\}$, since $D_q(t) = D_q(t) - D_{\alpha^N}(t)$.

1.0 COROLLARY. (a) $D_{(R+1)\alpha^n}(x) - D_{R\alpha^n}(x) = \displaystyle\sum_{k=R\alpha^n}^{(R+1)\alpha^n - 1} \psi_k(x) = AC_{\alpha^{-n}}(\{x\})$.

(b) $\sum_{k=Ra^n}^{Sa^n-1} \psi_k(x)\overline{\psi_k(y)} = 0$, *unless* $[\alpha^n x] = [\alpha^n y]$,

$$\text{or} \quad \sum_{Ra^n=k}^{Sa^n-1} \psi_k(x)\overline{\psi_k(y)} = AC_1([\alpha^n x] \ominus [\alpha^n y]).$$

(c) *If* $D(\{x\}) = N$ *then* $D_{a^n}(x) = 0$ *for* $n \geq N$.

Corresponding to Lemma 4 there are equivalent results with respect to the integral of $\psi_y(x)$.

DEFINITION 5. $P_b(t) = \int_0^b \psi_t(x)\,dx$.

LEMMA 5. (a) $P_b(t) = \alpha^n P_{ba^{-n}}(\alpha^n t)$

(b) $P_1(x) = P_1([x]) = C_1(x)$

(c) $P_{a^n}(x) = \alpha^n C_{a^{-n}}(x)$

(d) $P_b(t) = C_1(t) D_{[b]}(t) + \psi_{[b]}(t) P_{\{b\}}(t)$.

Proof. (a) This follows immediately by change of variable of integration.

(b) $P_1(x) = \int_0^1 \psi_x(y)\,dy = \int_0^1 \psi_{[x]}(y)\ \psi_{[y]}(x)\,dy = C_1([x])$ by orthonormality.

(c) $P_{a^n}(x) = \alpha^n P_1(\alpha^n x) = \alpha^n C_1(\alpha^n x) = \alpha^n C_{a^{-n}}(x)$.

(d) $\int_0^b \psi_t(y)\,dy = (\int_0^{[b]} + \int_{[b]}^b) \psi_t(y)\,dy$.

Now

$$\int_0^b \psi_t(y)\,dy = \sum_{k=0}^{b-1} \int_k^{k+1} \psi_t(y)\,dy = \sum_{k=0}^{b-1} \psi_k(t) \int_k^{k+1} \psi_{[t]}(y)\,dy = D_{[b]}(t)\, C_1(t),$$

and

$$\int_{[b]}^b \psi_t(y)\,dy = \psi_{[b]}(t) \int_{[b]}^b \psi_{[t]}(y)\,dy = \psi_{[b]}(t) \int_0^b \psi_{[t]}(y)\,dy$$

$$= \psi_{[b]}(t)\, P_{\{b\}}([t]).$$

2.0 COROLLARY. (a) $P_S(y) = C_1(y)D_S(y)$, $P_{Sa^n}(y) = \alpha^n C_{\alpha^{-n}}(y)D_S(\alpha^n y)$

(b) $P_{Sa^n}(y) - P_{Ra^n}(y) = \alpha^n C_{\alpha^{-n}}(y)(D_S(\alpha^n y) - D_R(\alpha^n y))$

(c) $\int_{Ra^n}^{(R+1)a^n} \psi_x(t)\overline{\psi_y}(t)\,dt = 0$ *unless* $[\alpha^n x] = [\alpha^n y]$

(d) $\int_0^a \psi_b(t \ominus x)\,dt = \overline{\psi_b}(x)P_a(b)$

(e) $|P_b(t)| \leq \alpha/2t$, *and if* $t \geq 1$, $|P_b(t)|$
$= |P_{\{b\}}(t)| \leq 1/2$.

Only the last part of this corollary needs proof. Take n such that $\alpha^{-n} \leq t < \alpha^{1-n} \leq 1$, and R so that $|R\alpha^{-n} - b| \leq \alpha^{-n}/2$. Then $P_{R\alpha^{-n}}(t) = 0$ and

$$|P_b(t)| = \left|\int_{R\alpha^{-n}}^{b} \psi_y(t)\,dy\right| \leq \alpha^{-n}/2 \leq \alpha/2t.$$

LEMMA 6. *If* $0 \leq a < \alpha^n$ *then*

$$\int_0^{\alpha^n} f(x)\,dx = \int_0^{\alpha^n} f(a \oplus x)\,dx = \int_0^{\alpha^n} f(a \ominus x)\,dx.$$

Proof. If E is a measurable subset of the interval $[0, \alpha^n)$ set $TE = (a \oplus x : x \in E)$. Now if E is any interval $k\alpha^p \leq x < (k+1)\alpha^p$, then with the exception of a denumerable set of x, TE is an interval $[t\alpha^p, (t+1)\alpha^p)$ and $T^{-1}E$ is an interval $[r\alpha^p, (r+1)\alpha^p)$. Thus $\mu(E) = \mu(T^{-1}E) = \mu(TE)$. Further, since $a < \alpha^n$, if E is in $[0, \alpha^n)$ so are TE and $T^{-1}E$. Now any open set may be expressed as a denumerable sum of non-overlapping such intervals, and thus by the standard argument for any open set E in $[0, \alpha^n)$, $\mu(E) = \mu(TE) = \mu(T^{-1}E)$. Finally one has, with the exception of a denumerable set $(x : f(a \oplus x) < c) = T^{-1}(x : f(x) < c)$ and the first part of the lemma is proved. Clearly the second part follows identically.

3.0 COROLLARY. $\int_{Ra^n}^{(R+1)a^n} f(x)\,dx = \int_{Sa^n}^{(S+1)a^n} f(a \oplus x)\,dx$, $0 \leq a < \alpha^n$.

Now for any function on $L(0,1)$ one can develop a Walsh-Fourier series in terms of $\psi_n(x)$. The properties of such expansions have been studied by Paley [4], Walsh [6] and Fine [2] for $\alpha = 2$, and by Chrestenson [1] for $\alpha \geq 2$. One has then, a Walsh-Fourier series, or W.F.S. given by

$$f(x) \sim \sum_{n=0}^{\infty} c_n \, \psi_n(x), \qquad \text{where } c_n = \int_0^1 f(x)\, \overline{\psi_n(x)}\, dx.$$

In a similar fashion one has the complex conjugate W.F.S.

$$f(x) \sim \sum_{n=0}^{\infty} c_n \, \overline{\psi_n(x)} \qquad \text{where } c_n = \int_0^1 f(x)\, \psi_n(x)\, dx.$$

It is clear that most of the criteria that imply convergence of the W.F.S. will imply convergence of the complex conjugate W.F.S. In particular one has the following, taken from (1) and (2).

4.0. The W.F.S. for $f(y)$ converges to $f(x)$ if

(1) $f(y)$ is B.V. in a neighbourhood of x and x is a point of continuity,

(2) $(f(x)-f(y))/(x-y)$ is integrable over an interval including x.

Note also, that if x is an α-adic rational then these conditions need only be right-hand conditions. One extra result is

(3) The α^nth partial sum of the W.F.S. converges a.e. to $f(x)$ as $n \longrightarrow \infty$.

Notice that by a simple re-definition of $f(x)$, these conditions, which are given for the interval $[0,1)$, hold for any interval $[[x],[x]+1)$, and that one has

LEMMA 7. $f(x) \sim \lim\limits_{n\to\infty} \int_0^1 f(x \ominus z) D_n(z)\, dz,$

$$f(x) \sim \lim_{n\to\infty} \int_0^1 f(x \oplus z) D_n(z)\, dz.$$

Proof. One has

$$f(x) \sim \lim_{n\to\infty} \int_{[x]}^{[x]+1} f(y) D_n(x \ominus y)\, dy,$$

and by 3.0, for $z = x \ominus y$,

$$\int_{[x]}^{[x]+1} f(y) D_n(x \ominus y)\, dy = \int_0^1 f(x \ominus z) D_n(z)\, dz.$$

For the second part the proof is repeated but with the complex conjugate W.F.S.

5.0 COROLLARY. $C_{R\alpha^n}(x) = \sum_{k=0}^{\infty} \psi_k(x) \int_0^{R\alpha^n} \psi_k(y)\,dy$

$$= \sum_{k=0}^{\infty} \overline{P_{R\alpha^n}}(k)\, \psi_k(x) \quad 0 \leq R\alpha^n < 1.$$

Note that by 2.0a the sum is actually finite.

2. L-transforms.

DEFINITION 6. (a) $T(f) = F(y) = \int_0^{\infty} f(x)\,\psi_y(x)\,dx$, the Walsh-Fourier transform.

(b) $T^{-1}(f) = F(y) = \int_0^{\infty} f(x)\,\overline{\psi_y(x)}\,dx$, the inverse Walsh-Fourier transform.

THEOREM 1. *If $f(x) \in L$, then*

$$\lim_{y \to \infty} \int_0^{\infty} f(x)\, \psi_y(x)\,dx = 0.$$

Proof. This is an immediate consequence of the fact that by 2.0b it is true for characteristic functions with α-adic rational end points.

6.0 COROLLARY. *Let $F(y) = T(f)$ for $f(y) \in L$, $s(x, \beta)$ the* $([\beta]-1)$st *partial sum of the complex conjugate* W.F.S. *for $f(y)$ over $[[x], [x]+1)$, and*

$$S(x, \beta) = \int_0^{\beta} \overline{\psi_x(y)}\, F(y)\,dy.$$

Then $\lim_{\beta \to \infty} S(x, \beta) - s(x, \beta) = 0$, uniformly in x.

Proof. Notice that if $x \neq t$ then $t \ominus x \geq 1$. Using 2.0 one has

$$S(x, \beta) = \int_0^{\infty} f(t)\,dt \int_0^{\beta} \overline{\psi_x(y)}\, \psi_t(y)\,dy = \int_0^{\infty} f(t)\, P_{\beta}(t \ominus x)\,dt$$

$$= \int_0^{\infty} f(t)\,dt\, \{C_1(t \ominus x)\, D_{[\beta]}(t \ominus x) + \psi_{[\beta]}(t \ominus x)\, P_{[\beta]}([t \ominus x])\}$$

$$=\int_{[x]}^{[x]+1} f(t)D_{[\beta]}(t \ominus x)\,dt + \int_{[\beta]}^{\beta} \overline{\psi_x(y)}\,dy \int_0^{\infty} f(t)\,\psi_t(y)\,dt$$

$$= s(x, \beta) + \int_{[\beta]}^{\beta} \overline{\psi_x(y)}\,dy \int_0^{\infty} f(t)\,\psi_y(t)\,dt.$$

Now by Theorem 1 the last term goes to zero independently of x.

6.1 COROLLARY. *Using the notation of* 6.0, $S(x, \beta) \longrightarrow f(x)$ *as* $\beta \longrightarrow \infty$ *if* $f(y)$ *satisfies any condition for convergence of the complex conjugate W.F.S. to* $f(x)$.

LEMMA 7. *If* $F(y) = T(C_{R\alpha^{-n}}(x))$, *then for* $\beta \geq \alpha^n$,

$$T^{-1}(F) = C_{R\alpha^{-n}}(x) = \int_0^{\infty} F(y)\,\overline{\psi_x(y)}\,dy = \int_0^{\beta} P_{R\alpha^{-n}}(y)\,\overline{\psi_x(y)}\,dy.$$

Proof. One has

$$F(y) = \int_0^{\infty} C_{R\alpha^{-n}}(x)\,\psi_y(x)\,dx = P_{R\alpha^{-n}}(y) = \alpha^{-n} C_{\alpha^n}(y)\,D_R(\alpha^{-n}y),$$

that $F(y) \in L$, and in fact $F(y) = 0$ for $y \geq \alpha^n$. Hence

$$T^{-1}(F) = \int_0^{\alpha^n} \overline{\psi_x(y)}\,P_{R\alpha^{-n}}(y)\,dy = \alpha^{-n} \sum_{k=0}^{R-1} \int_0^{\alpha^n} \overline{\psi_x(y)}\,\psi_k(\alpha^{-n}y)\,dy$$

$$= \alpha^{-n} \sum_{k=0}^{R-1} P_{\alpha^n}(k\alpha^{-n} \ominus x) = \sum_{k=0}^{R-1} C_{\alpha^{-n}}(k\alpha^{-n} \ominus x) = C_{R\alpha^{-n}}(x),$$

where Lemma 5 has been used extensively.

THEOREM 2. *If* $f(x) \in L$, $F(y) = T(f)$, *and* $F(y) \in L$, *then*

$$f(x) = \int_0^{\infty} F(y)\,\overline{\psi_x(y)}\,dy = T^{-1}(F).$$

Proof. Take $T \geq \alpha^n$, for $b = R\alpha^{-n}$. Then one has

$$\int_0^b f(y)\,dy = \int_0^{\infty} C_{R\alpha^{-n}}(y) f(y)\,dy = \int_0^{\infty} f(y)\,dy \int_0^T \overline{P_b}(t)\,\psi_y(t)\,dt$$

$$=\int_0^T \overline{P_b(t)}F(t)\,dt = \int_0^b \int_0^T \overline{\psi_t(y)}\,F(t)\,dtdy = \int_0^b \int_0^\infty \overline{\psi_t(y)}\,F(t)\,dtdy\,.$$

Now by continuity the equality holds for all b, and the result follows immediately.

Note that if $F(y) \notin L$ one still has

$$\int_0^{R\alpha^{-n}} f(y)\,dy = \int_0^{R\alpha^{-n}} \int_0^{\alpha^n} \overline{\psi_t(y)}\,F(t)\,dtdy\,.$$

DEFINITION 7. $J(a, b, x) = \int_0^a P_b(t \ominus x)\,dt$.

LEMMA 8. (a) $J(a, b, x) = (1 - C_{[x]+1}(a))(1 - C_1(b))$

$$+ C_1(a \ominus x)\int_{[a]}^a D_{[b]}(t \ominus x)\,dt + C_1(b)\int_{[b]}^b \overline{\psi_x(y)}\,D_{[a]}(y)\,dy$$

$$+ \overline{\psi_{[b]}}(x)P_{\{a\}}([b])P_{\{b\}}([a \ominus x])$$

(b) $\lim\limits_{a\to\infty} J(a, b, x) = 1$

(c) $|J(a, b, x)| \leq \alpha^2/2 + 1$

Proof. (a) $J(a, b, x) = \displaystyle\int_0^{[a]}\int_0^{[b]} + \int_{[a]}^a \int_0^{[b]} + \int_0^{[a]}\int_{[b]}^b$

$$+ \int_{[a]}^a \int_{[b]}^b \psi_y(t \ominus x)\,dydt.$$

By 2.0 $P_R(t \ominus x) = C_1(t \ominus x)D_R(t \ominus x)$. Thus, using 3.0, one gets

$$\int_0^{[a]} \int_0^{[b]} \psi_y(t \ominus x)\,dydt = \int_0^{[a]} C_1(t \ominus x)D_{[b]}(t \ominus x)\,dt$$

$$= C_{[a]}(x)\int_0^1 D_{[b]}(t)\,dt = C_{[a]}(x)(1 - C_1(b)) = (1 - C_{[x]+1}(a))(1 - C_1(b)).$$

One also has

$$\int_{[a]}^a \int_0^{[b]} \psi_y(t \ominus x)\,dydt = \int_{[a]}^a C_1(t \ominus x)D_{[b]}(t \ominus x)\,dt$$

$$= C_1(a \ominus x)\int_{[a]}^{a} D_{[b]}(t \ominus x)\,dt \text{ and } \int_0^{[a]}\int_{[b]}^{b} \psi_y(t \ominus x)\,dy\,dt$$

$$= \int_{[b]}^{b} \overline{\psi_y(x)}\, C_1(y) D_{[a]}(y)\,dy = C_1(b)\int_{[b]}^{b} \overline{\psi_y(x)} D_{[a]}(y)\,dy.$$

Finally

$$\int_{[a]}^{a}\int_{[b]}^{b} \psi_y(t \ominus x)\,dy\,dt = \int_{[a]}^{a} \psi_{[b]}(t \ominus x)\,dt \int_{[b]}^{b} \psi_{[t \ominus x]}(y)\,dy$$

$$= \psi_{[b]}(x) P_{\{a\}}([b]) P_{\{b\}}([a \ominus x]).$$

(b) By 2.0e the second and fourth terms of the equality just proved go to zero as $a \longrightarrow \infty$. For $b \geq 1$ the result is immediate. For $b < 1$ the first term of the equality vanishes and for the last term one has

$$\lim_{a \to \infty} \int_0^b \overline{\psi_y(x)} D_{[a]}(y)\,dy \sim \psi_x(z)$$

evaluated at $z = 0$, by Lemma 7a and the first condition of 4.0.

(c) From part (a) if $\alpha^{-n} \leq b < \alpha^{1-n} \leq 1$, one only need consider

$$\int_0^b \overline{\psi_y(x)} D_{[a]}(y)\,dy.$$

This is evaluated in two pieces. For the first one has

$$\int_0^{\alpha^{-n}} \overline{\psi_y(x)} D_{[a]}(y)\,dy = \int_0^1 C_{\alpha^{-n}}(y)\, \overline{\psi_y(x)} D_{[a]}(y)\,dy$$

$$= \alpha^{-n} \int_0^1 \overline{\psi_y(x)} D_{[a]}(y)\, D_{\alpha^n}(y)\,dy$$

which is bounded by 1. For the second piece, using Lemma 4c.

$$\left| \int_{\alpha^{-n}}^{a} \overline{\psi_y(x)} D_{[a]}(y)\,dy \right| \leq \int_{\alpha^{-n}}^{a} \alpha/2\alpha^{-n}\,dy \leq (\alpha^2 - \alpha)/2.$$

Thus for $b < 1$ one has $|J(a, b, x)| \leq \alpha^2/2 - \alpha/2 + 1$. In a similar fashion one can show the remaining inequality for $b \geq 1$.

7.0 COROLLARY. *If* $b \geq 1$ *then*

$$J(R, b, x) = 1 - C_{[x]+1}(R).$$

7.1 COROLLARY. *If* $a, b > x + 1$, $\beta \geq 1$, *then*

$$|J(a, \beta, x) - J(b, \beta, x)| \leq \frac{\alpha^2}{4\beta}\left(\frac{1}{[a \ominus x]} + \frac{1}{[b \ominus x]}\right).$$

Proof. One has

$$J(a, \beta, x) - J(b, \beta, x)$$

$$= \overline{\psi_{[\beta]}(x)}\left\{P_{\{a\}}([\beta])P_{\{\beta\}}([a \ominus x]) - P_{\{b\}}([\beta])P_{\{\beta\}}([b \ominus x])\right\}$$

and 2.0e yields the result.

7.2 COROLLARY.

If $\beta \geq 1$ *then* $|J(a, \beta, x) - J(b, \beta, x)| \leq \beta + 1$

If $\beta < 1$ *then* $|J(a, \beta, x) - J(b, \beta, x)| \leq \beta + \alpha.$

Proof. Let a^*, b^* be the integers nearest a and b. Then

$$J(a, \beta, x) - J(b, \beta, x) = J(a, \beta, x) - J(a^*, \beta, x) + J(b^*, \beta, x) - J(b, \beta, x)$$
$$+ J(a^*, \beta, x) - J(b^*, \beta, x).$$

Now the first two pairs are each bounded by $\beta/2$. The last pair is bounded by 1 for $\beta \geq 1$ (7.0) and by α for $\beta < 1$, this being shown by a proof similar to that for Lemma 8c, with $\alpha^{-n} \leq \beta < \alpha^{1-n}$.

LEMMA 9. *If* $g(x)$ *is integrable over any finite range and both real and imaginary parts tend monotonically to zero as* $x \longrightarrow \infty$ *from some point* x_0 *on, then one has the existence of the following as limits,*

(1) $\int_0^\infty g(t)\psi_y(t)\,dt$

(2) $\int_0^\infty g(t)P_\beta(t \ominus y)\,dt$

(3) $\int_0^\infty g(t)[t]P_\beta(t \ominus y)\,dt \quad \beta \geq 1,$

and if $B > x_0$ *the bounds*

$$(1) \quad \left|\int_B^\infty g(t)\psi_y(t)\,dt\right| \le \frac{4|g(B)|}{y}$$

$$(2) \quad \left|\int_B^\infty g(t)P_\beta(t\ominus y)\,dt\right| \le 4|g(B)|(\beta+\alpha)$$

$$(3) \quad \left|\int_B^\infty g(t)[t]P_\beta(t\ominus y)\,dt\right| \le \frac{2\alpha^2|g(B)|}{[\beta]}, \qquad \beta > \alpha y$$

Proof. After splitting the integrals into real and imaginary parts and using the second mean value theorem, then the existence of the first integral and its bound follow from 2.0e and of the second integral and its bound from 7.2.

For the third case note that since $\beta \ge 1$, one has, by 7.0, for $k > y+1$,

$$\int_k^{k+1} [t]P_\beta(t\ominus y)\,dt = k(J(k+1,\beta,y) - J(k,\beta,y)) = 0.$$

Thus for $a, b > \alpha y$ it is easy to show that

$$\frac{|\int_a^b [t]P_\beta(t\ominus y)\,dt| \le \alpha^2}{2[\beta]}.$$

The existence and bound of the third integral follow immediately.

THEOREM 3. *If* $f(x)$ *is integrable over any finite range, and satisfies any condition producing convergence of the complex conjugate* W.F.S. *to* $f(y)$ *then*

$$f(y) = \lim_{\beta\to\infty} \int_0^\infty f(x)P_\beta(x\ominus y)\,dx,$$

provided

$$(1) \quad \frac{f(x)}{1+x \in L}$$

or

$$(2) \quad \frac{f(x)}{[x]} \text{ is B.V. in } a, \infty \text{ for some } a > 0 \text{ and } \lim_{x\to\infty} \frac{f(x)}{[x]} = 0$$

or

(3) $\dfrac{1}{x}\displaystyle\int_1^x f(y)\,dy$ *is* B.V. *in* a, ∞ *for some* $a > 0$ *and has limit* 0 *as* $x \longrightarrow \infty$.

Proof. One has

$$\int_0^b f(t)P_\beta(t \ominus y)\,dt = \int_0^\beta \psi_y(x)\,dx \int_0^b f(t)\,\psi_x(t)\,dt$$

and if $b > y + 1$ this has limit $f(y)$ as $\beta \longrightarrow \infty$ by 6.1. It remains only to show that any of the three conditions implies

$$\int_b^\infty f(t)P_\beta(t \ominus y)\,dt \longrightarrow 0$$

as b and $\beta \longrightarrow \infty$. For Condition 1 one has by 2.0

$$\left|\int_b^\infty f(t)P_\beta(t \ominus y)\,dt\right| \leq \int_b^\infty |f(t)|\,\alpha/(t \ominus y)\,dt \quad \text{and} \quad f(t)/1+t \in L.$$

For Condition 2 $f(x)/[x]$ is the difference of functions tending to zero monotonically and the last bound of Lemma 9 suffices. Finally for Condition 3 let

$$g(x) = \frac{1}{x}\int_1^x f(y)\,dy \qquad\qquad x \geq \sigma > 0,$$

and one has $xg'(x) + g(x) = f(x)$. Now $xg'(x)$ satisfies Condition 1 and $g(x)$ Condition 2.

Working with the available inequalities it is possible to prove several theorems of the following type, but with weaker conditions on $f(x)$ and $g(x)$. The proof of the following is immediate with a change of order of integration.

THEOREM 4. *If* $f(x), g(x) \in L$, $T(f) = F(y)$, $T(g) = G(y)$, *then*

$$\int_0^\infty f(x)G(x)\,dx = \int_0^\infty F(x)g(x)\,dx.$$

3. The purpose of this section is to build up to the following theorem.

THEOREM 5. *If* $f(x) \in L$ *and* $T(f) = F(y)$, *then*

$$f(x) = \lim_{\beta\to\infty} \int_0^\beta (1-[u]/\beta) F(u)\, \overline{\psi_x(u)}\, du$$

provided

(1) $f(y)$ *is locally essentially bounded at* x (*essentially bounded in a neighborhood of* x)

(2) $\int_0^h |f(x+t) - f(x)|\, dt = o(h) \qquad h \longrightarrow 0.$

Further if x *is an* α*-adic rational Conditions* 1 *and* 2 *need only be right hand conditions.*

Theorem 5 is the best possible in the sense that Condition (2) holds almost everywhere and (1) cannot be completely removed. This follows from the following.

THEOREM 6. *There is a function* $f(x) \in L$ *satisfying* (2) *but not* (1) *of Theorem* 5 *for which the result of Theorem* 5 *does not hold.*

LEMMA 10. *If* $f(y) \in L$, *then for each* x

$$\lim_{\beta\to\infty} \int_0^\beta (1-[u]/\beta) F(u)\, \overline{\psi_x(u)}\, du - f(x)$$

$$= \lim_{\beta\to\infty} \int_{[x]}^{[x]+1} (f(y) - f(x))\, dy \int_0^{[\beta]} (1-[u]/\beta)\, \overline{\psi_x(u)}\, \psi_y(u)\, du.$$

Proof. For each $\beta \geq 1$ one has

$$\int_0^\beta (1-[u]/\beta) F(u)\, \overline{\psi_x(u)}\, du = \left\{ \int_0^{[x]} + \int_{[x]}^{[x]+1} + \int_{[x]+1}^\infty \right\} f(y)\, dy$$

$$\times \int_0^\beta (1-[u]/\beta)\, \overline{\psi_x(u)}\, \psi_y(u)\, du = I + J + K.$$

Since $y \leq [x]$, by 2.0

$$I = \int_0^{[x]} f(y)\,dy \int_0^{\beta} (1 - [u]/\beta)\, \overline{\psi_x(u)}\, \psi_y(u)\,du$$

$$= \int_0^{[x]} f(y)\,dy \left\{ \sum_{k=0}^{[\beta]-1} (1 - k/\beta) \int_k^{k+1} \overline{\psi_x(u)}\, \psi_y(u)\,du \right.$$

$$\left. + (1 - [\beta]/\beta) \int_{[\beta]}^{\beta} \overline{\psi_x(u)}\, \psi_y(u)\,du \right\}$$

$$= 0 + \int_0^{[x]} f(y) P_\beta(y \ominus x)(1 - [\beta]/\beta)\,dy = O(1/\beta) \int_0^{\infty} f(y)\,dy = O(1/\beta).$$

In a similar fashion $K = O(1/\beta)$ and

$$\int_{[x]}^{[x]+1} dy \int_0^{\beta} (1 - [u]/\beta)\, \overline{\psi_x(u)}\, \psi_y(u)\,du$$

$$= \int_{[x]}^{[x]+1} dy \left(\sum_{k=0}^{[\beta]-1} (1 - k/\beta) \overline{\psi_k(x)}\, \psi_k(y) + (1 - [\beta]/\beta) \overline{\psi_\beta(x)}\, \psi_\beta(y) \right)$$

$$= \sum_{k=0}^{[\beta]-1} (1 - k/\beta)\, \overline{\psi_k(x)} \int_{[x]}^{[x]+1} \psi_k(y) + O(1/\beta) = 1 + O(1/\beta).$$

Thus the proof of Theorem 5 reduces to showing

8.0 $$\lim_{\beta \to \infty} \int_0^1 f(t)\,dt \int_0^{[\beta]} (1 - [u]/\beta)\, \psi_t(u)\,du = 0,$$

where $f(t)$ is redefined appropriately and $f(0)$ is assumed to be zero. In the sections to follow it should be remembered that usually $0 \leq t < 1$.

DEFINITION 8. $L_n(t) = \sum_{k=0}^{n-1} k\, \psi_k(t)$

$$\phi(\beta, t) = \int_0^{[\beta]} (1 - [u]/\beta)\, \psi_t(u)\,du$$

$$F_k(t) = \frac{1}{k} \sum_{n=1}^{k} D_n(t).$$

LEMMA 11. (a) $\phi(\beta, t) = P_{[\beta]}(t) - L_{[\beta]}(t)/\beta$

(b) $\phi([\beta], t) = F_{[\beta]}(t)$.

Proof.

$$\phi(\beta, t) = \int_0^{[\beta]} \psi_t(u)\,du - \sum_{k=0}^{[\beta]-1} k/\beta \int_k^{k+1} \psi_t(u)\,du$$

$$= P_{[\beta]}(t) - L_{[\beta]}(t)/\beta$$

$$= D_{[\beta]}(t) - 1/\beta \left\{([\beta]-1) \sum_{k=0}^{[\beta]-1} \psi_k(t) - \sum_{k=0}^{[\beta]-1} ([\beta]-1-k)\,\psi_k(t)\right\}$$

$$= D_{[\beta]}(t) - 1/\beta \left\{([\beta]-1) D_{[\beta]}(t) - \sum_{k=0}^{[\beta]-1} \sum_{j=0}^{k-1} \psi_j(t)\right\}$$

$$= \frac{1-[\beta]+\beta}{\beta} D_{[\beta]}(t) + 1/\beta \sum_{k=0}^{[\beta]-1} D_k(t) = F_\beta(t)$$

where the last equality holds only if β is an integer.

For the purposes of what follows the useful part of this lemma is only the very simple first part. However it is of interest to point out that $F_k(t)$ is the Fejer kernal for the W.F.S. and that $C, 1$ summability of the transform will thus imply $C, 1$ summability of the W.F.S. (see [1]). However the converse is not immediately true, since $C, 1$ summability of the W.F.S. will only imply $C, 1$ summability of the transform for integral β. To proceed from summability for integral β to that for all β seems to require precisely the conditions of Theorem 5. At the same time, restricting $C, 1$ summability of the transforms to integral β will not ease the problem, because the present method of proof works equally well in both cases, and no change has been found as yet that alleviates the problem even for integral β.

LEMMA 12. $L_{A\alpha^n}(t) = D_A(\alpha^n t) L_{\alpha^n}(t) + \alpha^{2n} C_{\alpha^{-n}}(t) L_A(\alpha^n t)$.

Proof. $\displaystyle\sum_{k=0}^{A\alpha^n} k\,\psi_k(t) = \sum_{R=0}^{A-1}\sum_{k=0}^{\alpha^n-1} (k + R\alpha^n)\,\psi_{k+R\alpha^n}(t)$

$$= \sum_{R=0}^{A-1} R\alpha^n \psi_{R\alpha^n}(t) \sum_{k=0}^{\alpha^n-1} \psi_k(t) + \sum_{k=0}^{\alpha^n-1} k\psi_k(t) \sum_{R=0}^{A-1} \psi_R(\alpha^n t)$$

$$= \alpha^n L_A(\alpha^n t) D_{\alpha^n}(t) + L_{\alpha^n}(t) D_A(\alpha^n t).$$

LEMMA 13. *Let*

$$D(\beta) = N,\ t = t_1\alpha^{-A} + t_2\alpha^{-\beta} + \gamma,\ 0 < t_1 < \alpha,\ 0 \le t_2 < \alpha, 0 \le \gamma < \alpha^{-\beta}.$$

Then

(1) $|\phi(\beta, t)| \le \beta < \alpha^{N+1}$

(2) *If* $A \le N$ *then* $|\phi(\beta, t)| \le \alpha^A$

(3) *If* $A < B \le N$ *then* $|\phi(\beta, t)| \le \alpha^{A+B-N}$.

Proof. (1) is clear from the definition of $\phi(\beta, t)$. For (2) take Q an integer such that $|\beta - Q\alpha^A| \le \alpha^A/2$. Then

$$\phi(\beta, t) = \int_{Q\alpha^A}^{\beta} (1 - [u]/\beta)\,\psi_t(u)\,du - 1/\beta\, L_{Q\alpha^A}(t) + P_{Q\alpha^A}(t).$$

Now the integral is clearly bounded by

$$(\beta - Q\alpha^A)^2/\beta \le \alpha^{2A}/4\beta,\ P_{Q\alpha^A}(t) = 0,$$

and $L_{Q\alpha^A}(t) = D_Q(\alpha^A t) L_{\alpha^A}(t)$ which is bounded by $Q\alpha^{2A}/2$.

For Case (3) define Q and R,

$$|\beta - Q\alpha^A| \le \alpha^A/2,\ \ |Q\alpha^A - R\alpha^B| \le \alpha^B/2.$$

Then

$$\phi(\beta, t) = \left\{ \int_{R\alpha^B}^{Q\alpha^A} + \int_{Q\alpha^A}^{\beta} \right\} (1 - [u]/\beta) \psi_t(u)\, du + P_{R\alpha^B}(t) - 1/\beta\, L_{R\alpha^B}(t).$$

Now

$$L_{R\alpha^B}(t) = D_R(\alpha^B t) L_{\alpha^B}(t), \text{ and } L_{\alpha^B}(t) = L_{\alpha^A \alpha^{B-A}}(t) = D_{\alpha^{B-A}}(\alpha^A t) L_{\alpha^A}(t).$$

But

$$D_{\alpha^{B-A}}(\alpha^A t) = \alpha^{B-A} C_{\alpha^{B-A}}(\alpha^A t) = 0.$$

Thus the last two terms of the equality are zero, and again the second of the integrals is bounded by $\alpha^{2A}/4\beta$. Finally the first integral is given by

$$1/\beta \,|L_{Q\alpha^A}(t) - 1/\beta\, L_{R\alpha^B}(t)| = 1/\beta\, |L_{\alpha^A}(t)(D_Q(\alpha^A t) - D_{R\alpha^{B-A}}(\alpha^A t))|$$

$$\leq 1/\beta\, \alpha^{2A}/2\, |Q - R\alpha^{B-A}| \leq \alpha^{A+B}/2\beta \leq \alpha^{A+B-N}/2.$$

LEMMA 14. *If* $0 \leq g(t) < 1$, $g(t) = o(1)$ *as* $t \longrightarrow 0$, *and* $g(t)$ *is monotone increasing with* t, *then*

$$\lim_{\beta \to \infty} \int_0^1 g(t)\, \phi(\beta, t)\, dt = 0$$

Proof. Let $N = D(\beta)$ and define the following intervals.

$$I_{N+1} \qquad 0 \leq t < \alpha^{-N}$$

$I_{A,N+1}$ the $\alpha - 1$ intervals $R\alpha^{-A} \leq t < R\alpha^{-A} + \alpha^{-N}$

$I_{A,B}$ the $\alpha - 1$ intervals $R\alpha^{-A} + \alpha^{-B} \leq t < R\alpha^{-A} + \alpha^{1-B}$.

Now the integral $\int_0^1 g(t) \phi(\beta, t)\, dt$ is to be evaluated over these intervals and then summed, using the results of the previous lemma frequently.

Since $g(t) = o(1)$, we have

$$\left|\int_0^{\alpha^{-N}} g(t)\phi(\beta, t)\,dt\right| \le \int_0^{\alpha^{-N}} g(t)\alpha^{N+1}\,dt = o(1).$$

Now for $M \le A$ one has

$$\left|\int_{I_{A,B}} g(t)\phi(\beta,t)\,dt\right| \le g(\alpha^{1-M})\int_{I_{A,B}} |\phi(\beta,t)|\,dt$$

$$\le g(\alpha^{1-M})\alpha^{A+B-N}\int_{I_{A,B}} dt \le g(\alpha^{1-M})\alpha^{A+B-N}\alpha^{-B}(\alpha-1)^2 \le g(\alpha^{1-M})\alpha^{A-N+2}.$$

Thus

$$\left|\left(\int_{I_{A,N+1}} + \sum_{B=A+1}^{N}\int_{I_{A,B}}\right) g(t)\phi(\beta,t)\,dt\right|$$

$$\le g(\alpha^{1-M})(\alpha^{A-N+2} + (N-A)\alpha^{A-N+2}) = g(\alpha^{1-M})(N-A+1)\alpha^{A-N+2},$$

and summing over A from M to N yields

$$g(\alpha^{1-M})\sum_{A=M}^{N}(N-A+1)\alpha^{A-N+2} \le Rg(\alpha^{1-M})$$

where R is independent of M and N for sufficiently large M and N.

For the remaining intervals where $A < M$, an identical argument yields

$$\left|\int_{\alpha^{-M}\le t} g(t)\phi(\beta,t)\,dt\right| \le g(1)\sum_{A=1}^{M}(N-A+1)\alpha^{A-N+2} \le g(1)N\alpha^{M-N+3}.$$

Adding these results yields

$$\left|\int_0^1 g(t)\phi(\beta,t)\,dt\right| \le g(1)N\alpha^{M-N+3} + o(1) + Rg(\alpha^{1-M}),$$

and the proof is completed by choosing first M and then N sufficiently large.

9.0 COROLLARY. $\lim_{\beta\to\infty}\int_\rho^1 \phi(\beta,t)\,dt = 0 \qquad 0<\rho\le 1.$

9.1 COROLLARY. $\lim_{\beta\to\infty}\int_\rho^1 f(t)\phi(\beta,t)\,dt = 0, \qquad f(t)\in L(\rho,1).$

This corollary is an immediate result of 9.0 and a well known theorem (cf. [3] p. 231).

DEFINITION 9. A set E is of metric density k at x if

$$\lim_{a-b\to 0}\frac{|E\cap(a,b)|}{|(a,b)|} = k, \text{ where } a\le x\le b.$$

To allow for α-adic rationals this density can be restricted to only right-hand intervals. Thus if x is an α-adic rational it is to be understood that only intervals (x,b) are used.

LEMMA 15. *If* $f(t)\in L$ *and*

$$\int_0^h |f(x+t)-f(x)|\,dt = o(h),$$

then there exist $g(t)$ *such that* $0\le g(t)\le 1$, $g(t)$ *monotone decreasing to zero as* $t\longrightarrow 0$ *and* $E=(x+t\colon\ |f(x+t)-f(x)|\ge g(t))$ *has metric density zero at* x.

Proof. Set $E_n=(x+t\colon\ |f(x+t)-f(x)|\ge 2^{-n})$. Clearly E_n must have metric density zero at x. Now let

$$\mu(E_n,h)=\mu(E_n\cap(x-h,x+h)) \qquad \text{or}$$

$$\mu(E_n,h)=\mu(E_n\cap(x,x+h))$$

if x is an α-adic rational. Since E_n has density zero at x, for each n there is an h_n such that $\mu(E_n,h)\le |h|2^{-n}$, $|h|\le h_n$. Clearly it is possible to take h_n a power of α and $h_{n+1}<h_n$.

Now define $g(h)=2^{-n}$ if $h_{n+1}\le |h|<h_n$, $g(h)=1$ if $|h|\ge h_1$. Since $h_n\longrightarrow 0$ as $n\longrightarrow\infty$, $g(h)$ satisfies the requirements of the lemma, and it

remains to show that E has metric density zero at x.

Let I be any interval including x (right-hand only if x is an α-adic rational) and set $F = I \cap E$, $A_n = F \cap (x+h\colon \alpha^{-n} \leq h < \alpha^{1-n})$, $\mu(A_n) = a_n$.

For each n there is an m such that $h_{m+1} < \alpha^{1-n} \leq h_m$, so that by the construction of $g(h)$, $a_n \leq \alpha^{1-n}2^{-m} = \sigma(n)\alpha^{-n}$, where $\sigma(n)$ is monotone decreasing to zero as $n \longrightarrow \infty$.

Hence

$$\sum_{n=A}^{\infty} a_n \leq \sum_{n=A}^{\infty} \sigma(n)\alpha^{-n} \leq \sigma(A) \sum_{n=A}^{\infty} \alpha^{-n} = o(\alpha^{-A}).$$

Thus for $\alpha^{-A-1} < |h| \leq \alpha^{-A}$, $\mu(F, h) = o(\alpha^{-A}) = o(h)$, and F, and hence E, is of metric density zero at x.

Proof of Theorem 5. The problem has been reduced to considering

$$\int_0^1 f(y)\, \phi(\beta, y)\, dy.$$

Let I be the neighbourhood of 0 in which $f(y)$ is essentially bounded, the bound assumed to be 1. By Lemma 15 one has $g(t)$ such that $E = (t\colon |f(t)| \geq g(t))$ is of metric density zero at zero. Let $f(y) = f_1(y) + f_2(y) + f_3(y)$ where

$$f_1(g) = \begin{cases} f(y) & y \in I - E \\ 0 & y \in C(I-E) \end{cases} \qquad f_2(y) = \begin{cases} f(y) & y \in I \cap E \\ 0 & y \in C(I \cap E) \end{cases}$$

$$f_3(y) = \begin{cases} f(y) & y \in CI \\ 0 & y \in I. \end{cases}$$

The proof reduces to showing

$$\lim_{\beta \to \infty} \int_0^1 f_i(y)\, \phi(\beta, y)\, dy = 0.$$

For $i = 1$ this follows at once from Lemma 14, and for $i = 3$ from 9.1. For the case $i = 2$ define $E_n = (\alpha^{-n}, \alpha^{1-n})$, $A_n = I \cap E \cap E_n$ and $\mu(A_n) = a_n$. If $D(\beta) = N$ one has

$$\left| \int_0^1 f_2(y)\, \phi(\beta, y)\, dy \right| \leq \left| \sum_{n=1}^{N} \int_{A_n} \phi(\beta, y)\, dy \right| + \int_0^{\alpha^{-n}} |f_2(y)\, \phi(\beta, y)|\, dy$$

$$\leq \alpha^{N+1} \int_0^{\alpha^{1-N}} |f_2(y)|dy + \sum_{n=1}^{N} \int_{A_n} |\phi(\beta, y)|dy.$$

Now since $\int_0^h |f(y)|dy = o(h)$ one has

$$\alpha^{N+1} \int_0^{\alpha^{1-N}} |f_2(y)|dy = o(1)$$

as $N \longrightarrow \infty$. Since $I \cap E$ has metric density zero at zero $\sum_{n=B}^{\infty} a_n \leq k(B)\alpha^{-B}$, where $k(B)$ is taken monotone decreasing to zero. Hence $k(n)\alpha^{-n}$ is strictly decreasing and for each N there is an unique T such that $k(T)\alpha^{-T} \leq \alpha^{-N} < k(T-1)\alpha^{1-T}$. Now for any R, $T \leq R \leq N$, one has $a_R \leq k(R)\alpha^{-R} \leq k(T)\alpha^{-R}$, and hence

$$\sum_{R=T}^{N} \int_{A_R} |\phi(\beta, y)|dy \leq k(T) \sum_{R=T}^{N} \alpha^{-R}\alpha^{R} = (N-T+1)k(T).$$

For $R < T$ a procedure similar to that of Lemma 14 will yield

$$\sum_{R=1}^{T-1} \int_{A_R} |\phi(\beta, y)|dy \leq \sum_{R=1}^{T-1} (N-R+1)\alpha^{R+1-N}$$

$$\leq \alpha(\alpha-1)^{-2} \{\alpha^{T+1-N} + (N-T+1)\alpha^{T-N}(\alpha-1) + O(N)\alpha^{-N}\}.$$

Thus

$$\int_0^1 |f_2(y)\phi(\beta, y)|dy = o(1) + (N-T)k(T) + A\alpha^{T-N} + B(N-T+1)\alpha^{T-N}.$$

Now by choice of T one has $k(T) \leq \alpha^{T-N} < k(T-1)$ or

$$k(T)(N-T) \leq k(T)\log(1/k(T))/\log\alpha.$$

Hence one has in succession

$$\lim_{T\to\infty} k(T)(N-T) = 0, \quad \lim_{T\to\infty} \alpha^{T-N} = 0,$$

$$\lim_{T\to\infty} (N-T+1)\alpha^{T-N}, \quad \lim_{T\to\infty} (N-T+1)\alpha k(T-1) = 0.$$

Finally as $\beta \longrightarrow \infty$, so does N, and hence $T \longrightarrow \infty$, so that

$$\int_0^1 f_2(y)\, \phi(\beta, y)\, dy = o(1) \quad \text{as} \quad \beta \longrightarrow \infty.$$

10.0 COROLLARY. *If $f(y)$ is integrable over any finite interval, then for almost every x at which $f(y)$ is locally essentially bounded we have*

$$f(x) = \lim_{\beta \to \infty} \lim_{R \to \infty} \int_0^R f(y)\, \phi(\beta, y \ominus x)\, dy$$

$$= \lim_{\beta \to \infty} \int_{[x]}^{[x]+1} f(y)\, \phi(\beta, y \ominus x)\, dy.$$

Proof. For $[y] \neq [x]$, one has

$$\phi(\beta, y \ominus x) = \sum_{k=0}^{[\beta]-1} \int_k^{k+1} (1 - k/\beta)\, \overline{\psi_x(u)}\, \psi_y(u)\, du = 0.$$

Thus the inner limit exists, and now changing the order of integration completes the proof since the requirements of Theorem 5 are satisfied.

Proof of Theorem 6. Define $f(x) = \alpha^m$ if $\alpha^{-m} \le x < \alpha^{-m} + \alpha^{-2m}/m$, $f(x) = 0$ elsewhere, and $f(0) = 0$. Clearly $f(x) \in L$ and $\int_0^h f(x)\,dx = o(h)$. Now consider the sequence $\beta = \alpha^N$, for N even. Now if $t = t_1 \alpha^{-A} + t_2 \alpha^{-B} + \gamma$, as before, then for $A < B \le N$

$$\phi(\beta, t) = -1/\beta\, L_\beta(t) = -\alpha^{-N} L_{\alpha^B}(t) D_{\alpha^{N-B}}(\alpha^B t)$$

$$= -\alpha^{-N} D_{N-B}(\alpha^B t) D_{\alpha^{B-A}}(\alpha^A t) L_{\alpha^A}(t) = 0,$$

and for $A \le N < B$,

$$-\alpha^N \phi(\alpha^A \alpha^{N-A}, t) = L_{\alpha^N}(t) = D_{\alpha^{N-A}}(\alpha^A t) L_{\alpha^A}(t) + C_{\alpha^{-A}}(t) L_{\alpha^{N-A}}(\alpha^A t) \alpha^{2A}$$

$$= \alpha^{N-A} C_{\alpha^{A-N}}(\{\alpha^A t\}) L_{\alpha^A}(t) = \alpha^{N-A} L_{\alpha\alpha^{A-1}}(t)$$

$$= \alpha^{N-A} \{ D_\alpha(\alpha^{A-1}) L_{\alpha^{A-1}}(t) + C_{\alpha^{1-A}}(t) L_\alpha(\alpha^{A-1} t) \alpha^{2A-2} \}$$

$$= \alpha^{N-A}\alpha^{2A-2}L_\alpha(\alpha^{A-1}\ t) = \alpha^{N+A-2}\sum_{k=0}^{\alpha-1} k\psi_k^{(\alpha^{A-1}t)},$$

and thus for $A \leq N < B$, $\phi(\alpha^N, t) = \alpha^{A-2}B_A$.

Now take $A \geq (N-4)/2$, so that for $N > N_0$ $\alpha^{-2A}/A \leq \alpha^{-N}$. By the construction of $f(x)$ each interval where $f(x) \neq 0$ is entirely within an interval where $\phi(\beta, t) = \alpha^{A-2}B_1$. Again consider intervals $\alpha^{-n} \leq t < \alpha^{1-n}$. As in the proof of Theorem 5

$$\int_0^{\alpha-N} f(t)\phi(\beta, t)\,dt = o(1), \quad \text{as } N \longrightarrow \infty,$$

and

$$\sum_{n=(N-4)/2}^{N} \int_{A_n} f(t)\,\phi(\beta, t)\,dt$$

$$\leq \sum_{n=(N-4)/2}^{N} \alpha^n B_1\,\alpha^{n-2}\alpha^{-2n}/n = B_1\alpha^{-2}\sum_{n=(N-4)/2}^{N} 1/n.$$

For the remaining sets one has

$$\left|\sum_{n=1}^{(N-6)/2}\int_{A_n} f(t)\phi(\beta, t)\,dt\right| \leq \sum_{n=1}^{(N-6)/2} \alpha^n B_n\ \alpha^{n-2}\,\alpha^{1-N} \leq \alpha^{-5}\max_A |B_A|.$$

Now a quite simple computation shows $\max_A |B_A| \leq \alpha|B_1|$. Hence one has $\int_0^1 f(t)\phi(\beta, t)\,dt$ split into three pieces, the first of which is $o(1)$, the second tends to $B_1\alpha^{-2}\log 1/2$, and the third is bounded by $|B_1|\alpha^{-4}$. Hence the conclusion of Theorem 5 cannot hold.

The problem of showing that Theorem 5 holds almost everywhere, or of constructing a counter-example has not yet been solved.

4. L_p transforms. To a great extent the results of L_p transforms for $1 < p \leq 2$ can be proved similar to the proofs used, for example, by Titchmarsh [5]. For this reason the proofs will not be given, or given only briefly.

LEMMA 16. *For any finite set of α_i, $i = 0, 1, \cdots, n$, and any $N > 0$,*

$$\int_0^{\alpha^N}\left|\sum_{i=0}^{n} a_i\ \psi_x(i\alpha^{-N})\right|^2 dx = \alpha^N\sum_{i=0}^{n}|a_i|^2,$$

LEMMA 17. *For any finite set of* a_i, $i=0, 1, \cdots, n$, *and any* $N > 0$, *and* $1 < p < 2$

$$\int_0^{\alpha^N} \left| \sum_{i=0}^{n} a_i \psi_x(i\alpha^{-N}) \right|^q dx \le \alpha^N \left(\sum_{i=0}^{n} |a_i|^p \right)^{q/p}, \qquad \text{where } q = p/(p-1).$$

THEOREM 7. *If* $f(x) \in L_p$ $1 < p \le 2$, *then there is a function* $F(y) \in L_q$ *such that*

$$F(y) = \underset{B \to \infty}{\text{l.i.m.}}\ (q) \int_0^B f(x)\, \psi_y(x)\, dx \quad \text{and} \quad \|F\|_q \le \|f\|_p.$$

Proof. Let

$$F(x, a) = \int_0^a f(y)\, \psi_x(y)\, dy$$

and $n = [\alpha^N a]$. It is not difficult to show that over any finite interval $F(x, a)$ may be approximated uniformly by

$$\sum_{k=0}^{n-1} a_k\, \psi_x(k\alpha^{-N}) \quad \text{where } a_k = \int_{k\alpha^{-N}}^{(k+1)\alpha^{-N}} f(y)\, dy,$$

and hence, using Lemma 16 or 17

$$\int_0^D |F(x, a)|^q dx \le \left(\int_0^a |f(y)|^p dy \right)^{q/p}.$$

From this Theorem 7 is almost immediate.

THEOREM 8. *If* $f(x) \in L_2$,

$$T_2(f) = F(y) = \underset{B \to \infty}{\text{l.i.m.}}\ (2) \int_0^B f(x)\, \psi_y(x)\, dx,$$

then

$$f(x) = \text{l.i.m.}\ (2) \int_0^B F(y)\, \overline{\psi_x(y)}\, dy \quad \text{and} \quad \|F\|_2 = \|f\|_2.$$

Proof. From Lemma 16 one has $\overline{g(x)} = T_2(\overline{F})$ and $\|g\|_2 \le \|F\|_2 \le \|f\|_2$.

Now take p an α-adic rational, and take the limits in the following along sequences of integers, which is possible since the limits exist,

$$\int_0^p g(x)\,dx = \int_0^p dx \underset{B\to\infty}{\text{l.i.m.}} \int_0^B \overline{\psi_x(y)}\,dy \underset{C\to\infty}{\text{l.i.m.}} \int_0^C f(t)\,\psi_y(t)\,dt$$

$$= \lim_{B\to\infty} \int_0^p dx \lim_{C\to\infty} \int_0^C f(t)\,dt \int_0^B \overline{\psi_x(y)}\;\psi_t(y)\,dy$$

$$= \lim_{B\to\infty} \int_0^p dx \int_0^{[p]+1} f(t)\,dt \int_0^B \overline{\psi_x(y)}\;\psi_t(y)\,dy$$

$$= \lim_{B\to\infty} \int_0^p dx \int_0^B \overline{\psi_x(y)}\,dy \int_0^{[p]+1} f(t)\;\psi_y(t)\,dt = \int_0^p f(t)\,dt,$$

where the last step is justified by Theorem 2 after defining a new $f(t)$. Now continuity of the integrals yields the desired result.

THEOREM 9. *If* $f(x) \in L_p$, $1 < p \le 2$ *and* $T_p(f) = F(y)$, *then*

$$F(x) = \frac{d}{dx}\int_0^\infty f(y)\,P_x(y)\,dy \quad \textit{and} \quad f(x) = \frac{d}{dx}\int_0^\infty F(y)\,\overline{P_x(y)}\,dy.$$

Proof. The existence of the integrals follows since $P_x(y) \in L_n$ for $n > 1$. Then the first result follows from

$$\int_0^x F(y)\,dy = \lim_{B\to\infty} \int_0^x dy \int_0^B f(t)\;\psi_y(t)\,dt = \int_0^\infty f(t)\,P_x(t)\,dt.$$

For the second result set

$$G(x) = \int_0^\infty F(y)\,\overline{P_x(y)}\,dy.$$

It is simple to show that $G(x)$ is continuous. Then for an α-adic rational $x = k\alpha^{-n}$

$$\int_0^\infty F(y)\,\overline{P_x(y)}\,dy = \int_0^{\alpha^n} F(y)\,dy \int_0^x \overline{\psi_y(t)}\,dt$$

$$= \int_0^x dt \lim_{B\to\infty} \int_0^{\alpha^n} \overline{\psi_y(t)}\,dy \int_0^B f(u)\;\psi_y(u)\,du = \int_0^x f(u)\,du,$$

and again continuity of both sides finishes the proof.

LEMMA 18. *If* $f(x)$, $g(x) \in L_p$, $1 < p \leq 2$, *and* $F(y) = T_p(f)$, $G(y) = T_p(g)$ *then*

$$\int_0^\infty F(x)g(x)\,dx = \int_0^\infty f(x)G(x)\,dx.$$

LEMMA 19. *If* $p = 2$ *in Lemma* 18, *then*

$$\int_0^\infty F(x)\overline{G(x)}\,dx = \int_0^\infty f(x)\overline{g(x)}\,dx.$$

THEOREM 10. *If* $f(x) \in L_p$, $1 < p \leq 2$, $F(y) = T_p(f)$, *and* $f(x)$ *satisfies any condition producing convergence of the complex conjugate W.F.S. to* $f(y)$, *then*

$$f(y) = \lim_{B \to \infty} \int_0^B F(x)\,\overline{\psi_y(x)}\,dx.$$

Proof. Let $g(x,y) = \overline{\psi_x(y)}$ if $x < \beta$, $=0$ if $\beta \leq x$. Then $G(t,y) = T_p(g(x,y)) = P_\beta(t \ominus y)$, and by Lemma 18

$$\int_0^\infty F(x)\,g(x,y)\,dx = \int_0^\infty f(t)G(t,y)\,dt$$

$$= \int_0^\beta F(x)\psi_y(x)\,dx = \int_0^\infty f(t)P_\beta(t \ominus y)\,dt.$$

Now since $f(y)/(1+y) \in L$, Theorem 3 completes the proof.

THEOREM 11. *If* $f(x) \in L_p$ $1 < p \leq 2$, $F(y) = T_p(f)$, *then for almost every* x *at which* $f(y)$ *is locally essentially bounded*

$$f(x) = \lim_{B \to \infty} \int_0^B (1 - [u]/B)F(u)\,\overline{\psi_x(u)}\,du.$$

Proof. Theorem 5 combines with a method similar to that of Theorem 10 to yield the desired proof.

REFERENCES

1. H. E. Chrestenson, *A class of generalized Walsh functions*, Pacific J. Math. **5** (1955), 17-31.

2. N. J. Fine, *On the Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372-414.

3. M. J. McShane, *Integration*, Princeton 1944.

4. R. E. A. C. Paley, *A remarkable series of orthogonal functions*, Proc. Lond. Math. Soc. **34** (1932), 241-264.

5. E. C. Titchmarsh, *Theory of Fourier Integrals*, Oxford 1937.

6. J. L. Walsh, *A closed set of normal orthogonal functions*, Amer. J. Math. **45** (1923), 5-24.

UNIVERSITY OF OREGON AND
NAVAL ORDNANCE TEST STATION, CHINA LAKE, CALIFORNIA