A NOTE ON HELLY'S THEOREM

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1. Introduction. The aim of this note is to give a new elementary proof of Helly's theorem [1] on the intersection of convex sets in n dimensional Euclidean space E^n. Like other elementary proofs, our proof avoids the use of limit concepts and is thus valid for any n dimensional affine space with coordinates in a real number field. In § 3 we remark that Carathéodory's theorem on convex hulls may be derived from Helly's theorem. This is a reverse procedure of the one adopted by Rademacher and Schoenberg [2], and indicates the central position of Helly's theorem in the theory of convex bodies. We shall prove the following version of Helly's theorem.

HELLY'S THEOREM. *Let $C_1, \cdots, C_m$, $m > n$, be convex sets in E^n. If every $n+1$ of these sets have a point in common then there is a point common to all C_i, $i = 1, 2, \cdots, m$.*

Equivalently the theorem states that if

$$\bigcap_{i=1}^{m} C_i = \phi \ \text{(the void set)},$$

then there exist $k+1$ (with $k \le n$) sets $C_{i_1}, \cdots, C_{i_{k+1}}$ such that

$$C_{i_1} \cap \cdots \cap C_{i_{k+1}} = \phi .$$

Other versions of Helly's theorem refer, under suitable restrictions, to infinite sets of convex bodies. These are easily deduced from the above form. In these generalizations the completeness of the space is essential and it is impossible to avoid the limit concept in some form or another.

2. We shall first prove the following special case of Helly's theorem.

LEMMA 1. *Helly's theorem is valid in the special case when $C_1, \cdots, C_m$*

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are closed half-spaces of E^n.

Proof. The case $n = 1$ is simple. We proceed by induction and note that if we have the Lemma for some E^k it obviously remains true if some of the C_i are allowed to coincide with E^k or to be void sets. Let $C_1, \cdots, C_m$ be closed half-spaces of E^n defined by the hyperplanes $\pi_1, \cdots, \pi_m$ and assume

$$C_1 \cap \cdots \cap C_m = \phi . \tag{1}$$

We may assume that no C_i in (1) may be omitted without making the intersection nonvoid. C_1 is a closed half-space so $C_1 \supset \pi_1$ hence

$$\pi_1 \cap C_2 \cap \cdots \cap C_m = \phi,$$

that is

$$(\pi_1 \cap C_2) \cap \cdots \cap (\pi_1 \cap C_m) = \phi.$$

Now $\pi_1 \cap C_i$ is either a closed half-space of π_1 considered as an $n-1$ dimensional space, or (if π_1 and π_i are parallel) coincides with π_1 or the null-set. By virtue of the generalized induction hypothesis there are k, $k \leq n$, sets $\pi_1 \cap C_i$ having no point in common. Thus, after renumbering the sets if necessary:

$$(\pi_1 \cap C_2) \cap \cdots \cap (\pi_1 \cap C_{k+1}) = \pi_1 \cap C_2 \cap \cdots \cap C_{k+1} = \phi.$$

Denote $C_2 \cap \cdots \cap C_{k+1}$ by B then B is convex. We claim that either

(a) $B \cap \widetilde{C}_1 = \phi$ (where $\widetilde{C}_1$ is the complement of C_1 in E^n) or

(b) $B \cap C_1 = \phi$. Indeed, if both (a) and (b) were false there would exist two points P_1, P_2 with $P_1 \in B \cap \widetilde{C}_1$ and $P_2 \in B \cap C_1$ and the line segment $\overline{P_1 P_2}$ would have a point in common with π_1. As B is convex, $\overline{P_1 P_2} \subset B$ contradicting $B \cap \pi_1 = \phi$. Now case (a) is impossible, because it implies $\widetilde{C}_1 \cap C_2 \cap \cdots \cap C_m = \phi$ which together with (1) implies that

$$(\widetilde{C}_1 \cup C_1) \cap C_2 \cap \cdots \cap C_m = C_2 \cap \cdots \cap C_m = \phi$$

contrary to the assumption that none of the C_i in (1) could be omitted. Thus case (b) holds, that is, $C_1 \cap \cdots \cap C_{k+1} = \phi$; since $k \leq n$ the proof of the lemma is completed.

Proof of Helly's theorem. Let $C_1, \cdots, C_m$ be arbitrary convex sets in E^n

every $n+1$ of which have a nonempty intersection. Let $C_{i_1}, \cdots, C_{i_{n+1}}$ be any $n+1$ sets C_i and $P_{i_1, \cdots, i_{n+1}}$ any point in $C_{i_1} \cap \cdots \cap C_{i_{n+1}}$, denote by A the finite set of all these points (for this device compare [**1**]). The sets $C_i \cap A$ are finite sets every $n+1$ of which have a point in common. Put $B_i = H(C_i \cap A)$ where $H(S)$ stands for the convex hull of S. The convex hull of a finite set may be represented as the intersection of a finite number of closed half-spaces (for an elementary proof of this fact see [**3**]), thus $B_i = D_{i,1} \cap \cdots \cap D_{i,k_i}$, say. Let $D_1, \cdots, D_s$ be all the half-spaces appearing for all the B_i. To every D_j corresponds a certain B_i for which $D_j \supset B_i \supset C_i \cap A$ so that every $n+1$ of the D_j have a common point. By virtue of Lemma 1: $D_1 \cap \cdots \cap D_s \neq \phi$. Now

$$D_1 \cap \cdots \cap D_s = B_1 \cap \cdots \cap B_m$$

also $C_i \supset A \cap C_i$ so that by the convexity of C_i we have

$$C_i \supset H(C_i \cap A) = B_i$$

hence

$$\bigcap_{i=1}^{m} C_i \supset \bigcap_{i=1}^{m} B_i \neq \phi. \qquad \text{Q.E.D.}$$

3. Carathéodory's theorem states that the convex hull $H(S)$ where $S \subset E^n$ equals the union of the convex hulls $H(F)$ where F ranges over all sub-sets of S containing not more than $n+1$ points. It is easy to show that $H(S)$ equals the union of the convex hulls of all the finite sub-sets of S, so that the crucial point of Carathéodory's theorem lies in the following:

THEOREM. *Let $P_1, \cdots, P_k$, $k \geq n+1$, be points of E^n. Let $Q \in H(P_1, \cdots, P_k)$ then $n+1$ points $P_{i_1}, \cdots, P_{i_{n+1}}$ may be chosen so that $Q \in H(P_{i_1}, \cdots, P_{i_{n+1}})$.*

We shall deduce this result from Helly's theorem and the following easily established lemma.

LEMMA 2. *Let $Q \neq P_i$, $i = 1, \cdots, k$. Denote by π_i the hyperplane through P_i perpendicular to the direction $\overrightarrow{QP_i}$, let C_i be the closed half-space defined by π_i, which does not contain Q. A necessary and sufficient condition for $Q \in H(P_1, \cdots, P_k)$ is $C_1 \cap \cdots \cap C_k = \phi$.*

Proof of Carathéodory's theorem. We may suppose that $Q \neq P_i$, $i = 1, \cdots, k$.

By the lemma $\bigcap_{i=1}^{k} C_i = \phi$; by the special case of Helly's theorem $n+1$ half-spaces $C_{i_1}, \cdots, C_{i_{n+1}}$ may be chosen so that $\bigcap_{s=1}^{n+1} C_{i_s} = \phi$. Using again the lemma we conclude $Q \in H(P_{i_1}, \cdots, P_{i_{n+1}})$ Q.E.D.

REFERENCES

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