THE MEASURE RING FOR A CUBE OF ARBITRARY DIMENSION

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1. Introduction. From Maharam's theorem [2] on the structure of measure algebras it is very easy to obtain a unique characterization, in terms of cardinal numbers, of an arbitrary measure ring. An example of a measure ring is the ring of Hellinger types of (finite) measures on an additive class of sets. In this note, the cardinal numbers are computed which characterize the ring of Hellinger types of measures on the Baire subsets of a cube of arbitrary dimension.

2. Definitions. A lattice R is a *Boolean σ-ring* if (a) the family R_x of all subelements of any given element x form a Boolean algebra, and (b) any countable family of elements has a least upper bound. If, in addition, for each x in R there exists some countably additive finite-valued real function on R_x which is 0 only for the zero-element of R, then R is a *measure ring.*[1] A Boolean σ-ring with a largest element is a *Boolean σ-algebra.* A measure ring with a largest element is a *measure algebra.* The measure algebra of a finite measure μ is the Boolean σ-algebra of μ measurable sets modulo μ null sets.

A subset S of a Boolean σ-algebra R is a *σ-basis* if the smallest σ-subalgebra of R containing S is R itself. R is *homogeneous of order* α if, for every nonzero element x of R, the smallest cardinal number of a σ-basis of R_x is α. We observe that R cannot be homogeneous of finite nonzero order. If it is homogeneous of order 0, it is the two-element Boolean algebra.

Let α be an infinite cardinal, I the unit interval $[0, 1]$, and I^α the topological product of I with itself α times (the α-dimensional cube). $L^{(\alpha)}$ will denote the product Lebesgue measure on the Baire subsets of I^α, and $M^{(\alpha)}$ the measure algebra of $L^{(\alpha)}$. Then it is not hard to see that $M^{(\alpha)}$ is homogeneous of order α. Maharam has in fact shown that it is, essentially, the only measure algebra of order α.

[1] Our use of the terms 'measure ring', 'measure algebra', unlike Maharam's, refers only to the algebraic structure. In this sense two measure rings are isomorphic if they are isomorphic as σ-rings.

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3. Maharam's theorem and the characterization of measure rings.

THEOREM. (Maharam [2]). *Every measure algebra homogeneous of order* α *is isomorphic to* $M^{(\alpha)}$. *Every measure algebra is a direct product of countably many homogeneous measure algebras.*

Consider now a measure ring R. A nonzero element x of R is *homogeneous of order* α if the principal ideal R_x is so. The *cardinal function* p of R will associate with each cardinal α the smallest cardinal number p_α of a maximal disjoint family of elements of R which are homogeneous of order α.

THEOREM. *The measure ring R is determined to within isomorphism by its cardinal function* p.

Proof. If, for each r in an index set K, N_r is a measure algebra, we may define a σ-ring

$$N = \prod_{r \in K} N_r$$

consisting of all functions ϕ on K, such that (a) $\phi_r \in N_r$ for $r \in K$, (b) $\phi_r = 0$ for all but a countable number of indices r. Union, intersection and difference are defined in the obvious manner. N is actually a measure ring. Indeed, suppose that $\phi \in N$; and that the countable set of indices r for which $\phi_r \neq 0$ is arranged in a sequence $r_1, r_2, \cdots$. For each i, let m_i be a countably additive nonnegative finite-valued function on N_{r_i} having value 0 only for the zero-element of N_{r_i}. Then the function m, defined for subelements ψ of ϕ by

$$m(\psi) = \sum_i \frac{1}{2^i} \frac{m_i(\psi_{r_i})}{m_i(\phi_{r_i})},$$

is countably additive, nonnegative, and finite-valued and has value 0 only for $\psi = 0$.

Now for each cardinal α, let $\{x_{\alpha\beta}\}$ $(\beta < p_\alpha)$ be a maximal disjoint family of elements of R homogeneous of order α; let $R_{\alpha\beta}$ be the algebra of subelements of $x_{\alpha\beta}$. By the definition of homogeneity, the $x_{\alpha\beta}$ are all disjoint, and it is almost trivial to show that

$$R \cong \prod_{\substack{\alpha,\, \beta \\ \beta < p_\alpha}} R_{\alpha\beta}.$$

But, by Maharam's theorem, $R_{\alpha\beta} \cong M^{(\alpha)}$. Hence

$$R \cong \prod_{\alpha} [M^{(\alpha)}]^{p_\alpha},$$

which depends only on the cardinal function p.

4. The measure ring of Hellinger types for the cube. Let F be an additive class of sets. A finite measure μ on F is *absolutely continuous* with respect to a finite measure ν on F if all μ null sets are ν null sets. If μ and ν have the same null sets, they are of the same *Hellinger type*. The relation of absolute continuity furnishes an ordering of the Hellinger types under which, as is well known, the latter form a measure ring. In this ring, the ideal of all Hellinger types contained in the type of a measure μ is isomorphic with the measure algebra of μ.

We shall denote by $R^{(\alpha)}$ the measure ring of Hellinger types associated with the additive class $B^{(\alpha)}$ of Baire subsets of the cube I^α; and shall obtain the cardinal function characterizing $R^{(\alpha)}$.

LEMMA 1. $R^{(\alpha)} \cong R^{(\aleph_0)}$ *for any finite nonzero* α.

Proof. This follows from Kuratowski's result [**1**] that any two complete metric spaces X and Y having the same cardinal number are connected by a one-to-one Borel mapping of X onto Y whose inverse is also Borel.

In view of Lemma 1, we restrict ourselves to cardinals α which are infinite.

LEMMA 2. *The total number of measures on* $B^{(\alpha)}$ *is equal to or less than* c^α, *where* c *is the power of the continuum.*

Proof. I^α is the set of all functions ϕ on a set H, of cardinal number α, to $[0,1]$. Let P denote a basis for the topology of I^α consisting of all subsets of I^α of the form

$$\underset{\phi}{E}(\phi \in I^\alpha,\ \phi_\beta \in J_\beta \text{ for all } \beta \text{ in } G)$$

where G is a finite subset of H, and, for each β, J_β is an open interval in I with rational endpoints. Since α is infinite, it is easily seen that the cardinal number of P is α.

Now suppose μ and ν are two measures on $B^{(\alpha)}$ which coincide on P. I claim they must then be equal. If this is so, the lemma is proved, since the number of real functions on P is c^{α}.

Since P is closed under finite intersection, it is easily seen that μ and ν must coincide on finite unions of sets in P. Now let A be a closed Baire set in I^{α}; A must then be a countable intersection of open sets. But, between A and any open set containing it, we may by the compactness of I^{α}, place a finite union of sets in P. Passing to the limit, we find that μ and ν have the same value for A. Now by the regularity of μ and ν, their coincidence is assured for all Baire sets.

LEMMA 3. *For each cardinal γ which is either* 0 *or infinite and for which $\gamma \leq \alpha$, there are at least c^{α} disjoint measures on $B^{(\alpha)}$ whose measure algebras are homogeneous of order γ.*

Proof. With each point ϕ in I^{α} associate the measure μ_{ϕ} of a unit point mass at ϕ. Any two such points ϕ and ψ can be separated by Baire sets, hence the Hellinger types of μ_{ϕ} and μ_{ψ} are disjoint. Since there are c^{α} points in I^{α}, the lemma is true for $\gamma = 0$.

For any infinite cardinal γ, divide H, the index set of I^{α} (see proof of Lemma 2), into two disjoint parts M and N, of cardinal number γ and α respectively. Let J and K be the γ- and α-dimensional cubes with index sets M and N respectively. Since $M \cup N = H$, two points ϕ and ψ, in J and K respectively, define a point of I^{α} which we call $\phi \cup \psi$, for which $(\phi \cup \psi)_r = \phi_r$ or ψ_r according as $r \in M$ or $r \in N$. We fix ψ in K; and for each Baire subset A of I^{α}, let $T_{\psi}(A)$ be the subset of J consisting of those ϕ for which $\phi \cup \psi \in A$. I claim that T_{ψ} is a σ-homomorphism of $B^{(\alpha)}$ onto all Baire sets in J.

That T_{ψ} is a σ-homomorphism is evident. Now $B^{(\alpha)}$ is the smallest additive class of subsets of I^{α} which contains P (defined in the proof of Lemma 2). That every Baire set A in I^{α} maps into a Baire set in J will follow if sets in P go into Baire sets; but the latter is evident. A corresponding argument in J shows that all Baire sets in J are maps of Baire sets in I^{α}. Hence the claim made for T_{ψ} is correct.

Now let $L^{(\gamma)}$ denote the product Lebesgue measure defined on the Baire sets of J. For each ψ in K, and each Baire set A in I^{α}, put

$$\mu_{\psi}(A) = L^{(\gamma)}[T_{\psi}(A)].$$

Evidently the measure algebras of μ_{ψ} and $L^{(\gamma)}$ are isomorphic; the former is therefore homogeneous of order γ. Further, if $\psi_1, \psi_2 \in K$, $r \in N$, $\psi_1(r) \neq \psi_2(r)$, and A is the Baire set of all ϕ in I^α with $\phi(r) = \psi_1(r)$, we see that

$$\mu_{\psi_1}(I^\alpha \sim A) = 0, \quad \mu_{\psi_2}(A) = 0.$$

It follows that μ_{ψ_1} and μ_{ψ_2} are disjoint whenever $\psi_1, \psi_2 \in K$ and $\psi_1 \neq \psi_2$. The fact that there are c^α elements in K completes the proof of the lemma.

THEOREM. *If α is an infinite cardinal, the cardinal function p characterizing the measure ring $R^{(\alpha)}$ of Hellinger types of measures on the Baire subsets of the α-dimensional cube is given by:*

$$P_\gamma = \begin{cases} c^\alpha \text{ if } \gamma \text{ is a } 0 \text{ or infinite cardinal with } \gamma \leq \alpha; \\ 0 \text{ otherwise.} \end{cases}$$

Proof. Since P, a σ-basis of $B^{(\alpha)}$ (see proof of Lemma 2), is of cardinal number α, we must have $p_\gamma = 0$ for $\gamma > \alpha$. If γ is 0 or infinite, and $\gamma \leq \alpha$, Lemmas 1 and 2 prove the existence of a maximal disjoint family of elements of $R^{(\alpha)}$ homogeneous of order γ, of cardinal number exactly c^α. That c^α is the smallest possible cardinal number of such a family, follows easily from the fact that no element of $R^{(\alpha)}$ intersects more than a countable number of disjoint elements of $R^{(\alpha)}$.

REFERENCES

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