CONTINUOUS SPECTRA AND UNITARY EQUIVALENCE

C. R. PUTNAM

1. Introduction. In the differential equation

$$(1) \qquad (px')' + (\lambda + f(t))x = 0 ,$$

let λ denote a real parameter and let $p(t)$ (>0) and $f(t)$ be continuous real-valued functions on $0 \leqq t < \infty$. Suppose that (1) is of the limit-point type, so that (1) and a linear homogeneous boundary condition

$$(2_\alpha) \qquad x(0)\cos\alpha + x'(0)\sin\alpha = 0 , \qquad 0 \leqq \alpha < \pi ,$$

determine a boundary value problem with a spectrum $S = S_\alpha$ on the half-line $0 \leqq t < \infty$; cf. [7]. The continuous spectrum C_α (if it exists) is determined by a continuous monotone nondecreasing basis function $\rho_\alpha(\lambda)$. It is known that the set of cluster points, S', of S_α is independent of α, [7, p. 251]; the question as to whether the corresponding assertion for C_α is also true was raised by Weyl [7, 7. 252] but is still undecided.

Consider the self-adjoint operators $H_\alpha = \int \lambda dE_\alpha(\lambda)$ (all of which are extensions of the same symmetric operator) belonging to the various boundary value problems determined by (1) and (2_α); cf. for example, [2]. The object of this note is to shown that any two H_α possessing purely continuous (hence, in view of the above remark concerning S', necessarily identical) spectra are unitarily equivalent, at least if certain conditions concerning the nature of the sets C_α and the basis functions $\rho_\alpha(\lambda)$ are met. In fact there will be proved the following.

THEOREM (*). *Suppose that there exist two (distinct) values α_1 and α_2 ($0 \leqq \alpha_k < \pi$) such that, for each of the two boundary value problems determined by (1) and $(2_{\alpha k})$, the following three conditions are satisfied:*

(i) $S_{\alpha k} \neq (-\infty, \infty)$,

(ii) *the point spectrum is empty, and*

(iii) $\rho_{\alpha k}(\lambda)$ *is absolutely continuous. Then $H_{\alpha 1}$ and $H_{\alpha 2}$ are unitarily equivalent.*

The condition (i) of (*) surely holds if, for instance, f is bounded or even bounded from below on $0 \leqq t < \infty$. It should be noted however that every (real) λ belongs to an S_α for some α (depending on λ); [1].

For other results on the continuous spectra of boundory value pro-

Received April, 1956. This work was supported by the National Science Foundation research grant NSF-G481.

blems with absolutely continuous basis functions (on certain intervals), see [4].

The proof of (*) in § 2 will depend upon the following result of M. Rosenblum [5] concerning perturbations of self-adjoint operators: Let the self-adjoint operators $A_k=\int \lambda dE(\lambda)$ (for $k=1, 2, 3$) satisfy $A_1-A_2=A_3$. Suppose, in addition, that A_3 is completely continuous and such that $\int|\lambda|dE_3(\lambda)$ has a finite trace while (E_1x, y) and (E_2x, y) are absolutely continuous functions of λ for arbitrary x and y in Hilbert space. Then A_1 and A_2 are unitarily equivalent.

2. Proof of (*). In the sequel, the index α_k will be replaced by k. It is clear from the assumptions that there exists some real value $\lambda=\lambda^*$ not belonging to S_k for $k=1, 2$. Consequently, since $f(t)$ can be replaced in (1) by $f(t)+\lambda^*$, it can be supposed without loss of generality that $\lambda=0$ is not in either of the sets S_k. Then the operators H_k^{-1}, where

$$H_k^{-1}=\int_\lambda^{-1} dE_k(\lambda)=\int dF_k(\lambda) \qquad (F_k(\lambda)=E_k(\lambda^{-1}))$$

are bounded, self-adjoint integral operators with kernels $G_k(s, t)$ on $0\leq s, t<\infty$; cf. for example, [2], [7]. Furthermore,

$$G_1(s, t)-G_2(s, t)=cg(s)g(t) ,$$

where c denotes a constant and $g(t)$ is a function of class $L^2[0, \infty)$; cf. [7, p. 251]. Thus $(H_1^{-1}-H_2^{-1})x$ is a multiple of g for every element x of class $L^2[0, \infty)$. Hence $H_1^{-1}-H_2^{-2}$ is a multiple of a one-dimensional projection operator; in particular, $H_1^{-}-H_2^{-1}$, corresponding to A_3, satisfies the trace condition on that operator mentioned at the end of § 1.

In view of the assumptions (ii) and (iii) of (*), it follows from the formulas relating the basis functions $\rho_k(\lambda)$ to the projections $E_k(\lambda)$ (cf., for example, [2]) that $||E(\lambda)x||$ is an absolutely continuous function of λ for every x in the Hilbert space; therefore $(E_k(\lambda)x, y)$, hence also $(F_k(\lambda)x, y)$, is absolutely continuous for every pair x, y of this space. According to the above mentioned theorem of Rosenblum, it now follows that the operators H_k^{-1} (hence also the H_k) are unitarily equivalent, and the proof of (*) is now complete.

3. Consider the special case of (1) in which $f\equiv 0$. It is readily seen that there are no eigenvalues for either of the boundary value problems determined by $x''+\lambda x=0$ and the respective boundary conditions $x(0)=0$ and $x'(0)=0$. (These boundary conditions correspond to $\alpha=0$,

$\pi/2$ in (2^{α}); in a somewhat more general connection, cf. [3, p. 792]). Thus, in each case, there is a purely continuous spectrum consisting of the half-line $0 \leqq \lambda < \infty$. Moreover, the basis functions, which, in this instance, are even known explicitly [6, p. 59] are absolutely continuous. Consequently, Theorem (*) is applicable and shows that the self-adjoint operators belonging to the above mentioned boundary value problems are unitarily equivalent.

REFERENCE

1. P. Hartman and A. Wintner, *An oscillation theorem for continuous spectra*, Proc. Nat. Acad. Sci., **33** (1947), 376–379.

2. K. Kodaira, *The eigenvalue problem for ordinary differential equations of the second order and Heisenberg's theory of S-matrices*, Amer. J. Math., **71** (1949), 921–945.

3. C. R. Putnam, *An application of spectral theory to a singular calculus of variations problem*, Amer. J. Math., **70** (1948), 780–803.

4. ———, *On the continuous spectra of singular boundary value problems*, Can. J. Math., **6** (1954), 420–426.

5. M. Rosenblum, *Perturbations of the continuous spectrum and unitary equivalence*, Technical Report No. 12, for Office of Ordnance Research, Department of Mathematics, University of California, December 1955.

6. E. C. Titchmarsh, *Eigenfunction expansions associated with second-order differential equations* (Oxford, 1946).

7. H. Weyl, *Ueber gewöhnliche Differentialgleichungen mit Singularitäten und die zugehörigen Entwicklungen willkürlicher Funktionen*, Math. Ann., **68** (1910), 222–269.

PURDUE UNIVERSITY