BIORTHOGONAL SYSTEMS IN BANACH SPACES

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1. Introduction. We shall be interested, in this paper, in the following question: Given a biorthogonal system (x_n, f_n) in a separable Banach space B, under what conditions can one assert that the sequence $\{x_n\}$ constitutes a basis? The system (x_n, f_n) is called a biorthogonal system if

$$x_n \in B, \ f_n \in B^* \quad \text{and} \quad f_n(x_m) = \delta_{nm} .$$

We shall assume throughout the paper that $\|x_n\|=1$ and the sequence $\{x_n\}$ is fundamental. When the sequence $\{x_n\}$ constitutes a basis it will be called *regular* otherwise *irregular*.

2. Irregular systems. Let $\{x_n\}$ be an irregular sequence. (For example the trigonometric functions for $C(-\pi, \pi)$). The following definitions will be used.

$$\varphi_n(x) = \sum_{i=1}^{n} f_i(x) x_i$$

$$|||x||| = \sup \{ \|\varphi_n(x)\|, n=1, 2, 3, \cdots \}$$

Compare [4]

$$E_0 = \{x \mid \lim_{n\to\infty} \varphi_n(x) = x\}$$

$$E_1 = \{x \mid \ |||x||| < \infty\}$$

$$E_2 = \{x \mid \lim_{n\to\infty} \| \varphi_n(x) \| = \infty\}$$

$$E_3 = \{x \mid \ |||x||| = \infty\} .$$

We have $E_0 \subset E_1$ and $E_2 \subset E_3$. For regular systems $E_0 = E_1 = B$ and $E_2 = E_3 = \phi$ where ϕ is the null set. The system is regular if and only if the sequence $\{\|\varphi_n\|\}$ is bounded [2], and if the sequence $\{\|\varphi_n\|\}$ is not bounded the set

$$\bigcap_{n=1}^{\infty} \{x \mid \ \|\varphi_n(x)\| \leqq K\}$$

is nowhere dense [2], hence for irregular systems the set

$$E_1 = \bigcup_{k=1} \bigcap_{n=1} \{x \mid \ \|\varphi_n(x)\| \leqq K\}$$

is of the first category. Also $E_3 = B - E_1$ is dense and of the second

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category. In the case of regular systems there exists a number $K \geqq 1$ such that if $\|x\|=1$ then $1 \leqq |||x||| \leqq K$. The existence of such a bound, K, is equivalent to the equiboundedness of $\{\|\varphi_n(x)\|\}$ $\|x\|=1$ and therefore for irregular systems for any number a, there exists a point x such that $\|x\|=1$ and $|||x|||>a$, moreover such a point might be found in the linear manifold generated by $\{x_n\}$. (Equiboundedness of $\{\|\varphi_n(x)\|\}$ on a dense subset of the unit sphere would imply equiboundedness on the unit sphere.) It is interesting to note that for every number $a \geqq 1$ there exists a point x such that $\|x\|=1$ and $|||x|||=a$. There exists a point y_n satisfying

$$y_n=\sum_{i=1}^{n} a_i x_i, \quad \|y_n\|=1, \quad |||y_n|||>a\,.$$

On the other hand $\|x_1\|=1$ and $|||x_1|||=1$. Let $0 \leqq t \leqq 1$, then $(1-t)x_1+ty_n \neq 0$. Define

$$g_\nu(t)=\left\|\varphi_\nu\left(\frac{(1-t)x_1+ty_n}{\|(1-t)x_1+ty_n\|}\right)\right\| \qquad \nu=12\cdots n$$

The functions $g_\nu(t)$ are continuous in t, and so is $g(t)$ where

$$g(t)=\sup\{g_\nu(t)\,|\,1 \leqq \nu \leqq n\}\,.$$

$$g(0)=1$$

and

$$g(1)=\sup\{\|\varphi_\nu(y_n)\|\,1 \leqq \nu \leqq n\}=|||y_n|||>a\,.$$

There exists a number t_0 such that

$$0 \leqq t_0 < 1 \quad \text{and} \quad g(t_0)=a\,.$$

This following generalization of Baire's theorem [1] will be used: *Let $\{u_n(x)\}$ be a sequence of real valued continuous functions defined on a metric space c, and $\lim u_n(x)=u(x)$, $|u_n(x)| \leqq M$, then the set of points of discontinuity of u is of the first category.*

THEOREM 1. *The set E_2 is of the first category.*

Proof. Define the functions $u_n(x)$ by

$$u_n(x)=\frac{\|\varphi_n(x)\|}{1+\|\varphi_n(x)\|}\,.$$

We have $0 \leqq u_n(x) \leqq 1$ and if $x \in E_0 \cup E_2$ then

$$\lim u_n(x)=u(x)$$

where $u(x)=1$ for $x \in E_2$ and

$$u(x)=\frac{||x||}{1+||x||} \text{ for } x \in E_0 .$$

If E_2 is a set of the second category then there exists at least one point of continuity of u. Let us denote such a point by x_0.

The set E_0 is dense in B. Let $\{y_n\}$ by a sequence of points in E_0 with $\lim y_n=x_0$, then

$$u(x_0)=\lim u(y_n)=\lim\frac{||y_n||}{1+||y_n||}=\frac{||x_0||}{1+||x_0||}$$

The set E_2 is dense in B. If $x \in E_2$ and $y \in E_0$ then $x+y \in E_2$. Let $\{z_n\}$ be a sequence of points in E_2 with $\lim z_n=x_0$, then $u(z_n)=1$ and

$$\frac{||x_0||}{1+||x_0||}=u(x_0)=\lim u(z_n)=1$$

which is absurd.

THEOREM 2. *Let S be a subset of B such that each point $x \in S$ is the limit of some sequence $\{y_n\}$, $y_n \in B$, and the sequence $\{|||y_n|||\}$ is bounded, then S is of the first category.*

Proof. Define the functions $v_n(x)$ by

$$v_n(x)=\frac{|||\varphi_n(x)|||}{1+|||\varphi_n(x)|||}$$

then $0 \leq v_1(x) \leq v_2(x) \leq \cdots < 1$. If $x \in B$ let $\lim v_n(x)=v(x)$. $v(x)=1$ for $x \in E_3$ and the set E_3 is dense, hence $v(x)=1$ at every point of continuity of v. Let x be a point of continuity of v and $\{z_n\}$ a sequence with $\lim z_n=x$, then

$$\lim v(z_n)=v(x)=1$$

therefore the sequence $\{|||z_n|||\}$ is unbounded. Thus the set S is contained in the set of points of discontinuity of v which is a set of the first category by Baire's theorem.

3. General criteria for regularity. From Theorems 1 and 2 we derive the following criteria.

THEOREM 3. *A necessary and sufficient condition for the regularity of the system (x_n, f_n) is.*

$$\sup \{\|\varphi_n(x)\|, n=1, 2, \cdots\}=\infty$$

implies

$$\lim \|\varphi_n(x)\|=\infty . \quad (or \ E_2=E_3) .$$

Proof. If the system is regular, then $E_2=E_3=\phi$. On the other hand, if the system is irregular E_2 is of the first category and E_3 of the second category.

Let

$$\psi_n(x)=\sum_{i=1}^{n} a_i^n x_i$$

denote the point nearest to x on the subspace spanned by

$$\{x_1, x_2, x_3, \cdots, x_n\} .$$

THEOREM 4. *The system (x_n, f_n) is regular if and only if the sequence $\{|||\psi_n(x)|||\}$ is bounded for each x.*

Proof. If the system is regular, then there exists a positive number K, such that $|||x|||\leqq K\|x\|$. Then

$$|||\psi_n(x)|||\leqq K\|\psi_n(x)\|\leqq K(\|x\|+\|x-\psi_n(x)\|)\leqq K2\|x\| ,$$

hence the condition is necessary. Sufficiency is clear by Theorem 2.

4. Biorthogonal systems in Hilbert spaces. In this section we assume that B is a Hilbert space. In order to use Theorem 4 let us compute $|||\psi_n(x)|||$. $\psi_n(x)=\sum_{i=1}^{n} a_i^n x_i$ and the coefficient a_i^n may be computed from the equation

$$(x-\sum_{i=1}^{n} a_i^n x_i, x_k)=0 \qquad k=1, 2, \cdots, n$$

or

$$(x, x_k)=\sum_{i=1}^{n} a_i^n (x_i, x_k) \qquad \text{see [5]}.$$

We introduce the following notation

$$(x_i, x_k)=c_{ik}$$

$$C_n=(c_{ik}) \quad 1\leqq i\leqq n \quad 1\leqq k\leqq n$$

$$((x, x_i), (x, x_2), \cdots, (x, x_n))=(\gamma)_n$$

$$(a_1^n, a_2^n, \cdots, a_n^n)=(a)_n$$

Then

$$(\gamma)_n=(a)_nC_n \quad \text{or} \quad (a)_n=(\gamma)_nC_n^{-1}$$

since C_n^{-1} exists. Now

$$\|\sum_{i=1}^{j} a_i^n x_i\|^2=\sum_{i,k=1}^{j} a_i^n \overline{a_k^n} c_{ik}=(a)_n E_j^n C_n E_j^n (a)_n^*$$

where E_j^n is the matrix $(e_{l,m})$ with

$$e_{l,m}\begin{cases}1 & l=m\leqq j\\ 0 & \text{otherwise}\end{cases}$$

$C_n^*=C_n$ and $(a)_n=(\gamma)_nC_n^{-1}$ hence

$$\|\sum_{j=1}^{j} a_i^n x_i\|^2=(\gamma)_n C_n^{-1}E_j^n C_n E_j^n C_n^{-1}(\gamma)_n^*$$

Orthogonalizing the sequence $\{x_n\}$ by Schmidt's process we get the sequence $\{y_n\}$ with

$$x_1=y_1 \quad x_k=\sum_{\alpha=1}^{k} d_{k,\alpha} y_\alpha$$

where

$$d_{k,\alpha}=\begin{cases}(x_k,\ y_\alpha) & \alpha\leqq k\\ 0 & \alpha>k\end{cases}$$

and $d_{11}=1$ see [3].
Let D_n denote the triangular matrix

$$(d_{k,\alpha}) \quad 1\leqq\alpha\leqq n \quad 1\leqq k\leqq n\,.$$

$$(x_i,\ x_j)=\sum_{\alpha} d_{i,\alpha}\overline{d_{j,\alpha}} \text{ or } C_n=D_nD_n^*\,.$$

The matrix D_n can be computed from this relation.

Let $$(\delta)_n=((x,\ y_1),\ (x,\ y_2),\ \cdots,\ (x,\ y_n))$$

$$(x,\ x_k)=\sum_{\alpha=1}^{k}(x,\ y_\alpha)\overline{d}_{k,\alpha} \text{ or } (\gamma)_n=(\delta)_nD_n^*$$

and hence

$$\|\sum_{i=1}^{j} a_i^n x_i\|^2=(\gamma)_nC_n^{-1}E_j^nC_nE_j^nC_n^{-1}(\gamma)_n^*=(\delta)_nD_n^*C_n^{-1}E_j^nC_nE_j^nC_n^{-1}D_n(\delta)_n^*$$

$$=(\delta)_n(D_n^{-1}E_n^jD_n)(D_n^*E_j^n(D_n^*)^{-1})(\delta)_n^*$$

Let $A_j^n=D_n^{-1}E_j^nD_n$ then

$$|||\varphi_n(x)||| = \max\{(\delta)_n A_j^n (A_j^n)^* (\delta)_n^* \,|\, 1 \leqq j \leqq n\}$$

The triangular matrix A_j^n is an operator defined on the Hilbert space. If

$$x = \sum_{i=1}^{\infty} \delta_i y_i$$

then

$$A_j^n(x) = (\delta_1, \cdots, \delta_n) A_j^n .$$

By Theorem 4 and the above computation the system is regular if and only if for each x

$$\sup\{||A_j^n(x)|| \,|\, 1 \leqq j \leqq n \; n = 12 \cdots\} < \infty$$

or by the uniform boundedness theorem.

THEOREM 5. *The system is regular if and only if the double sequence* $\{||A_j^n||\}$ *is bounded, or, in other words, if and only if the set of characteristic roots of* $A_j^n(A_n^n)^*$ *is bounded.*

We shall use Theorem 3 to derive the following theorems.

THEOREM 6. *The system* (x_i, f_i) *is regular and* $\sum_{i=1}^{\infty} |f_i(x)|^2 < \infty$ *if and only if for every* $x \in B$ *there exists a real number* $\alpha = \alpha(x)$ *such that*

$$2\Re\{\sum_{i=1}^{n} \sum_{j=1}^{i-1} f_i(x) \overline{f_j(x)} c_{ij}\} > \alpha \tag{1}$$

Proof. If the system is regular and

$$\sum_{i=1}^{\infty} |f_i(x)|^2 < \infty \text{ then}$$

$$\begin{aligned} ||x||^2 - \sum |f_i(x)|^2 &= \sum_{i \neq j} f_i(x) f_j(x) c_{ij} \\ &= 2\Re\{\sum_{j<i} f_i(x) \overline{f_j(x)} c_{ij}\} \end{aligned}$$

Therefore the necessity of condition (1) is verified. Assume that condition (1) is satisfied then

$$\begin{aligned} ||\varphi_{n+p}(x)||^2 &= \sum_{ij=1}^{n+p} f_i(x) \overline{f_j(x)} c_{ij} \\ &= ||\varphi_n(x)||^2 + \sum_{i=n+1}^{n+p} |f_i(x)|^2 + 2\Re\{\sum_{i=n+1}^{n+p} \sum_{j=1}^{i-1} f_i(x) \overline{f_j(x)} c_{ij}\} \\ &\geqq ||\varphi_n(x)||^2 + 2\alpha \end{aligned}$$

Therefore $\sup_n \|\varphi_n(x)\| = \infty$ implies $\lim \|\varphi_n(x)\| = \infty$. According to Theorem 3 the system is regular.
Moreover

$$\sum_{i=1}^{n} |f_i(x)|^2 = \|\varphi_n(x)\|^2 - 2\Re\{\sum_{j<i\leqq n} c_{ij} f_i(x) f_j(x)\}$$
$$\leqq |||x|||^2 - \alpha < \infty .$$

An immediate consequence is the following. The system is regular if $\sum_{i\neq j}|c_{ij}| < \infty$ and the sequence $\{\|f_i\|\}$ is bounded.

Professor R. C. James called my attention to the fact that this may be proved directly and without the assumption of boundedness of the sequence $\{\|f_i\|\}$ as follows. We may assume without loss of generality that $\sum_{i\neq j}|c_{ij}| = r < 1$

$$|\sum_{i\neq j}^{n} a_i \bar{a}_j c_{ij}| \leqq \max |a_i \bar{a}_j| \cdot r \leqq \sum_{i=1}^{n} |a_i|^2 r$$

$$\|\sum_{i=1}^{n} a_i x_i\|^2 = \sum_{i=1}^{n} |a_i|^2 + \sum_{i\neq j}^{n} a_i \bar{a}_j c_{ij} \leqq 2 \sum_{i=1}^{n} |a_i|^2$$

Hence

$$\|\sum_{i=1}^{n+p} a_i x_i\|^2 = \sum_{i=1}^{n+p} |a_i|^2 + \sum_{i\neq j}^{n+p} a_i \bar{a}_j c_{ij}$$
$$\geqq \sum_{i=1}^{n+p} |a_i|^2 (1-r) \geqq \sum_{i=1}^{n} |a_i|^2 (1-r)$$
$$\geqq \frac{1-r}{2} \left\|\sum_{i=1}^{n} a_i x_i\right\|^2$$

and by [4] the system is regular.

Using the same method as in Theorem 6 we arrive at the following.

THEOREM 7. *The system is regular if and only if for each x*

$$\inf_{n,p} \Re\{\sum_{i=1}^{n} \sum_{j=n+1}^{p} f_i(x) f_j(x) c_{ij}\} > -\infty \tag{2}$$

Proof.

$$\|\sum_{i=1}^{n+p} f_i(x) x_i\|^2 = \|\sum_{i=1}^{n} f_i(x) x_i\|^2 + \|\sum_{i=n+1}^{n+p} f_i(x) x_i\|^2$$
$$+ 2\Re\{(\sum_{i=1}^{n} f_i(x) x_i ,\ \sum_{j=n+1}^{n+p} f_j(x) x_j)\} .$$

If condition (2) is satisfied then according to Theorem 3 the system is regular. If the system is regular then

$$|(\sum_{i=1}^{n} f_i(x)x_i, \sum_{j=n+1}^{n+p} f_j(x)x_j)| \leqq 2|||x|||^2.$$

As a simple application we note the following.

If $c_{ij}=0$ when $|i-j|>N$ then the system is regular if and only if the sequence $\{||f_i||\}$ is bounded.

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