CURVATURE IN HILBERT GEOMETRIES

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For every pair of points, p and q, interior to a simple, closed, convex curve C in the Euclidean plane, the line $\xi = p \times q$ cuts C in a pair of points u and v. If C has at most one segment then the Hilbert distance from p to q, defined by

$$h(p, q) = \left| \log\left(\frac{up}{uq} \cdot \frac{vq}{vp} \right) \right| ,$$

is a proper metric (where up denotes the Euclidean distance from u to p), and is invariant under projective transformations. The geometry induced on the interior of C is a Hilbert geometry, and the Hilbert lines are carried by Euclidean lines [2].

We shall be concerned here with curvature at a point defined in a qualitative rather than a quantitative sense (cf. [1, p 237]).

Definition 1. The *curvature at* p is *positive* or *negative* if there exists a neighborhood U of p such that for every x, y in U we have

$$2\ h(\bar{x}, \bar{y}) \geqq h(x, y) ,$$

respectively

$$2\ h(\bar{x}, \bar{y}) \leqq h(x, y) ,$$

where $\bar{x}$, $\bar{y}$ are the Hilbert midpoints respectively of the segments from p to x and p to y. If there is neither positive nor negative curvature at a point then the curvature is *indeterminate* at that point. This qualitative curvature is clearly a projective invariant.

In order to state our result we need one more concept.

Definition 2. A point p is a *projective center* of C if there exists a projective transformation, π, of the plane so that πp is the affine center of πC.

A projective center is characterized by the following. Let ξ be a line through p, and let $\xi \cap C = \{u, v\}$, and let p'_ξ be the harmonic conjugate of p with respect to u and v. Finally, let L_p be the locus of all p'_ξ. Then p is a projective center if and only if L_p is a straight line.

Conic sections are characterized by the fact that every point in their interior is a projective center [3]. We can now state our main result, which solves a problem of H. Busemann [1, Problem 34, p. 406].

Theorem. *If p is a point of determinate curvature then it is* *a projective center of C. In particular, if the curvature is determinate everywhere then C is an ellipse and the Hilbert geometry is hyperbolic.*

We first establish some lemmas.

LEMMA 1. *For any point p, interior to C, there exists* a *line η* (possibly the line at infinity) *which intersects L_p in at least two points and does not intersect C.*

Proof. There is at least one chord of C which is bisected by p. If ξ_1 is the line of such a chord then ξ_1 intersects L_p at q_1 on the line at infinity. If L_p has a second point at infinity then the line at infinity satisfies the lemma. If L_p has only one point at infinity then L_p is a connected curve. It cannot lie within the strip formed by the two supporting lines of C which are parallel to ξ_1 for then it would intersect C. There is therefore a point q_2 of L_p outside this strip and the line $\eta = q_1 \times q_2$ satisfies the lemma.

COROLLARY. *For every p in the interior of C there exists a projective transformation, π, so that πC is a closed, convex curve, and so that πp is the midpoint of two mutually perpendicular chords of πC whose endpoints are points of differentiability of πC.*

Proof. Since all but a denumerable set of points of C are points of differentiability, we may choose the line η of Lemma 1 so that $\eta \cap L_p$ contains p'_{ξ_1} and p'_{ξ_2} and so that C is differentiable at its points of intersection with ξ_1 and ξ_2. Now let π_1 be a projective transformation which maps η into the line at infinity, and let π_2 be an affine transformation which maps $\pi_1 \xi_1$ and $\pi_1 \xi_2$ into perpendicular lines. Then $\pi = \pi_2 \pi_1$ has the required properties.

LEMMA 2. *If a chord of C, of* (Euclidean) *length $2k$, has p for its midpoint and if q is a neighboring point on the chord at* (Euclidean) *distance ds from p, then $dS = (2/k)\, ds + O(ds^3)$, where $dS = h(p, q)$.*

Proof. If the endpoints of the chord are u and v, and the order of the points on the chords is u, p, q, v, then, by definition,

$$\begin{aligned} dS &= \log\left(\frac{up+pq}{up}\right)\left(\frac{vp}{vp-pq}\right) = \log\left(\frac{k+ds}{k}\right)\left(\frac{k}{k-ds}\right) \\ &= \log\left(1+\frac{ds}{k}\right) - \log\left(1-\frac{ds}{k}\right) \\ &= \left[\frac{ds}{k} - \frac{1}{2}\left(\frac{ds}{k}\right)^2 + \frac{1}{3}\left(\frac{ds}{k}\right)^3 - \cdots\right] \\ &\qquad - \left[-\frac{ds}{k} - \frac{1}{2}\left(\frac{ds}{k}\right)^2 - \frac{1}{3}\left(\frac{ds}{k}\right)^3 - \cdots\right] \end{aligned}$$

$$= \frac{2}{k} ds + O(ds^3) .$$

LEMMA 3. *Let (r, θ) be polar coordinates whose pole p is an interior point of C at which the curvature is determinate. If C is differentiable at the ends of two perpendicular chords which bisect each other at p, then C satisfies the "one-sided" differential relations*

$$(1) \qquad \begin{aligned} \frac{d}{d\theta}\left(\frac{\csc 2\theta}{r^2}\right)\Big|_{\theta_0^+} &= \frac{d}{d\theta}\left(\frac{\csc 2\theta}{r^2}\right)\Big|_{(\theta_0+\pi)^+} \\ \frac{d}{d\theta}\left(\frac{\csc 2\theta}{r^2}\right)\Big|_{\theta_0^-} &= \frac{d}{d\theta}\left(\frac{\csc 2\theta}{r^2}\right)\Big|_{(\theta_0+\pi)^-} \end{aligned}$$

for all θ_0.

Proof. We first introduce Cartesian coordinates, with origin p, so that the y-axis intersects C at points of second order differentiability, and so that the axes do not coincide with the two given chords bisected by p. The curve C is then given by an "upper" arc $y=y_1(x)$ and a "lower" arc $y=-y_2(x)$. Let the bisected chords lie on the lines $\xi_1: y=ax$ and $\xi_2: y=(1/a)x$ respectively. Let $b_1=(dx, a\,dx)$ and $c_1=(2\,dx, 2a\,dx)$ on ξ_1, and $b_2=(dx, -(1/a)\,dx)$ and $c_2=(2\,dx, -(2/a)\,dx)$ on ξ_2, where dx is positive and chosen so that b_1, b_2, c_1, and c_2 lie inside C. Assume that p is a point of negative curvature. Then.

$$(2) \qquad 2\, h(m_1, m_2) \leqq h(c_1, c_2).$$

where m_i is the Hilbert midpoint of the segment from p to c_i.

To show that $h(m_i, b_i)=O(dx^3)$, we define $dS_1=h(p, c_1)$ and $ds_1=pc_1$. With $2k$ representing the Euclidean length of the chord on ξ_1, it follows from Lemma 2 that $dS_1=(2/k)\,ds_1+O(ds_1^3)$, and hence that

$$(3) \qquad h(p, m_1)=\frac{1}{2} dS_1=\frac{1}{k} ds_1+O(ds_1^3) .$$

Also, from Lemma 2 and the relation $ds_1=2\,pb_1$, it follows that

$$(4) \qquad h(p, b_1)=\frac{2}{k} pb_1+O[(pb_1)^3]=\frac{1}{k}(ds_1)+O(ds_1^3) .$$

Since $h(m_1, b_1)=|h(p, m_1)-h(p, b_1)|$, equations (3) and (4) imply that $h(m_1, b_1) =O(ds_1^3)$. But $ds_1=dx(1+a^2)^{1/2}=O(dx)$, hence $h(m_1, b_1)=O(dx^3)$. Similarly, $h(m_2, b_2)=O(dx^3)$, and therefore

$$(5) \qquad h(m_1, b_1)+h(m_2, b_2)=O(dx^3) .$$

From the triangle inequality,

$$h(m_1, m_2) \geqq h(b_1\, b_2) - h(m_1, b_1) - h(m_2, b_2)\ . \tag{6}$$

This, together with (5), yields

$$h(m_1, m_2) \geqq h(b_1, b_2) - O(dx^3)\ , \tag{7}$$

and from (1) and (7) we obtain

$$2\ h(b_1, b_2) < h(c_1, c_2) + O(dx^3)\ . \tag{8}$$

We now wish to calculate the distances in (8). First, we have

$$\begin{aligned} h(b_1, b_2) &= h[(dx, a\, dx),\ (dx, -\frac{1}{a}dx)] \\ &= \log\left[\frac{y_1(dx) + \frac{1}{a}dx}{y_1(dx) - a\, dx} \cdot \frac{y_2(dx) + a\, dx}{y_2(dx) - \frac{1}{a}dx}\right] \\ &= \log\left[1 + \frac{dx}{a\, y_1(dx)}\right] + \log\left[1 + \frac{a\, dx}{y_2(dx)}\right] \\ &\qquad - \log\left[1 - \frac{a\, dx}{y_1(dx)}\right] - \log\left[1 - \frac{dx}{a\, y_2(dx)}\right]\ . \end{aligned} \tag{9}$$

Using the Maclaurin expansion of the logarithms, and collocting first and second degree terms, we obtain

$$\begin{aligned} h(b_1, b_2) &= dx\left(a + \frac{1}{a}\right)\left[\frac{1}{y_1(dx)} + \frac{1}{y_2(dx)}\right] \\ &\quad + \frac{dx^2}{2}\left(a^2 - \frac{1}{a^2}\right)\left[\frac{1}{y_1^2(dx)} - \frac{1}{y_2^2(dx)}\right] + O(dx^3)\ . \end{aligned} \tag{10}$$

Because both of the functions $y_1(x)$ and $y_2(x)$ are convex and have second derivatives at $x=0$, they can be represented in the form

$$y_i(dx) = y_i(0) + y_i'(0)dx + O(dx^2)\ , \qquad i = 1, 2, \tag{11}$$

and hence

$$\begin{aligned} \frac{1}{y_i(dx)} &= \frac{1}{y_i(0)} - \frac{y_i'(0)}{y_i^2(0)}dx + O(dx^2) \\ \frac{1}{y_i^2(dx)} &= \frac{1}{y_i^2(O)} + O(dx)\ . \end{aligned} \tag{12}$$

The substitution of (12) in (10) gives

$$\begin{aligned} h(b_1, b_2) &= dx\left(a + \frac{1}{a}\right)\left[\frac{1}{y_1} - \frac{y_1' dx}{y_1^2} + \frac{1}{y_2^2} - \frac{y_1' dx}{y_2^2}\right] \\ &\quad + \frac{dx^2}{2}\left(a^2 - \frac{1}{a^2}\right)\left(\frac{1}{y_1^2} - \frac{1}{y_2^2}\right) + O(dx^3)\ , \end{aligned} \tag{13}$$

where $y_i = y_i(0)$. Hence

$$(14)\qquad 2\,h(b_1,b_2) = 2\,dx\Big(a+\frac{1}{a}\Big)\Big[\frac{1}{y_1}+\frac{1}{y_2}-\frac{y_1'dx}{y_1^2}-\frac{y_2'dx}{y_2^2} + \frac{dx}{2}\Big(a-\frac{1}{a}\Big)\Big(\frac{1}{y_1^2}-\frac{1}{y_2^2}\Big)\Big]+O(dx^3)\,.$$

By the substitution of 2 dx for dx we obtain

$$(15)\qquad h(c_1,c_2) = 2\,dx\Big(a+\frac{1}{a}\Big)\Big[\frac{1}{y_1}+\frac{1}{y_2}-\frac{2\,y_1'\,dx}{y_1^2}-\frac{2\,y_2'\,dx}{y_2^2} + dx\Big(a-\frac{1}{a}\Big)\Big(\frac{1}{y_1^2}-\frac{1}{y_2^2}\Big)\Big]+O(dx^3)\,.$$

Substituting this and (14) in (8) we have

$$(16)\qquad \begin{aligned} &2dx\Big(a+\frac{1}{a}\Big)\Big[\frac{1}{y_1}+\frac{1}{y_2}-\frac{y_1'dx}{y_1^2}-\frac{y_2'dx}{y_2^2} + \frac{dx}{2}\Big(a-\frac{1}{a}\Big)\Big(\frac{1}{y_1^2}-\frac{1}{y_2^2}\Big)\Big] \\ &< 2\,dx\Big(a+\frac{1}{a}\Big)\Big[\frac{1}{y_1}+\frac{1}{y_2}-\frac{2\,y_1'\,dx}{y_1^2}-\frac{2\,y_2'\,dx}{y_2^2} + dx\Big(a-\frac{1}{a}\Big)\Big(\frac{1}{y_1^2}-\frac{1}{y_2^2}\Big)\Big]+O(dx^3)\,. \end{aligned}$$

By dividing both sides of this inequality by $2dx(a+1/a)$, and then rearranging the terms, we obtain

$$(17)\qquad dx\Big(\frac{y_1'}{y_1}+\frac{y_2'}{y_2}\Big)-\frac{dx}{2}\Big(a-\frac{1}{a}\Big)\Big(\frac{1}{y_1^2}-\frac{1}{y_2^2}\Big) < O(dx^2)\,.$$

Division of both sides of (17) by dx yields a new inequality whose right side is $O(dx)$ but whose left side is independent of dx. From this it follows that

$$(18)\qquad \frac{y_1'}{y_1}+\frac{y_2'}{y_2}-\frac{1}{2}\Big(a-\frac{1}{a}\Big)\Big(\frac{1}{y_1^2}-\frac{1}{y_2^2}\Big) \leqq 0\,.$$

Consider now a reflection in the y-axis taking C into a curve $\overline{C}$ which is divided by the x-axis into an "upper" arc $z=z_1(x)$ and a "lower" arc $z=-z_2(x)$. With the lines $z=(1/a)$ and $z=-ax$ playing the roles of ξ_1 and ξ_2, and with $\bar{b}_1$, $\bar{c}_1$, $\bar{b}_2$, $\bar{c}_2$ defined respectively by $(dx, (1/a)\,dx)$, $(2\,dx, (2/a)\,dx)$, $(dx, -a\,dx)$, and $(2\,dx, -2a\,dx)$, a repetition of the former argument leads to

$$(19)\qquad \frac{z_1'}{z_1^2}+\frac{z_2'}{z_2^2}-\frac{dx}{2}\Big(\frac{1}{a}-a\Big)\Big(\frac{1}{z_1^2}-\frac{1}{z_2^2}\Big) \leqq 0\,.$$

Since $z_i = y_i$ and $z_i' = -y_i'$, (19) is also

$$-\frac{y_1'}{y_1^2}-\frac{y_2'}{y_2^2}+\frac{dx}{2}\left(a-\frac{1}{a}\right)\left(\frac{1}{y_1^2}-\frac{1}{y_2^2}\right)\leqq 0\ . \tag{20}$$

Combining the opposite inequalities (18) and (20), we obtain

$$\frac{y_1'}{y_1^2}+\frac{y_2'}{y_2^2}-\frac{1}{2}\left(a-\frac{1}{a}\right)\left(\frac{1}{y_1^2}-\frac{1}{y_2^2}\right)=0\ . \tag{21}$$

Since (21) is an equality, it is clear that the same result would have been obtained if all preceding inequalities has been reversed. In other words (21) holds if p is a point of determinate curvature.

To express (21) in polar coordinates, let the polar axis be ξ_1 and let θ_0 designate the angle between the polar axis and the upper half-line of the y-axis. The angles of inclination to the x-axis of the tangent lines to C at $(0, y_1)$ and $(0, y_2)$ are α_1 and α_2 respectively and the clockwise angles from the radius vectors to the tangent lines at these points are ω_1 and ω_2. From the standard relationships between polar and Cartesian coordinates, it follows that

$$\begin{aligned} y_1'(0)&=\tan\alpha_1=-\cot\omega_1=\left[-\frac{1}{r}\frac{dr}{d\theta}\right]\theta_0 \\ y_2'(0)&=-\tan\alpha_2=\cot\omega_2=\left[\frac{1}{r}\frac{dr}{d\theta}\right]\theta_0+\pi \end{aligned} \tag{22}$$

Also, by definition, $a=\cot\theta_0$ so $\frac{1}{2}\left(a-\frac{1}{a}\right)=\cot 2\theta_0$. Substituting this and (22) in (21) we obtain

$$\left[-\frac{1}{r^3}\frac{dr}{d\theta}\right]_{\theta_0}+\left[\frac{1}{r^3}\frac{dr}{d\theta}\right]_{\theta_0+\pi}-(\cot 2\theta_0)\left[\frac{1}{r^2(\theta_0)}-\frac{1}{r^2(\theta_0+\pi)}\right]=0\ , \tag{23}$$

and hence

$$\left[\frac{1}{r^3}\frac{dr}{d\theta}+\frac{1}{r^2}\cot 2\theta\right]_{\theta_0}=\left[\frac{1}{r^3}\frac{dr}{d\theta}+\frac{1}{r^2}\,2\theta\right]_{\theta_0+\pi}. \tag{24}$$

Multiplying both sides of (24) by $2\,csc 2\theta_0=2\,csc 2(\theta_0+\pi)$ we have

$$\frac{d}{d\theta}\frac{(csc\ 2\theta)}{r^2}\bigg|_{\theta_0}=\frac{d}{d\theta}\frac{(csc\ 2\theta)}{r^2}\bigg|_{\theta_0+\pi}. \tag{25}$$

Since (25) involves only first derivatives, it holds for all θ_0 for which r is differentiable at both θ_0 and $\theta_0+\pi$. Since the one-sided derivative exists everywhere, we get the desired relations in (1), for all θ_0, from the semi-continuity of the one sided derivative.

Proof of the Theorem. According to the corollary of Lemma 1 there is always a projective transformation such that, after the transformation,

p satisfies the conditions of Lemma 3. From (1) we obtain

$$\int_{\theta_0}^{\theta} d\left(\frac{csc\, 2\theta}{r^2}\right)=\int_{\theta_0+\pi}^{\theta+\pi} d\left(\frac{csc\, 2\theta}{r^2}\right), \tag{26}$$

where the integrals are Stieltjes intergrals and the interval (θ_0, θ) does not contain a multiple of $\pi/2$. Hence

$$\frac{1}{r^2(\theta)}=\frac{1}{r^2(\theta+\pi)}+k_j \sin 2\theta,\ k_j=\text{constant} \tag{27}$$

where $$(j-1)\frac{\pi}{2}\leqq\theta\leqq j\frac{\pi}{2}, (j=1, 2, 3, 4)\ .$$

Since r is differentiable at the points for which $\theta=0, \pi/2, \pi, 3\pi/2$, we obtain from (27), upon differentiation at these points, the relations $k_1=k_2=k_3=k_4$. On the other hand, if we replace θ by $\theta+\pi$ in (27) we obtain the relations $k_1=-k_3$, and $k_2=-k_4$. In other words, $k_j=0$ and $r(\theta)=r(\theta+\pi)$. Since this shows p to be a metric center, it was initially a projective center.

The last statement in the theorem is well known (see [3] and e.g. [2, p.164]).

If a Hilbert metric is defined in the interior of an n-dimensional, convex surface S, the definitions for curvature and projective centers are unchanged. The metric for the space induces, on any plane through an interior point p, a two-dimensional Hilbert geometry. If p is a point of determinate curvature, it is a two-dimensional projective center for every plane through it. Since the L_p locus for every plane section is a line, it is easily seen that the total L_p locus must be a plane and hence that p is a projective center of S. If curvature is determinate everywhere then S is an ellipsoid and the geometry is hyperbolic.

It seems probable that a Hilbert geometry can contain no points of positive curvature.

References

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