LOCALLY COMPACT DIVISION RINGS

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Let K be a division ring with a non-discrete topology T with respect to which both the additive group K^+ and the multiplicative group K^* of K are locally compact topological groups.[1] If m is Haar measure for K^+ and $a \in K$, the function $m'(E) = m(aE)$ is clearly an invariant Borel measure for K^+. Hence there exists a real number $\phi(a)$ such that $m'(E)=\phi(a)m(E)$ for all Borel subsets E of K^+. The real-valued function ϕ on K (which is essentially the Radon-Nikodym derivative of m with respect to left-invariant Haar measure on K^*) evidently has the first two of the following three properties.

(1) $\phi(a)\geqq 0$; $\phi(a)=0$ if and only if $a=0$.

(2) $\phi(ab)=\phi(a)\phi(b)$.

(3) There exists $M>0$ such that $\phi(a)\leqq 1$ implies $\phi(1+a)\leqq M$.

We shall show that ϕ satisfies (3) also, i. e., is a valuation for K, and that the topology T_ϕ for K defined by ϕ coincides with T.[2] The classification of K then follows from known results.

LEMMA 1. *ϕ is continuous.*

Proof. Let ε be a positive number and let E be a compact set of positive measure. By the regularity of Haar measure we may choose an open set U containing E such that $m(U)-m(E)<\varepsilon m(E)$. Choose a neighborhood V of 1 with $V=V^{-1}$ and $V\cdot E\subset U$. Then for x in V, $\phi(x)=m(xE)/m(E)\leqq m(U)/m(E)<1+\varepsilon$; since $x^{-1}\in V$, $\phi(x)=(\phi(x^{-1}))^{-1}>(1+\varepsilon)^{-1}$. Hence $1-\varepsilon<\phi(x)<1+\varepsilon$ and the continuity of ϕ on K^* follows,[2*] Now choose an open set U with $m(U)<\varepsilon m(E)$ and a neighborhood V of 0 with $V\cdot E\subset U$. Then for a in V, $\phi(a)=m(aE)/m(E)\leqq m(U)/m(E)<\varepsilon$ and ϕ is continuous at 0.

LEMMA 2. *$S=\{a\in K: \phi(a)\leqq 1\}$ is compact.*

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[1] Continuity of the inverse multiplicative operation need not be assumed; cf. the concluding remark. The continuity of multiplication implies that $a\to -a=(-1)$. a is continuous.

[2] This idea was suggested by some work of Tate, [12].

[2*] Cf. Halmos [3, §60.6, p. 265].

Proof. Let C be a compact neighborhood of 0 and choose a neighborhood V of 0 such that $V\cdot C\subset C$. Let $a\in V\cap C$ such that $0<\phi(a)<1$. If $a^nS\subset C$ holds for no $n=1, 2, \cdots$, we select for each n an $s_n\in S$ such that $a^ns_n\notin C$. Since $\phi(a^k)\to 0$ and all the a^k lie in the compact set C, $a^k\to 0$ and hence $a^ks_n\in C$ for sufficiently large k. We may therefore choose $k_n\geqq n$ such that $a^{k_n}s_n\notin C$ but $a^{k_n+1}s_n\in C$. Then the sequence $\{a^{k_n}s_n\}$ of elements of the compact set $a^{-1}C$ has a cluster point c in $a^{-1}C$. Hence $\phi(a^{k_n}s_n)=\phi(a)^{k_n}\phi(s_n)\leqq\phi(a)^{k_n}$ has $\phi(c)$ as a cluster point by the continuity of ϕ; thus $\phi(c)=0$ and $c=0$, which contradicts $a^{k_n}s_n\notin C$. It follows that S is a subset of the compact set $a^{-n}C$ for some n and so, being closed by virtue of the continuity of ϕ, is compact.

COROLLARY. *ϕ is a valuation.*

Proof. $\phi(1+S)$, the continuous image of the compact set $1+S$, is bounded.

LEMMA 3. $T_\phi=T$.

Proof. Let $V\in T-\{\phi\}$, $a\in V$ and $B_n=\{b\in K:\ \phi(b-a)<2^{-n}\}$. Suppose we can choose $b_n\in B_n$ with $b_n\notin V$ for each $n=1, 2, \cdots$. But then the points b_n-a, all of which lie in the compact set S, have a cluster point c in S which must be 0 since $\phi(c)=0$. Hence $b_n\to a$ contrary to our assumption and it follows that $T\subset T_\phi$. Since the opposite inclusion is an immediate consequence of the continuity of ϕ, the proof is complete.

If K is connected[3], it is the real, complex or quaternion field (Pontrjagin [10]); in particular, ϕ is archimedean. Conversely, if ϕ is archimedean, the theorem of Ostrowski [8, p. 278] asserts that the center of K is either the real or complex field and so K, not being totally disconnected, is connected.[5]

If K is totally disconnected, ϕ is non-archimedean (and conversely, according to the above) and results due to van Dantzig [2], Hasse [4], Hasse and Schmidt [5], Jacobson and Taussky [6] and Jacobson [7] assert that K is of one of the following three types;[4]

(i) the completion of an algebraic number field at a finite prime,

(ii) the completion of an algebraic function field in one variable

[3] K is either connected or totally disconnected: if the component C of 0 contains $a\notin 0$ then $ba^{-1}C$ is a connected set containing 0 and $b\in K$.

[4] Otobe [9] shows that $a\to a^{-1}$ need not be assumed to be continuous; cf. our final remark in this connection.

[5] Alternatively, if K is connected, it is not difficult to show that ϕ is archemedian; then K is a vector space over the reals (Ostrowski) with ϕ as a norm, hence is the real, complex or quaternion field (Arens [1] Tornheim [13]), proving Pontrjagin's theorem.

over a finite field H,

(iii) a division ring D obtained from a field F of type (ii) by redefining $x. a=a^{\sigma}. x$, $a \in H$, σ a fixed non-trivial automorphism of H, the elements of D and F being regarded as power series $\sum_{i=n}^{\infty} a_i x^i$ in an indeterminate x over H with coefficients in H.

REMARK. Continuity of $a \rightarrow a^{-1}$ need not be assumed, for it appears in the connected case only in the proof that K is not compact in the proof of the Pontrjagin theorem [11, p. 173, Theorem 45.]. If K were compact, $\phi(a)=m(aK)/m(K)\leqq 1$ for all $a \in K$. But, as in the proof of the continuity of ϕ at 0 in Lemma 1, we can find $a \in K$ such that $0<\phi(a)<1$; then $\phi(a^{-1})>1$ and it follows that K is not compact. If K is totally disconnected we have only to apply to T, K^* the following unpublished theorem of A. M. Gleason: Let G be a group with a totally disconneted topology T under which the group operation is continuous from $G\times G$ to G. Then T, G is a topological group.

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