ASYMPTOTIC EXPRESSIONS FOR $\sum n^a f(n) \log^r n$

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In this paper some asymptotic expressions for sums of the type

$$\sum n^a f(n) \log^r n ,$$

where $f(n)$ is a number theoretic function, are presented. (The summations extend over $1 \leq n \leq x$ unless otherwise noted.) The method applied is to obtain the Laplace transformation,

$$\mathscr{L}\{F(t)\} = \int_0^\infty e^{-st} F(t) dt = f(s)$$

of the sum and then use a Tauberian theorem either from Doetsch [2] or its modification for a pole at points other than the origin, or from Delange [1] to obtain the asymptotic relation. If $f(n)$ is non-negative, then $F(t)$ is a non-negative, non-decreasing function and hence satisfies the conditions for the Tauberian theorems. In many cases the closed form of a Dirichlet series involving the functions are known, and in this case the relation

$$\mathscr{L}\Big\{ \sum_{1 \leqq n \leqq e^t} n^a f(n) \log^r n \Big\} = (-1)^r s^{-1} (d/ds)^r \sum_1^\infty n^{a-s} f(n)$$

can be used. The functions chosen for discussion and the Dirichlet series involving them can be found in Hardy and Wright [3], Landau [4], [5], or Titchmarsh [7]. We present first a few illustrations of the method and then a more extensive collection of results is presented at the end in a table.

First we choose $\sigma_k(n)$ as an example of a simpler type. Since

$$\sum_1^\infty n^{-s} \sigma_k(n) = \zeta(s)\zeta(s-k) ,$$

we have

$$\mathscr{L}\Big\{ \sum_{1 \leqq n \leqq e^t} n^{b-1-k} \sigma_k(n) \log^r n \Big\} = f(s) = (-1)^r s^{-1} (d/ds)^r \{\zeta(s+1-b)\zeta(s+1-b+k)\}.$$

For $k > 0$ the pole where $\Re s$ is greatest is at $s = b$ if $b \geq 0$. At that pole, since

$$\zeta^{(m)}(s+1-b) \sim (-1)^m m!\, (s-b)^{-m-1} ,$$

the Laplace transformation of the sum has the form

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$$f(s) \sim b^{-1}\zeta(1+k)r\,!\,(s-b)^{-r-1} .$$

Now if $b > 0$, then by modifying Doetsch [2, p. 517] for poles not at the origin or from Delange [1, p. 235] we obtain

$$\sum_{1 \leq n \leq e^t} n^{b-1-k}\sigma_k(n) \log^r n \sim b^{-1}\zeta(1+k)e^{bt}t^r ,$$

or, if $x = e^t$

$$\sum n^{b-1-k}\sigma_k(n) \log^r n \sim b^{-1}\zeta(1+k)x^b \log^r x .$$

If $b = 0$, then

$$f(s) \sim \zeta(1+k)r\,!\,s^{-r-2} ,$$

so that form Doetsch [2, p. 517] after substituting $x = e^t$ we obtain

$$\sum n^{-1-k}\sigma_k(n) \log^r n \sim (r+1)^{-1}\zeta(1+k) \log^{r+1} x .$$

The expressions for $\sigma(n)$ can be obtained by setting $k = 1$.

For $k = 0$, $\sigma_k(n)$ becomes $d(n)$ which will be covered as a special case of $d_k(n)$.

For $k < 0$ the pole where $\Re s$ is greatest is at $s = b - k$ so that for $b > k$

$$f(s) \sim (b-k)^{-1}\zeta(1-k)r\,!\,(s-b+k)^{-r-1} .$$

Hence

$$\sum n^{b-1-k}\sigma_k(n) \log^r n \sim (b-k)^{-1}\zeta(1-k)x^{b-k} \log^r x , \qquad \text{for } b > k ;$$
$$\sum n^{-1}\sigma_k(n) \log^r n \sim (r+1)^{-1}\zeta(1-k) \log^{r+1} x , \qquad \text{for } b = k .$$

By analogy, since

$$\sum_1^\infty n^{-s}\phi(n) = \zeta(s-1)/\zeta(s) ,$$

then

$$\sum n^{b-2}\phi(n) \log^r n \sim \{b\zeta(2)\}^{-1}x^b \log^r x , \qquad \text{for } b > 0 ;$$
$$\sum n^{-2}\phi(n) \log^r n \sim \{(r+1)\zeta(2)\}^{-1} \log^{r+1} x , \qquad \text{for } b = 0 .$$

If $\chi_k(n)$ represents a character, mod k, then the Dirichlet series can be represented by

$$\sum_1^\infty n^{-s}\chi_k(n) = L_k(s)$$

so that if χ_k is a principal character then $L_k(s)$ has a pole at $s = 1$ and

$$\sum n^{b-1}\chi_k(n) \log^r n \sim \phi(k)(kb)^{-1}x^b \log^r x , \qquad \text{for } b > 0 ;$$
$$\sum n^{-1}\chi_k(n) \log^r n \sim \phi(k)\{(r+1)b\}^{-1} \log^{r+1} x , \qquad \text{for } b = 0 .$$

The Dirichlet series involving $d_k(n)$ yields a power of the ζ-function, i.e.

$$\sum_1^\infty n^{-s} d_k(n) = \zeta^k(s) ,$$

so that for $k > 0$

$$\mathscr{L}\Big\{ \sum_{1 \le n \le e^t} n^{b-1} d_k(n) \log^r n \Big\} = (-1)^r s^{-1} (d/ds)^r \zeta^k(s + 1 - b) .$$

Now the Laplace transform can be written to show the behavior at the pole at $s = b$,

$$f(s) \sim (r + k - 1)! \, \{b(k-1)!\}^{-1} (s - b)^{-r-k} .$$

Thus

$$\sum n^{b-1} d_k(n) \log^r n \sim \{b(k-1)!\}^{-1} x^b \log^{r+k-1} x , \qquad \text{for } b > 0 ;$$
$$\sum n^{-1} d_k(n) \log^r n \sim \{(r+k)(k-1)!\}^{-1} \log^{r+k} x , \qquad \text{for } b = 0 .$$

Special cases can be obtained for $k = 1, 2$, since $d_1(n) = 1$ and $d_2(n) = \sigma_0(n) = d(n)$.

In an analogous manner we can obtain from

$$\sum_1^\infty n^{-s} d(n^2) = \zeta^3(s)/\zeta(2s)$$

the expressions

$$\sum n^{b-1} d(n^2) \log^r n \sim \{2b\zeta(2)\}^{-1} x^b \log^{r+2} x , \qquad \text{for } b > 0 ;$$
$$\sum n^{-1} d(n^2) \log^r n \sim \{2(r+1)\zeta(2)\}^{-1} \log^{r+3} x , \qquad \text{for } b = 0 .$$

Certain of the common number-theoretic functions have not been considered and do not appear in the table (in particular $\mu(n)$, $\lambda(n)$, and $\chi_k(n)$ for non-principal characters) because the sum $F(t)$ fails to satisfy the non-decreasing hypothesis for the Tauberian theorems. $\lambda(n)$ has the additional bad characteristic as shown by the poles of the closed from of the Dirichlet series

$$\sum_1^\infty n^{-s} \lambda(n) = \zeta(2s)/\zeta(s)$$

in that the pole of the numerator is on the line $\Re s = 1/2$ which is critical for the determinator, and thus this is not the pole where $\Re s$ is greatest as required by the theorem from Delange.

Results which he has obtained for the case $r = 0$ and the functions $\sigma(n)$, $\sigma_k(n)$, $d(n)$, and $\phi(n)$, treated by a different method, have been communicated to me in advance of their publication by Mr. Swetharanyam [6].

Table

Asymptotic expressions for $\sum n^a f(n)\log^r n$

General term of the sum	Asymptotic Expressions $b > 0$	$b = 0$
$n^{b-1-k}\sigma_k(n)\log^r n$ $(k > 0)$	$b^{-1}\zeta(1+k)x^b\log^r x$	$(r+1)^{-1}\zeta(1+k)\log^{r+1}x$
$n^{b-1}\sigma_k(n)\log^r n$ $(k < 0)$	$(b-k)^{-1}\zeta(1-k)x^{b-k}\log^r x$ $(b > k)$	$(r+1)^{-1}\zeta(1-k)\log^{r+1}x$ $(b = k)$
$n^{b-2}\sigma(n)\log^r n$	$b^{-1}\zeta(2)x^b\log^r x$	$(r+1)^{-1}\zeta(2)\log^{r+1}x$
$n^{b-1}d_k(n)\log^r n$	$\{b(k-1)!\}^{-1}x^b\log^{r+k-1}x$	$\{(r+k)(k-1)!\}^{-1}\log^{r+k}x$
$n^{b-1}d(n)\log^r n$	$b^{-1}x^b\log^{r+1}x$	$(r+2)^{-1}\log^{r+2}x$
$n^{b-1}\log^r n$	$b^{-1}x^b\log^r x$	$(r+1)^{-1}\log^{r+1}x$
$n^{b-1}\Lambda(n)\log^r n$	$b^{-1}x^b\log^r x$	$(r+1)^{-1}\log^{r+1}x$
$n^{b-2}\phi(n)\log^r n$	$\{b\zeta(2)\}^{-1}x^b\log^r x$	$\{(r+1)\zeta(2)\}^{-1}\log^{r+1}x$
$n^{b-1}q_k(n)\log^r n$	$\{b\zeta(k)\}^{-1}x^b\log^r x$	$\{(r+1)\zeta(k)\}^{-1}\log^{r+1}x$
$n^{b-1}\lvert\mu(n)\rvert\log^r n$	$\{b\zeta(2)\}^{-1}x^b\log^r x$	$\{(r+1)\zeta(2)\}^{-1}\log^{r+1}x$
$n^{b-1}2^{\omega(n)}\log^r n$	$\{b\zeta(2)\}^{-1}x^b\log^{r+1}x$	$\{(r+2)\zeta(2)\}^{-1}\log^{r+2}x$
$n^{b-1}d(n^2)\log^r n$	$\{2b\zeta(2)\}^{-1}x^b\log^{r+2}x$	$\{2(r+3)\zeta(2)\}^{-1}\log^{r+3}x$
$n^{b-1}d^2(n)\log^r n$	$\{6b\zeta(2)\}^{-1}x^b\log^{r+3}x$	$\{6(r+4)\zeta(2)\}^{-1}\log^{r+4}x$
$\dfrac{\sigma_a(n)\sigma_d(n)\log^r n}{n^{1+a+d-b}}$ $(a > 0)$ $(d > 0)$	$\dfrac{\zeta(1+a+d)\zeta(1+a)\zeta(1+d)}{b\zeta(2+a+d)}x^b\log^r x$	$\dfrac{\zeta(1+a+d)\zeta(1+a)\zeta(1+d)}{(r+1)\zeta(2+a+d)}\log^{r+1}x$
$\dfrac{\sigma_a(n)d(n)\log^r n}{n^{1+a-b}}$ $(a > 0)$	$\dfrac{\zeta^2(1+a)}{b\zeta(2+a)}x^b\log^{r+1}x$	$\dfrac{\zeta^2(1+a)}{(r+2)\zeta(2+a)}\log^{r+2}x$
$n^{b-2}a(n)\log^r n$	$2(3b)^{-1}x^b\log^r x$	$2\{3(r+1)\}^{-1}\log^{r+1}x$
$n^{b-1}\chi_k(n)\log^r n$	$\phi(k)(kb)^{-1}x^b\log^r x$	$\phi(k)\{k(r+1)\}^{-1}\log^{r+1}x$
$n^{b-1}r(n)\log^r n$	$4b^{-1}L_4(1)x^b\log^r x$	$4(r+1)^{-1}L_4(1)\log^{r+1}x$
$n^{b-1}\Lambda(n)\chi_k(n)\log^r n$	$b^{-1}x^b\log^r x$	$(r+1)^{-1}\log^{r+1}x$
$n^{b-2}\phi(n)\chi_k(n)\log^r n$	$\phi(k)\{kbL_k(2)\}^{-1}x^b\log^r x$	$\phi(k)\{(r+1)kL_k(2)\}^{-1}\log^{r+1}x$
$n^{b-1}2^{\omega(n)}\chi_k(n)\log^r n$	$4\phi(k)\{3kb\zeta(2)\}^{-1}x^b\log^{r+1}x$	$4\phi(k)\{3k(r+2)\zeta(2)\}^{-1}\log^{r+2}x$
$n^{b-1}\{\pi(n)-\pi(n-1)\}\log^r n$ $(r > 0)$	$b^{-1}x^b\log^{r-1}x$	$r^{-1}\log^r x$

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