A FIXED POINT THEOREM FOR CHAINED SPACES

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1. **Introduction.** There are a number of theorems in the literature of the following type: if a topological space is acyclic in the sense of containing no simple closed curve, and if other appropriate conditions are satisfied then the space has the fixed point property, that is, each continuous function f of the space into itself admits a solution of the equation $x = f(x)$. For example, if the space is compact metric and locally connected (i.e., a dendrite) then it has the fixed point property. There are many generalizations of this theorem. Appropriate to this discussion are of those of Borsuk [1], Plunkett [2], Wallace [3], the author [5] and [6], and Young [8]. A common characteristic of these generalizations is their requirement, explicit or implicit, of rather strong unicoherence conditions. But it is clear that many relatively simple acyclic spaces possessing the fixed point property are not unicoherent. As an example consider the following sets in the Cartesian plane:

$$A = \{(x, y) : 0 < x \leq 1, y = \sin(\pi/x)\} ,$$
$$B = \{(0, y) : -2 \leq y \leq 1\} ,$$
$$C = \{(x, -2) : 0 \leq x \leq 1\} ,$$
$$D = \{(1, y) : -2 \leq y \leq 0\} .$$

The continuum $M = A \cup B \cup C \cup D$ is not unicoherent but it is arcwise connected, acyclic, and has the fixed point property. It is the purpose of this note to formulate and prove a fairly general result which includes this and related examples. In so doing we shall generalize the theorems of Borsuk and Young cited above. As in our earlier papers the methods used here are order-theoretic in character. Section 2 is devoted to the partial order stucture of the spaces to be considered, and may be regarded as an addendum to [4], [6] and [7].

2. **Chained spaces.** Throughout all spaces to be considered are Hausdorff. By a *topological chain* or, more simply, a *chain*, we mean a continuum (=compact connected set) which has exactly two non-cutpoints. These two points are, of course, endpoints and a chain is simply the natural analogue of an arc in spaces which are not assumed to be metric. A space is *topologically chained* or *chained* provided each two distinct points lie in some chain. Obviously each two distinct points of a chained space are the endpoints of some chain. If a space has the property that each two distinct points are the endpoints of at most one chain, then it is said to be *acyclic*. In this case the unique chain whose

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endpoints are x and y is denoted $[x, y]$. It is convenient to define $[x, x]$ to be the set whose only element is x.

Acyclic chained spaces have an inherent partial order structure which facilitates their study. By a *partial order* on a set we mean a binary, reflexive, transitive relation $\leqq$ between elements of the set which, in addition, satisfies the rule

$$x \leqq y \text{ and } y \leqq x \text{ implies } x = y.$$

If $x \leqq y$ but $x \neq y$ we write $x < y$, and if P is a partially ordered set we define

$$L(x) = \{y \in P : y \leqq x\} \ , \quad M(x) = \{y \in P : x \leqq y\} \ .$$

In order to characterize acyclic chained spaces we recall a related theorem from [7]. A *dendritic* space is a connected and locally connected space in which each two distinct points can be separated by the omission of some third point.

THEOREM 1. *A necessary and sufficient condition that a locally connected space be dendritic is that it admit a partial order satisfying*

(i) *$L(x)$ and $M(x)$ are closed sets for each point x,*

(ii) *if $x < y$ then there exists z such that $\times < z$ and $z < y$,*

(iii) *for each x and y the set $L(x) \cap L(y)$ is nonempty, compact and simply ordered,*

(iv) *for each x the set $M(x) - x$ is open.*

Although many chained spaces are not locally connected (e.g., the space M of § 1) they can be made locally connected by properly altering the topology. This change of topology preserves the original chain structure of the space, and functions which are continuous in the orginal topology remain continuous in the new one. This technique appears to have originated with Young [8]. If X is a Hausdorff space let us say that a *chain component* of X is any subset of X which is maximal with respect to being chained. The *chain topology* is that topology which results from taking the chain components of open sets of the given topology as a basis for the chain topology. It is easily seen (and was proved in [8]) that any space is locally connected in its chain topology.

LEMMA 1. *An acyclic chained space is dendritic with respect to its chain topology.*

Proof. Let x and y be distinct points of the acyclic chained space X and let $z \in [x, y] - x \cup y$. Since X is acyclic no chain in $X - z$ contains both x and y, and therefore z separates x and y in the chain topology. Since X is connected and locally connected in the chain topology it is dendritic.

From Theorem 1 and Lemma 1 we infer that each acyclic chained space is endowed with an intrinsic partial order structure which can aptly be called the *chain cutpoint ordering*. It can be described in the following way (compare with [7]). Select an element e and define $x \leqq y$ if and only if $x \in [e,y]$. We now prove that the chain cutpoint ordering characterizes the acyclic spaces.

THEOREM 2. *A necessary and sufficient condition that the Hausdorff space X be acyclic and chained is that it be dendritic in its chain topology.*

Proof. The necessity was established in Lemma 1. To prove the sufficiency of the condition let X be a space which is dendritic in its chain topology. By Theorem 1 X admits a partial order which satisfies (i) – (iv) relative to the chain topology. If x and y are distinct points of X then by (ii) and (iii) they are contained in a continuum $L(x) \cup L(y)$ and by Theorem 3 of [7] that continum is a tree. Since a tree is chained, so is X. If two distinct chains C_1 and C_2 have common endpoints, let A_1 be a component of $C_1 - C_2$, x and y the endpoints of $\bar{A}_1$, and $\bar{A}_2$ the minimal subchain of C_2 which joins x and y. Obviously no point can separate x and y in the chain topology, for it would have to lie in $A_1 \cap \bar{A}_2 = 0$. Since this is a contradiction we conclude that X is acyclic.

3. A condition on rays. Let X be a space and $e \in x$. A *ray of X with endpoint e* is the union of a maximal nest of chains which have e as a common endpoint. Thus, in a Euclidean space a half line emanating from the origin is a ray in this sense. In the example of § 1 the set A is a ray of M with endpoint (1,0).

If R is a ray with endpoint e in the space X and $x \in R$, let $A(R,x)$ be the closure of $(R - [e, x]) \cup x$, We then define

$$K_R = \bigcap \{A(R, x) : x \in R\} .$$

In a Euclidean space a ray R consisting of a half line emanating from the origin has $K_R = 0$. However, in the example of § 1 the set A has K_A equal to a closed line segment.

The crux of our fixed point argument is the following. If $f: X \to X$ is continuous where X is acyclic and chained, we examine the points x such that $x \leqq f(x)$. Either there is a "last" such point in a restricted order-theoretic sense, in which case that point is fixed by a continuity argument, or else such points form a ray R. Then we can show that $f(K_R) \subset K_R$, so that the fixed point property follows provided each K_R has the fixed point property.

We begin by formalizing this condition on rays.

(F_A) If *R is a ray with endpoint a then* K_R *has the fixed point property.*

In the example of § 1 let $a = (1, -2)$. Then there are two rays with endpoint a, $B \cup C$ and $A \cup D$. Since $K_{B \cup C}$ is a point and $K_{A \cup D}$ is a line segment the space M satisfies (F_a).

THEOREM 3. *If X is an arcwise connected space in which the union of any nest of arcs is contained in an arc then X is acyclic and X satisfies (F_a) for each $a \in X$.*

Proof. Since the union of any nest of arcs is contained in an arc, X is acyclic; and if R is a ray then $\overline{R}$ is evidently an arc so that K_R is a point.

The substance of Young's fixed point theorem [8] is that the spaces of Theorem 3 have the fixed point property; hence, Theorem 5 below is truly a generalization.

THEOREM 4. *If X is an arcwise connected, hereditarily unicoherent continuum then X satisfies (F_a) for each $a \in X$.*

Proof. We note that each subcontinuum of X is arcwise connected, for if x and y are elements of the subcontinuum Y and $[x, y] - Y$ is not empty then $[x, y] \cup Y$ would not be unicoherent. Now if R is a ray of X then K_R, being the intersection of a nest of continua, is a continuum and hence is itself arcwise connected and hereditarily unicoherent. Borsuk's theorem [1] asserts that such sets have the fixed point property.

This result demonstrates that all continua statisfying the hypothesis of Borsuk's fixed point theorem are included in Theorem 5.

If A and B are subsets of a partially ordered set with $A \subset B$ then A is *cofinal* in B provided for each $b \in B$ there exists $a(b) \in A$ such that $b \leqq a(b)$.

THEOREM 5. *Let X be a topologically chained acyclic space and suppose there exists $e \in X$ such that (F_e) is satisfied. Then X has the fixed point property.*

Proof. We give X the chain cutpoint ordering with minimal element e and let $f : X \to X$ be a continuous function. Consider the family $\mathscr{S}$ of all pairs (S, S') satisfying the following six conditions:

(i) *S is a nonempty simply ordered subset of X,*

(ii) *S and S' are connected,*
(iii) *S' is cofinal in S,*
(iv) e $\in S$,
(v) $x \leqq f(x)$ *for each* $x \in S$,
(vi) $S \cup f(S')$ *is simply ordered.*

Obviously the pair (e, e) is a member of $\mathscr{S}$. We can partially order $\mathscr{S}$ by defining $(S_\gamma, S'_\gamma) \prec (S_\delta, S'_\delta)$ if and only if $S_\gamma \subset S_\delta$ and $S_\delta \cup f(S'_\gamma)$ is simply ordered. If $\mathscr{N} = \{(S_\gamma, S'_\gamma)\}$ is a $\prec$ − simple subfamily of $\mathscr{S}$ and $S = \bigcup \{S_\gamma\}$, $S' = \bigcup \{S'_\gamma\}$ then it is clear that $(S, S') \in \mathscr{S}$ and that (S, S') is a $\prec$ − upper bound of $\mathscr{N}$. Thus Zorn's lemma can be applied; let (S_0, S'_0) be a $\prec$ − maximal member of $\mathscr{S}$.

If $x_0 = \sup S_0$ exists we assert that $x_0 \leqq f(x_0)$. For suppose there is $t \in S_0$ such that $f(x_0)$ is not a successor of t. We may assume $t < f(t)$; if $T = [t, x_0]$ then $f(T)$ is a tree and t separates $f(t)$ and $f(x_0)$ in $f(T)$. If W is the component of $f(T) - t$ which contains $f(x_0)$ then W is a neighborhood of $f(x_0)$ in the relative topology of $f(T)$ and hence there is $q \in S_0$, $t < q < x_0$ such that $f(q) \in W$. But this implies that $f(q)$ is not a successor of q, a contradiction. Therefore, $t \leqq f(x_0)$ for each $t \in S_0$ and hence $x_0 \leqq f(x_0)$. If $x_0 < f(x_0)$ let U be a connected neighborhood of $f(x_0)$ *relative to the chain topology* such that $\overline{U} \subset X - x_0$. Then there exists $x_1 \in X - \overline{U}$ such that $x_0 < x_1 < f(x_0)$ and $f([x_0, x_1]) \subset U$. But then each point $p \in [x_0, x_1]$ satisfies $p \leqq f(p)$; letting $S_1 = S_0 \cup [x_0, x_1]$, $S'_1 = x_1$, it is apparent that $(S_0, S'_0) \prec (S_1, S'_1)$ in contradiction of the maximality of S_0. Hence $x_0 = f(x_0)$.

On the other hand if S_0 has no supremum it is a ray R with endpoint e and it remains only to show that $f(K_R) \subset K_R$. By (vi) and the fact that $S_0 = R$ is a ray we have $f(S'_0) \subset R$. Moreover, $A(R, x) \subset \overline{S'_0}$ for each $x \in S'_0$ and hence $K_R \subset \overline{S'_0}$. Therefore, $f(K_R) \subset \overline{R}$. Now suppose $f(y) \in \overline{R} - K_R$ for some $y \in K_R$. Let V be a neighborhood of $f(y)$ such that $\overline{V}$ and K_R are disjoint; then $\overline{V}$ and $A(R, x)$ are disjoint for some $x \in R$ and there exists $a \in R - [e, x]$ such that $f(a) \in V$. Moreover, it is clear by (ii) and (iii) that a may be so chosen that $a \in S'_0$ and hence $f(\mathrm{a}) \in A(R, x)$, a contradiction. Therefore, $f(K_R) \subset K_R$ and the proof is complete.

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