FAITHFUL*-REPRESENTATIONS OF NORMED ALGEBRAS

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1. Introduction. Let B be a complex Banach algebra with an involution $x \to x^*$ in which, for some $k > 0$, $\|xx^*\| \geqq k\|x\|\|x^*\|$ for all x in B. Kaplansky [8, p. 403] explicitly made note of the conjecture that all such B are symmetric. An equivalent formulation is the conjecture that all such B are B^*-algebras in an equivalent norm. In 1947 an affirmative answer had already been provided by Arens [1] for the commutative case. We consider in § 2 the general (non-commutative) case. It is shown that the answer is affirmative if k exceeds the sole real root of the equation $4t^3 - 2t^2 + t - 1 = 0$. This root lies between .676 and .677. In any case these algebras are characterized spectrally as those Banach algebras with involution for which self-adjoint elements have real spectrum and there exists $c > 0$ such that $\rho(h) \geqq c\|h\|$, h self-adjoint (where $\rho(h)$ is the spectral radius of h).

A basic question concerning a given complex Banach algebra B with an involution is whether or not it has a faithful*-representation as operators on a Hilbert space. In § 3 we give a necessary and sufficient condition entirely in terms of algebraic and linear space notions in B. This is that $\rho(h) = 0$ implies $h = 0$ for h self-adjoint and that $R \cap (-R) = (0)$. Here R is the set of all self-adjoint elements linearly accessible [11, p. 448] from the set of all finite sums of elements of the form x^*x. This is related to a previous criterion of Kelley and Vaught [10] which however involves topological notions (in particular, the assumption that the involution is continuous).

If B is semi-simple with minimal one-sided ideals a simpler discussion of *-representations (§ 5) is possible even if B is incomplete. For example if B is primitive then B has a faithful*-representation if and only if $xx^* = 0$ implies $x^*x = 0$. The incomplete case has features not present in the Banach algebra case. In the former case, unlike the latter, a*-representation may be discontinuous. A class of examples is provided in § 5.

2. Arens*-algebras. Let B be a complex normed algebra with an involution $x \to x^*$. An *involution* is a conjugate linear anti-automorphism of period two. Elements for which $x = x^*$ are called *self-adjoint* (s. a.) and the set of s. a. elements is denoted by H. Let $\mathfrak{H}$ be a Hilbert space and

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$\mathfrak{E}(\mathfrak{H})$ be the algebra of all bounded linear operators on $\mathfrak{H}$. By a*-representation of B we mean a homomorphism $x \to T_x$ of B into some $\mathfrak{E}(\mathfrak{H})$ where T_{x^*} is the adjoint of T_x. A^*-representation which is one-to-one is called *faithful*.

We shall be mainly, but not exclusively, interested in the case where B is complete (a Banach algebra). In § 2 we shall assume throughout that B is a Banach algebra with an involution $x \to x^*$.

As in [5, p. 8] we set $x \circ y = x + y - xy$ and say that x is quasi-regular with quasi-inverse y if $x \circ y = y \circ x = 0$. The quasi-inverse of x is unique, if it exists, and is denoted by x'. As, for example, in [16, p. 617] we define the *spectrum* of x, $sp(x)$, to be the set consisting of all complex numbers $\lambda \neq 0$ such that $\lambda^{-1}x$ is not quasi-regular, plus $\lambda = 0$ provided there does not exist a subalgebra of B with an identity element and containing x as an invertible element. (The treatment of zero as a spectral value plays no role below.) The *spectral radius* $\rho(x)$ if x is defined to be $\sup |\lambda|$ for $\lambda \in sp(x)$.

We say that B is an *Arens**-algebra* [1] if there exists $k > 0$ such that $\| x^*x \| \geqq k \| x \| \| x^* \|$, $x \in B$. As usual, we say that B is a *B*-algebra* if $\| x^*x \| = \| x \|^2$, $x \in B$.

2.1. LEMMA. *Let B an Arens*-algebra with $\| xx^* \| \geqq k \| x \| \| x^* \|$, $x \in B$. Then for each* s. a. *element h, $\rho(h) \geqq k \| h \|$ and $sp(h)$ is real.*

That the spectrum of a s. a. element h is real is shown in [1, p. 273]. By use of the inequality $\| h^{2^n} \| \geqq k \| h^{2^{n-1}} \|^2$ as in [16, p. 626] it follows that $\rho(h) \geqq k \| h \|$. We shall show (Theorem 2.4) that the spectral conditions of Lemma 2.1 imply that B is an Arens*-algebra.

2.2. LEMMA. *Suppose that for each* s. a. *element h, $\rho(h) \geqq c \| h \|$ and $sp(h)$ is real, where $c > 0$. Let h be* s. a., *$sp(h) \subset [-a, b]$ where $a \geqq 0$, $b \geqq 0$ and let $r > 0$. Then*

(1) *$\| (-t^{-1}h)' \| < r$ if $t > (1 - cr)b/cr$ and $t > (1 + cr)a/cr$,*

(2) *$\| (t^{-1}h)' \| < r$ if $t > (1 - cr)a/cr$ and $t > (1 + cr)b/cr$.*

Note that (2) follows from (1) as applied to the element-h. By [18, Theorem 3.4] the involution is continuous on B. Therefore h generates a closed*-subalgebra B_0. Let $\mathfrak{M}$ be the space of regular maximal ideals of B_0. For $t > a$ set $u = (-t^{-1}h)'$. By [8, Theorem 4.2], $u \in B_0$. It is readily seen that u is s. a. Since $-t^{-1}h + u + t^{-1}hu = 0$ we have, for each $M \in \mathfrak{M}$, $u(M) = h(M)/(t + h(M))$. By, [8, p. 402] the spectrum of h is the same whether computed in B or in B_0 so that $-a \leqq h(M) \leqq b$. Since $\lambda/(t + \lambda)$ is an increasing function of λ we see that $-a/(t - a) \leqq u(M) \leqq b/(t + b)$. Now $\rho(u) = \sup | u(M) |$, $M \in \mathfrak{M}$. Therefore, since u is s.a.,

$$c \| u \| \leqq \rho(u) \leqq \max [a/(t - a), b/(t + b)] \ . \tag{2.1}$$

From formula (2.1), $\| u \| < r$ if $a/(t-a) < cr$ and $b/(t+b) < cr$. This yields (1).

Note that, under the given hypotheses, $c \leqq 1$.

2.3. LEMMA. *Let x and y be quasi-regular. Then $x+y$ is quasi-regular if and only if $x'y'$ is quasi-regular.*

The formulas $x' \circ (x+y) \circ y' = x'y'$ and $x + y = x \circ (x'y') \circ y$ yield the desired result. Let $r > 0$. If $\| x' \| < r$ and $\| y' \| < r^{-1}$ it follows from Lemma 2.3 and [12, p. 66] that $(x+y)'$ exists.

Consider the situation of Lemma 2.2 and let h_k be s. a., $k = 1, 2$ where $N = \max(\rho(h_1), \rho(h_2))$. By Lemma 2.2, $\| (t^{-1}h_k))' \| < 1$ and $\| (-t^{-1}h_k)' \| < 1$ if $t > (1+c)N/c$. Then, by Lemma 2.3,

$$sp(h_1 + h_2) \subset [-(1+c)N/c, (1+c)N/c] \ . \tag{2.2}$$

Suppose next that $sp(h_k) \subset [0, \infty)$, $k = 1, 2$. Then $\| (t^{-1}h_k)' \| < 1$ if $t > (1+c)N/c$ and $\| (-t^{-1}h_k)' \| < 1$ if $t > (1-c)N/c$. Then by Lemma 2.3,

$$sp(h_1 + h_2) \subset [-(1-c)N/c, (1+c)N/c] \ . \tag{2.3}$$

2.4. THEOREM. *Suppose that for each* s. a. *element h, $\rho(h) \geqq c \| h \|$ and $sp(h)$ is real, where $c > 0$. Then B is an Arens*-algebra with $\|xx^*\| \geqq k \| x \| \| x^* \|$, $x \in B$, where k can be chosen to be $c^5/(1+c)(1+2c^2)$.*

Let $x = u + iv$ where u and v are s. a. Then $x^*x = u^2 + v^2 + i(uv - vu)$, $xx^* = u^2 + v^2 + i(vu\text{-}uv)$ and $xx^* + x^*x = 2u^2 + 2v^2$. We next compare $\rho(u^2) = [\rho(u)]^2$ and $\rho(v^2)$ with $\rho(xx^*)$. For this purpose we may suppose that $\rho(u) \geqq \rho(v)$ for otherwise we can replace x by $ix = -v + iu$. If $\lambda \neq 0$ then $\lambda \in sp(xx^*)$ if and only if $\lambda \in sp(x^*x)$. Thus $\rho(xx^*) = \rho(x^*x)$. By (2.2), $sp(xx^* + x^*x) \subset [-(1+c)\rho(xx^*)/c, (1+c)\rho(xx^*)/c]$. Now $2u^2 = xx^* + x^*x - 2v^2$. Let $r > 0, t > 0$. By Lemma 2.2,

$$\| [t^{-1}(xx^* + x^*x)]' \| < r, t > (1+cr)(1+c)\rho(xx^*)/c^2r \ . \tag{2.4}$$

Since $sp(-2v^2) \subset (-\infty, 0]$ and $\rho(2v^2), \leqq \rho(2u^2)$, by Lemma 2.2 we have, for $t > 0$,

$$\| [t^{-1}(-2v^2)]' \| < r^{-1}, t > (r-c)\rho(2u^2)/c \ . \tag{2.5}$$

we select $c < r < 2c$. For such r, Lemma 2.3 and formulas (2.4) and (2.5) show that $[t^{-1}(2u^2)]'$ exists if $t > \max\{(1+cr)(1+c)\rho(xx^*)/c^2r, (r-c)\rho(2u^2)/c\}$. Now $(r-c)/c < 1$ and $sp(2u^2) \subset [0, \infty)$. Therefore, letting $r \to 2c$, we have

$$\rho(2u^2) \leqq (1+2c^2)(1+c)\rho(xx^*)/(2c^3) \ . \tag{2.6}$$

On the other hand $\| x \| \leqq \| u \| + \| v \| \leqq [\rho(u) + \rho(v)]/c \leqq 2\rho(u)/c$ and $\| x^* \| \leqq 2\rho(u)/c$. Therefore, by (2.6), (2.7) $$\| x \| \, \| x^* \| \leqq 4\rho(u^2)/c^2 \leqq (1 + 2c)(1 + c)\rho(xx^*)/c^5 .$$

But $\rho(xx^*) \leqq \| xx^* \|$. This together with (2.7) completes the proof.

2.5. COROLLARY. *Under the hypotheses of Theorem* 2.4, *the norm of the involution as an operator on B does not exceed* $(1 + c)(1 + 2c^2)/c^5$.

In (2.7) we may replace $\| x \| \, \| x^* \|$ by $\| x^* \|^2$ and $\rho(xx^*)$ by $\|x\| \, \|x^*\|$. This gives $\| x^* \| \leqq (1 + c)(1 + 2c^2) \| x \|/c^5$.

We denote by $P(N)$ the set of $x \in B$ such that $sp(x^*x) \subset [0, \infty)$ $(sp(x^*x) \subset (-\infty, 0])$.

2.6. LEMMA. *For an Arens*-algebra B the following are equivalent.*

(a) *B is a B^*-algebra in an equivalent norm.*

(b) $N = (0)$.

(c) $P = B$.

Suppose that $N = (0)$. Let $y \in B$. Since the involution on B is continuous, the element y^*y generates a closed*-subalgebra B_0. Let $\mathfrak{M}$ be the space of regular maximal ideals of B_0. By [1, p. 279] the commutative algebra B_0 is *-isomorphic to $C(\mathfrak{M})$. Also $sp(y^*y)$ is real. Hence there exist $u, v \in B_0$ such that $u(M) = \sup(y^*y(M), 0)$ and $v(M) = -\inf(y^*y(M), 0)$, $M \in \mathfrak{M}$. Then u and v are s. a., $y^*y = u - v$ and $uv = 0$. As in [14, p. 281], $(yv)^*(yv) = -v^3$ so that $yv = 0$ by hypothesis. Then $v = 0$ and $sp(y^*y) \subset [0, \infty)$.

A theorem of Gelfand and Neumark [13] asserts that if B is semi-simple, has a continuous involution, is symmetric ($B = P$) and has an identity then there exists a faithful*-representation $x \to T_x$ of B. This theorem is also valid when B has no identity [4, Theorem 2.16]. In our situation, B is semi-simple [18, Lemma 3.5] and the involution is continuous. Thus a faithful*-representation exists. This representation is bi-continuous by [18, Corollary 4.4].

That (a) implies (b) follows from the well-known fact that any B^*-algebra is symmetric [14, p. 207 and p. 281].

The equation $4t^3 - 2t^2 + t - 1 = 0$ has exactly one real root a. This root a lies between .676 and .677.

2.7. THEOREM. *Suppose that for each* s. a. *element h, $\rho(h) \geqq c \| h \|$ and $sp(h)$ is real, where $c > 0$. Then there is an equivalent norm for B in which B is a B^*-algebra if $c > a$.*

Suppose that $sp(x^*x) \subset (-\infty, 0]$. By Lemma 2.6 it is sufficient to show that $x = 0$. Suppose that $x \neq 0$. By Theorem 2.4 it is clear that $x^*x \neq 0$ and $\rho(x^*x) \neq 0$. Set $x = u + iv$ where u and v are s. a. As in the proof of Theorem 2.4, $xx^* + x^*x = 2u^2 + 2v^2$ and we may assume that $\rho(u) \geqq \rho(v)$. Since $sp(u^2) \subset [0, \infty)$, $sp(v^2) \subset [0, \infty)$ formula 2.3 shows that $sp(2u^2 + 2v^2) \subset [-(1 - c)\rho(2u^2)/c, (1 + c)\rho(2u^2)/c]$. Let $r > 0, t > 0$. From Lemma 2.2,

$\| [-t^{-1}(2u^2 + 2v^2)]' \| < r$ if $t > (1 - cr)(1 + c)\rho(2u^2)/(c^2r)$ and $t > (1 + cr)(1 - c)\rho(2u^2)/(c^2r)$.

We write $x^*x = 2u^2+2v^2+(-xx^*)$. By Lemma 2.2, $\| [-t^{-1}(-xx^*)]'\| < r^{-1}$ if $t > 0$ and $t > (r - c)\rho(x^*x)/c$ since $sp(-xx^*) \subset [0, \rho(x^*x)]$. By Lemma 2.3, $(-t^{-1}x^*x)'$ exists if $t > \max \{(1+cr)(1-c)\rho(2u^2)/c^2r, (1-cr)(1+c)\rho(2u^2)/c^2r, (r - c)\rho(x^*x)/c\}$. Since $sp(x^*x) \subset (-\infty, 0]$, $\rho(x^*x)$ cannot exceed this maximum. Now select r, $1 \leqq r < 2c$ which is possible since $c > a$. Then $(r - c)/c < 1$ and $(1 + cr)(1 - c) \geqq (1 - cr)(1 + c)$. Therefore $\rho(x^*x) \leqq (1 + cr)(1 - c)\rho(2u^2)/c^2r$. Letting $r \to 2c$ we obtain

$$\rho(x^*x) \leqq (1 + 2c^2)(1 - c)\, \rho(2u^2)/2c^3 . \tag{2.8}$$

Next we express $-2u^2 = 2v^2 + (-xx^* - x^*x)$. By formula (2.3), $sp(-xx^* - x^*x) \subset [-(1 - c)\rho(x^*x)/c, (1 + c)\rho(x^*x)/c]$. Recall that $\rho(2v^2) \leqq \rho(2u^2)$. Repeating the above reasoning we see that for $r > 0, t > 0$, $(-t^{-1}(-2u^2))'$ exists for $t > \max \{1-cr)(1+c)\rho(x^*x)/c^2r, (1+cr)(1-c)\rho(x^*x)/c^2r, (r - c)\rho(2u^2)/c\}$. But $sp(-2u^2) \subset (-\infty, 0]$. Then by the argument above we obtain

$$\rho(2u^2) \leqq (1 + 2c^2)(1 - c)\rho(x^*x)/2c^3 . \tag{2.9}$$

From formulas (2.8) and (2.9) we see that $(1 + 2c^2)(1 - c) \geqq 2c^3$ or $4c^3 - 2c^2 + c - 1 \leqq 0$. This gives $c \leqq a$ which is impossible by hypothesis.

Thus if $c > a$ we have $N=(0)$. We subsequently show (Corollary 2.11) that, in any case, N and P are closed in an Arens*-algebra B.

Following Rickart [16, p. 625] we say that B is an *A^*-algebra* if there exists in B an auxiliary normed-algebra norm $| x |$ (B need not be complete it this norm) such that, for some $c > 0$, $| x^*x | \geqq c | x |^2$. He raises the question of whether every A^*-algebra has a faithful*-representation.

2.8. COROLLARY. *An A^*-algebra B where $| x^*x | \geqq c | x |^2$, $x \in B$, in the auxiliary norm has a faithful*-representation if $c > a$.*

Observe that $| x^* || x | \geqq c | x |^2$ so that $| x^* | \leqq c^{-1} | x |$, $x \in B$. Thus the involution on B is continuous in the topology provided by the norm $| x |$. Let B_0 be the completion of B in the norm $| x |$. We extend the function $| x |$ from B to B_0 by continuity. Likewise the involution $x \to x^*$ can be extended by continuity to provide a continuous involution $y \to y^*$ on B_0. We then have $| y^*y | \geqq c | y |^2$, $y \in B_0$. As in [16, p. 626] we obtain $\rho(h) \geqq c | h |$ for h s. a. in B_0 where $\rho(h)$ is the spectral radius computed for h as an element of the Banach algebra B_0, $\rho(h) = \lim | h^n |^{1/n}$. Also $| y^*y | \geqq c^2 | y^* | |y|$, $y \in B_0$, so that B_0 is an Arens*-algebra. Hence, by Lemma 2.1, the spectrum of each s. a. element of B_0 is real. By Theorem 2.7, B_0 is a B^*-algebra in an equivalent norm. Therefore B has the desired faithful*-representation.

We have no information on the truth or falsity of Theorem 2.7 for $c \leqq a$.

To prove Theorem 2.7 without restriction on the size of c one can assume without loss of generality that B has an identity. For suppose that B has no identity, $\| x^*x \| \geqq k \| x^* \| \| x \|$, $x \in B$. Adjoin an identity e to B to form the algebra B_1 with the norm defined in B_1 by the rule

$$\| \lambda e + x \| = \sup_{\substack{\|y\|=1 \\ y \in B}} \| \lambda y + xy \| .$$

Then B_1 is a Banach algebra with the involution $(\lambda e + x)^* = \bar{\lambda} e + x^*$[**1**, p. 275]. By changing in minor ways arguments in [**14**, p. 207] we see that B_1 is an Arens*-algebra. There is a constant K such that $\| x^* \| \leqq K \| x \|$, $x \in B$. Choose $0 < r < 1$. Given $\lambda e + x \in B_1$ there exists $y \in B$, $\| y \| = 1$, such that

$$\begin{aligned} r^2 \| \lambda e + x \|^2 < \| \lambda y + xy \|^2 &\leqq K \| (\lambda y + xy)^* \| \| \lambda y + xy \| \\ &\leqq K k^{-1} \| y^*(\lambda e + x)^*(\lambda e + x) y \| \\ &\leqq K^2 k^{-1} \| (\lambda e + x)^*(\lambda e + x) \| . \end{aligned}$$

Then

$$\| (\lambda e + x)^*(\lambda e + x) \| \geqq k K^{-2} \| \lambda e + x \|^2 \geqq (kK^{-2})^2 \| \lambda e + x \| \| (\lambda e + x)^* \|.$$

We use this fact later.

Some results on spectral theory in Arens*-algebras were obtained by Newburgh [**15**]. In a B^*-algebra $\rho(x)$ is a continuous function on the set H of s.a. elements since $\rho(h) = \| h \|$, $h \in H$. This property holds for all Arens*-algebras.

2.9. THEOREM. *In any Arens*-algebra, $\rho(x)$ is a continuous function on H.*

We assume that $\rho(h) \geqq c \| h \|$ and $sp(h)$ is real, $h \in H$. We shall use the following principle [**12**, p. 67]. If y' exists and $\| z \| < (1 + \| y' \|)^{-1}$ then $(y + z)'$ exists.

Let $h \in H$, $h \neq 0$. Select $t > \rho(h)$ and set $u = (t^{-1}h)'$. We proceed as in the proof of Lemma 2.2. Let B_0 be the closed*-subalgebra generated by h and let $\mathfrak{M}$ be its space of regular maximal ideals. Then $u \in B_0$. Since $t^{-1}h \circ u = 0$ we obtain, for each $M \in \mathfrak{M}$, $u(M) = h(M)/(h(M) - t)$. Since $\lambda/(\lambda - t)$ is a decreasing function of λ, $\sup | u(M) |$ can be majorized by $\rho(h)/(t - \rho(h))$. Then $(1 + \|u\|)^{-1} \geqq (1 + c^{-1}\rho(u))^{-1} \geqq c(t - \rho(h))/(ct + (1 - c)\rho(h)) = a(t)$, say.

Therefore $t^{-1}h + t^{-1}h_1$ is quasi-regular if $\| t^{-1}h_1 \| < a(t)$ or if

$$ct^2 - c[\rho(h) + \| h_1 \|]t - (1 - c)\rho(h) \| h_1 \| > 0 . \tag{2.10}$$

We apply this to $h_1 \in H$, $\| h_1 \| < \rho(h)$. The larger zero d of the left hand side of (2.10) is given by

$$(2.11) \qquad 2d = \rho(h) + \| h_1 \| + [(\rho(h) - \| h_1 \|)^2 + 4c^{-1}\rho(h) \| h_1 \|]^{1/2} .$$

The radical term of (2.11) is majorized by $\rho(h) - \| h_1 \| + 2(c^{-1}\rho(h)\| h_1 \|)^{1/2}$. Hence $d \leqq \rho(h) + (c^{-1}\rho(h) \| h_1 \|)^{1/2}$. Thus $t \notin sp(h + h_1)$ if $t > \rho(h) + (c^{-1}\rho(h) \| h_1 \|)^{1/2}$. Likewise $t \notin sp(-h - h_1)$ under the same condition. This shows that

$$(2.12) \qquad \rho(h + h_1) \leqq \rho(h) + (c^{-1}\rho(h) \| h_1 \|)^{1/2} .$$

provided $h_1 \in H$ and $\| h_1 \| < \rho(h)$.

Note that $\rho(h + h_1) \geqq c \| h + h_1 \| \geqq c (\| h \| - \| h_1 \|) \geqq c(\rho(h) - \| h_1 \|)$. Therefore if $\| h_1 \| < c(\rho(h) - \| h_1 \|)$ or equivalently if $\| h_1 \| < c\rho(h)/(1+c)$ we have $\| h_1 \| < \rho(h + h_1)$. We may then apply the above analysis to the pair of s. a. elements $(h + h_1)$, $-h_1$, to obtain (if $\| h_1 \| < c\rho(h)/(1 + c)$)

$$(2.13) \qquad \rho(h) \leqq \rho(h_1 + h_2) + (c^{-1}\rho(h + h_1) \| h_1 \|^{1/2} .$$

From (2.12), $\rho(h + h_1) \leqq [c^{-1/2} + 1]\rho(h)$. Inserting this estimate in the radical term of (2.13) we obtain

$$(2.14) \qquad \rho(h) \leqq \rho(h + h_1) + (c^{-1} + c^{-3/2})^{1/2}(\rho(h) \| h_1 \|)^{1/2}$$

Combining (2.12) and (2.14) we obtain

$$| \rho(h + h_1) - \rho(h) | \leqq [(c^{-1} + c^{-3/2})\rho(h) \| h_1 \|]^{1/2}$$

provided $\| h_1 \| < c\rho(h)/(1 + c)$.

This show that $\rho(x)$ is continuous on H at $x = h$. Clearly we have continuity on H at $x = 0$.

For x s.a. in an Arens*-algebra let $[a(x), b(x)]$ be the smallest closed interval containing $sp(x)$.

2.10. COROLLARY. *For an Arens*-algebra B, $a(x)$ and $b(x)$ are continuous functions of x on H.*

As remarks above indicate, there is no loss of generality in supposing that B has an identity e. Let h be s.a. Choose $\lambda > 0$ such that $sp(\lambda e + h) \subset [1, \infty)$. Let $h_n \to h$, where each h_n is s.a., and choose $0 < \varepsilon < 1$. We have $\rho(\lambda e + h) = b(\lambda e + h) = \lambda + b(h)$. By the "spectral continuity theorem" (see e. g. [15, Theorem 1]) for all n sufficiently large $sp(\lambda e + h_n) \subset (1 - \varepsilon, b(\lambda e + h) + \varepsilon)$. Also for all n sufficiently large $|\rho(\lambda e + h_n) - \rho(\lambda e + h)| < \varepsilon$ by Theorem 2.9. Since, for such n, $sp(\lambda e + h_n) \subset (0, \infty)$, then $\lambda + b(h_n) = \rho(\lambda e + h_n) \to \lambda + b(h)$. Therefore $b(h_n) \to b(h)$. A similar argument shows that $a(h_n) \to a(h)$.

2.11. COROLLARY. *For an Arens*-algebra B, N and P are closed sets.*

This follows directly from the continuity of the involution on B and Corollary 2.10. Likewise the set H^+ of all s.a. elements whose spectrum is non-negative is closed.

3. **Faithful*-representations.** Let B be a Banach algebra with an involution $x \to x^*$. Our aim here is to give necessary and sufficient conditions for B to possess a faithful*-representation. Our criterion (Theorem 3.4) is in terms of algebraic and linear space properties of B. A criterion of Kelley and Vaught [10] is largely topological in nature. To discuss this we first prove a simple lemma. We adopt the following notation. Let R_0 be the collection of all finite sums of elements of B of the form x^*x. Let $R = \{x \in H \mid$ there exists $y \in R_0$ such that $ty + (1-t)x \in R_0,\ 0 < t \leqq 1\}$. In the notation of Klee [11, p. 448], $R = \operatorname{lin} R_0$ (computed in the real linear space H, the union in H of R_0 and the points of H linearly accessible from R_0). Let P be the closure in B of R_0. If B has an identity e and the involution is continuous then H is closed, e is an interior point of R_0 [10] and $R = P$ [11, p. 448]. If B has no identity or if the involution is not assumed continuous we see no relation, in general, between R and P other than $R \subset P$.

3.1. LEMMA. *Suppose that B has a continuous involution $x \to x^*$ and an identity e. Then there is an equivalent Banach algebra norm $\| x \|_1$ where $\| x^* \|_1 = \| x \|_1$, $x \in B$, and $\| e \|_1 = 1$.*

We first introduce an equivalent norm $\| x \|_0$ in which $\| x^* \|_0 = \| x \|_0$, $x \in B$, by setting $\| x \|_0 = \max(\| x \|, \| x^* \|)$. Let $L_x(R_x)$ be the operator on B defined by left (right) multiplicaton by x; $L_x(y) = xy$ and $R_x(y) = yx$. Let $\| L_x \|$ be the norm of L_x as an operator on B where the norm $\| y \|_0$ is used for B. $\| R_x \|$ is defined in the same way. We set $\| x \|_1 = \max(\| L_x \|, \| R_x \|)$. Then $\| x + y \|_1 \leqq \| x \|_1 + \| y \|_1$ and $\| xy \|_1 \leqq \| x \|_1 \| y \|_1$. Clearly $\| x \|_1 \leqq \| x \|_0$. Moreover $\| L_x \| \geqq \| x \|_0 / \| e \|_0$ and the norms $\| x \|_0$ and $\| x \|_1$ are equivalent. Trivially $\| e \|_1 = 1$. Also

$$\| L_{x^*} \| = \sup_{\| y \|_0 = 1} \| x^* y \|_0 = \sup_{\| y \|_0 = 1} \| y^* x \|_0 = \| R_x \| .$$

Then $\| x^* \|_1 = \max(\| L_{x^*} \|, \| R_{x^*} \|) = \max(\| L_x \|, \| R_x \|) = \| x \|_1$.

In view of Lemma 3.1 the result [10, p. 51] of Kelley and Vaught in question may be expressed as follows.

3.2. THEOREM. *Let B be a Banach algebra with an identity and an involution $x \to x^*$. Then B has a faithful*-representation if and only if * is continuous and $P \cap (-P) = (0)$.*

As it stands this criterion breaks down if B has no identity. For let $B = C([0, 1])$ with the usual involution $x \to x^*$ and norm. Let B_0 be the algebra obtained from B by keeping the norm and involution but defining all products to be zero. Then* is still continuous and $P \cap (-P) = (0)$. But B_0 has no faithful*-representation, for otherwise B_0 would be semi-simple [16, p. 626].

As in [4] we call the involution $x \to x^*$ in B *regular* if, for h s.a., $\rho(h) = 0$ implies $h = 0$. By [4, Lemma 2.15]. * is regular if and only if every maximal commutative *-subalgebra of B is semi-simple. Also every maximal commutative*-subalgebra of B is closed [4, Lemma 2.13].

By a *positive linear functional* f on B we mean a linear functional such that $f(x^*x) \geqq 0$, $x \in B$. The functional f is not assumed to be continuous. If B has an identity then [13, p. 115], $f(h)$ is real for h s.a. and $f(x^*) = \overline{f(x^*)}$. Trivial examples show this to be false, in general. However, from the positivity of f, $f(x^*y)$ and $f(y^*x)$ are complex conjugates which is the fact really needed for the introduction of the inner product in Theorem 3.4.

3.3 LEMMA. *Let the involution on B be regular. Then*

(1) *a positive linear f satisfies the inequalities*

$$f(y^*hy) \leqq f(y^*y) \| h \|, y \in B, h \in H, \tag{3.1}$$

$$f(y^*x^*xy) \leqq f(y^*y) \| x^*x \|, x, y \in B, \tag{3.2}$$

(2) *if B has an identity e, any* $h \in H$, $\| e - h \| \leqq 1$ *has a* s.a. *square root and, moreover, any positive linear functional is continuous on H.*

Suppose first that B has an identity e, $\| e - h \| \leqq 1$, h s.a. In the course of the proof of [4, Theorem 2.16] it was shown that h has a s.a. square root. Next do not assume that B has an identity. Let B_1 be the Banach algebra obtained by adjoining an identity e to B. Consider the power series $(1 - t)^{1/2} = 1 - t/2 - t^2/8 \cdots$. Let $h \in B$, h s.a. and $\| h \| \leqq 1$. Then the expansion $-h/2 - h^2/8 - \cdots$ converges to an element $z \in B$. Let B_0 be a maximal abelian*-subalgebra of B containing h. As noted above, B_0 is a semi-simple Banach algebra. The involution is continuous on B_0 ([16, Corollary 6.3]). Therefore z is s.a. Also $(e + z)^2 = e - h$. Let $y \in B$ and set $k = y + zy$. Then $k^*k = (y^* + y^*z)(y + zy) = y^*(e + z)^2y = y^*y - y^*hy$. For any positive linear functional f on B, $f(k^*k) \geqq 0$ which yields (3.1). Formula (3.2) is a special case.

Suppose that B has an identity e. If we set $y = e$ in (3.1) we obtain $| f(h) | \leqq f(e) \| h \|$ which shows that f is continuous on H.

3.4. THEOREM. *B has a faithful*-representation if and only if* * *is regular and* $R \cap (-R) = (0)$.

Suppose that B has a faithful*-representation $x \rightarrow T_x$ as operators on a Hilbert space $\mathfrak{H}$. Let h be s.a. and $\rho(h) = 0$. Then $\rho(T_h) = 0$. As T_h is a s.a. operator on a Hilbert space, $T_h = 0$ and therefore $h = 0$. Thus the involution is regular. Let $x \in R \cap (-R)$ and let f be a positive linear functional on B. Then clearly $f(y) \geqq 0$, $y \in R_0$. From the definition of R there exists $y \in R_0$ such that $tf(y) + (1 - t)f(x) \geqq 0$, $0 < t \leqq 1$. It follows that $f(x) \geqq 0$ and hence $f(x) = 0$. Let $\xi \in \mathfrak{H}$ and set $f(x) = (T_x\xi, \xi)$. Then $(T_x\xi, \xi) = 0$ for all $\xi \in \mathfrak{H}$. Since T_x is a s.a. operator we see that $T_x = 0$ and $x = 0$.

Suppose now that* is regular and $R \cap (-R) = (0)$. We show first that the regularity of the involution makes available a general representation procedure of Gelfand and Neumark [13].

Let f be a positive linear functional on B. Let $I_f = \{x | f(x^*x) = 0\}$. I_f is a left ideal of B. Let π be the natural homomorphism of B onto B/I_f. Since $f(x^*y) = f(y^*x)$, $\mathfrak{H}'_f = B/I_f$ is a pre-Hilbert space if we define $(\pi(x), \pi(y)) = f(y^*x)$. As in [13, p. 120] we associate with $y \in B$ an operator A_y^f on $\mathfrak{H}'_f$ defined by $A_y^f[\pi(x)] = \pi(yx)$. Formula (3.2) yields

$$(3.3) \qquad || A_y^f[\pi(x)] ||^2 = f(x^*y^*yx) \leqq || y^*y || \, || \pi(x) ||^2 .$$

Thus A_y^f is a bounded operator with norm not exceeding $|| y^*y ||^{1/2}$. It may then be extended to T_y^f, a bounded operator on the completion $\mathfrak{H}_f$ of $\mathfrak{H}'_f$. The mapping $x \to T_x^f$ is a *-representation of B with kernel $\{y \in B \mid yx \in I_f$, for all $x \in B\} = K$. Note that $K^* = K$.

Now take the direct sum $\mathfrak{H}$ of the Hilbert spaces $\mathfrak{H}^f$ as f ranges over all positive linear functionals on B ([13, p. 95]). Since $|| T_y^f || \leqq || y^*y ||^{1/2}$ by (3.3) and this estimate is independent of f, the direct sum ([13, p. 113]) $x \to T_x$ of the representations $x \to T_x^f$ yields a*-representation of B as bounded operators on $\mathfrak{H}$ with kernel $\{y \in B \mid yx \in \cap I_f$, for all $x \in B\}$. If B has an identity, the kernel is the reducing ideal of B ([13, p. 130]), namely $\cap I_f$.

Supppose first that B has an identity e. The set R_0 has the property that $x, y \in R_0, \lambda, \mu \geqq 0$ imply $\lambda x + \mu y \in R_0$. By Lemma 3.3, $R_0 \supset \{x \in H \mid || e - x || \leqq 1\}$. Thus e is an interior point of R_0. By the theory of convex sets in normed linear spaces, R is the closure in H of R_0 and R is a closed cone in H ([11, p. 448]).

Let f be a positive linear functional on B. By Lemma 3.2, f is continuous on H. Also $f(w) \geqq 0$, $w \in R$. Let H' be the conjugate space of H and $G = \{g \in H' \mid g(w) \geqq 0, w \in R\}$. It is easy to see ([10, p. 48]) that G, the dual cone of R, is the set of linear functionals on H which are the restrictions to H of positive linear functional on B. There is no loss generality in assuming that $|| e || = 1$. Let $x \in H$. By [10, Lemma 1.3], dist $(-x, R) = \sup \{g(x) \mid g \in G, g(e) \leqq 1\}$.

We show that $R \cap (-R) = H \cap (\cap I_f)$. Let $y \in H$, $y \in \cap I_f$. For any fixed f, $T_y^f = 0$ and $(T_y^f \xi, \xi) = 0$, $\xi \in \mathfrak{H}_f$. Then $(\pi(yx), \pi(x)) = 0$ for all $x \in B$ in the notation used above. Therefore $f(x^*yx) = 0$, $x \in B$. Setting $x = e$ we see that $f(y) = 0$. Then by the distance formula, $-y \in R$. Likewise $y \in R$. Suppose conversely that $y \in R \cap (-R)$. It is easy to see that for each $z \in B$, $z^*R_0z \subset R_0$. Therefore $z^*Rz \subset R$. Hence $z^*yz \in R \cap (-R)$, $z \in B$. From the distance formula, $\sup \{f(z^*yz) \mid f \text{ positive}, f(e) \leqq 1\} = 0 = \sup \{f(-z^*yz) \mid f \text{ positive}, f(e) \leqq 1\}$. Hence $f(z^*yz) = 0$ for each positive linear functional. Then $(T_y^f \pi(z), \pi(z)) = 0$ for all z whence $T_y^f = 0$. Therefore $T_y = 0$ and $y \in H \cap (\cap I_f)$.

This proves the theorem in case B has an identity. Suppose that B has no identity. Let B_1 be the algebra obtained by adjoining an identity e to B. We extend the involution to B_1 by setting $(\lambda e + x)^* = \bar{\lambda} e + x^*$. The involution on B_1 is regular [4, Lemma 2.14]. Let R_0' and R' be the sets R_0 and R respectively computed for the algebra B_1. By the above it is sufficient to show that $R \cap (-R) = (0)$ implies $R' \cap (-R') = (0)$. Suppose that $R \cap (-R) = (0)$.

Let $x, y \in B$. Then $y^*(\lambda e + x)^*(\lambda e + x)y = (\lambda y + xy)^*(\lambda y + xy)$. This shows that $y^*R_0'y \subset R_0$ which implies $y^*R'y \subset R$. Note also that B is semi-simple [18, Lemma 3.5] which implies that $zB = (0)$, or $Bz = (0)$, $z \in B$, can hold only for $z = 0$.

Suppose that $\lambda e + x \in R' \cap (-R')$ where $x \in B$ and λ is a scalar. We derive a contradiction from $\lambda \neq 0$. For every $y \in B$, $y^*(\lambda e + x)y \in R \cap (-R)$. Setting $u = -x/\lambda$ we have $y^*(e-u)y = 0$ or $y^*y = y^*uy$ for all $y \in B$. Then

$$h^2 = huh,\ h \text{ s.a.} \tag{3.4}$$

Let h_1 and h_2 be s.a. Then $(h_1 + h_2)^2 = (h_1 + h_2)u(h_1 + h_2)$. From (3.4) we obtain

$$h_1h_2 + h_2h_1 = h_1uh_2 + h_2uh_1 . \tag{3.5}$$

Also $(h_1 - ih_2)(h_1 + ih_2) = (h_1 - ih_2)u(h_1 + ih_2)$ From (3.4) we get

$$h_2h_1 - h_1h_2 = h_2uh_1 - h_1uh_2 \tag{3.6}$$

From (3.5) and (3.6) we see that $h_1h_2 = h_1uh_2$. Consequently for h_k s.a., $k = 1, 2, 3, 4$, we see that $(h_1 + ih_2)(h_3 + ih_4) = (h_1 + ih_2)u(h_3 + ih_4)$. In other words

$$zw = zuw,\ z, w \in B . \tag{3.7}$$

From (3.7) $(z - zu)w = 0$ for all $w \in B$ so that $z = zu$ for each z. Hence u is a right identity for B. Likewise from $z(w - uw) = 0$ for all $z \in B$ we see that u is an identity for B. But this is impossible since we are considering the case where B has no identity.

We now have $x \in R' \cap (-R')$. Then $y^*xy = 0$ for all $y \in B$. Therefore $hxh = 0$, h s.a. Also for h_k s.a., $k = 1, 2$, $(h_1 + h_2)x(h_1 + h_2) = 0$ so that $h_2xh_1 + h_1xh_2 = 0$. Also $(h_1 - ih_2)x(h_1 + ih_2) = 0$ so that $h_1xh_2 - h_2xh_1 = 0$. Therefore $h_1xh_2 = 0$. It follows that $zxw = 0$ for all $z, w \in B$. This implies that $x = 0$ and completes the proof.

4. Preliminary ring theory. Let R be a semi-simple ring with minimal one-sided ideals. For a subset A of R let $\mathfrak{L}(A) = \{x \in R \mid xA = (0)\}$ and $\mathfrak{R}(A) = \{x \in R \mid Ax = (0)\}$. Consider a two-sided I of R. If $x \in R(I)$, $y \in R$, $z \in I$ then $zy \in I$, $z(yx) = 0$ so that $\mathfrak{R}(I)$ is a two-sided ideal of R. Therefore $\mathfrak{R}(I)I$ is an ideal. But $[\mathfrak{R}(I)I]^2 = (0)$. Thus, by semi-simplicity, $\mathfrak{R}(I)I = (0)$

and $\mathfrak{R}(I) \subset \mathfrak{L}(I)$. Likewise we have $\mathfrak{L}(I) \subset \mathfrak{R}(I)$ and thus $\mathfrak{R}(I) = \mathfrak{L}(I)$. Let S be the socle [5, p. 64] of R. This is the algebraic sum of the minimal left (right) ideals of R. S is a two-sided ideal. Therefore $\mathfrak{L}(S) = \mathfrak{R}(S)$. This set we denote by $S^{\perp}$. Note that $S \cap S^{\perp} = (0)$.

We call an idempotent e of R a *minimal idempotent* if $e R$ is a minimal right ideal.

4.1. LEMMA. (a) *Let I be a left (right) ideal of R, $I \neq (0)$. Then I contains no minimal left (right) ideal of R if and only if $I \subset S^{\perp}$.*

(b) *$R/S^{\perp}$ is semi-simple. If S_0 is the socle of $R/S^{\perp}$ then $S_0^{\perp} = (0)$.*

Let $I \neq (0)$ be a left ideal of R. Suppose that $I \subset S^{\perp}$. Then I cannot contain a minamal left ideal J of R for any such J would be contained in $S \cap S^{\perp}$. Next suppose that $I \not\subset S^{\perp}$. We must show that I contains a minimal left ideal of R. There exists a minimal idempotent e such that $e I \neq (0)$. Choose $u \in I$ such that $eu \neq 0$. By semi-simplicity and the minimality of eR, $eR = euR$. Thus there exists $z \in R$ such that $euz = e$. Since $(euz)^2 = e$, we have $j \neq 0$ where $j = zeu$. Note that $j^2 = j$. As $u \in I$ we have $Rj \subset I$. To see that Rj is the desired minimal ideal it is sufficient to see that jRj is a division ring [5, p. 65].

Note that $jz = zeuz = ze \neq 0$. Then $Rze = Re$ so that there exists $v \in R$ where $vze = e$. Then $vj = vzeu = eu$ and $vjz = e$.

We assert that $jx_1j = jx_2j$ if and only if $eux_1ze = eux_2ze$. · For if $jx_1j = jx_2j$, multiply on the left by v and on the right by z and use the relations $vj = eu$ and $jz = ze$. If $eux_1ze = eux_2ze$ multiply on the left by z and on the right by u and use $zeu = j$.

Therefore the mapping $\tau : \tau(jxj) = euxze$ is a well-defined one-to-one mapping of jRj into eRe. The mapping is onto. For let $ewe \in eRe$. Then $ewe = euzwvze = \tau(jzwvj)$. τ is clearly additive. But also $\tau[(jxj)(jyj)] = \tau(jxjyj) = euxjyze = (euxze)(euyze) = \tau(jxj)\tau(jyj)$. Therefore τ is a ring isomorphism of jRj onto eRe. Since eRe is a division ring so is jRj.

Let J be the radical of $R/S^{\perp}$ and π be the natural homomorphism of R onto $R/S^{\perp}$. Suppose that $J \neq 0$. Then $\pi^{-1}(J) \supset S^{\perp}$ and $\pi^{-1}(J) \neq S^{\perp}$. By (a), $\pi^{-1}(J)$ contains a minimal idempotent e of R. We then have $\pi(e) \in J$, $\pi(e) \neq 0$. This is impossible since the radical of a ring contains no non-zero idempotents.

Let S_0 be the socle of $R/S^{\perp}$ and e be a minimal idempotent of R. Clearly $\pi(e) \neq 0$ and π is one-to-one on eRe. Then $\pi(e)\pi(R)\pi(e)$ is a division ring so that, since $R/S^{\perp}$ is semi-simple, $\pi(e) \in S_0$. Let $\pi(x) \in S_0^{\perp}$. Then $\pi(ex) = 0$ so that $ex \in S^{\perp} \cap S = (0)$. Hence $x \in S^{\perp}$ and $\pi(x) = 0$.

The following result is due to Rickart [17, Lemma 2.1.]:

4.2. LEMMA. *Let A be any ring. Let $x \to x^*$ be a mapping of A onto A such that $x^{**} = x$, $(xy)^* = y^*x^*$ and $xx^* = 0$ implies $x = 0$. Then any* *minimal right (left) ideal I of A can be written in the form $I=eA(I=Ae)$ where $e^2 = e \neq 0, e^* = e$.*

We improve this result by relaxing the conditions on $x \to x^*$ but at the expense of assuming the ring to be semi-simple.

4.3. LEMMA. *Let R be semi-simple with minimal one-sided ideals. Let $x \to x^*$ be a mapping of R onto R satisfying $x^{**} = x$ and $(xy)^* = y^*x^*$. Then the following statements are equivalent.*

(1) *Every minimal right ideal is generated by a* s.a. *idempotent.*

(2) *Every minimal left ideal is generated by a* s.a. *idempotent.*

(3) *$jj^* \neq 0$ for each minimal idempotent j of R.*

(4) *$xx^* = 0$ implies $x \in S^\perp$*

We say that the idempotent e is s.a. if $e^* = e$. Note that $x \to x^*$ is one-to-one and $0^*=0$. As a preliminary we show that j^* is a minimal idempotent if j^* is a minimal idempotent. The ideal $I = jR$ is a minimal right ideal. Then $I^* = Rj^*$ is a left ideal $\neq (0)$. Suppose $I^* \supset K \neq (0)$, $I^* \neq K$ where K is a left ideal of R. By semi-simplicity there exists $x \in K$ such that $x^2 \neq 0$. Then $I^* \supset Rx \neq (0)$, $I^* \neq Rx$. This implies that $I \supset x^*R \neq (0)$, $I \neq x^*R$. This is impossible. Therefore I^* is a minimal left ideal and j^* is a minimal idempotent. It is clear from this argument that (1) and (2) imply each other.

Assume (1). Let j be a minimal idempotent, $I = Rj$ a minimal left ideal. We can write $I = Re$ where e is a s.a. idempotent. Then for some $v \in R$, $vj = e$. But $e = ee^* = vjj^*v$. Therefore $jj^* \neq 0$. Thus (1) implies (3).

Assume (3). Suppose that $xx^* = 0, x \neq 0$. Let $I = Rx$. Then $I \neq (0)$. Suppose that I contains a minimal left ideal Rj of R where j is a minimal idempotent. We can write $j = yx, y \in R$. Then $0 \neq jj^* = yxx^*y^* = 0$. This shows that I contains no minimal left ideal of R. By Lemma 4.1, $I \subset S^\perp$. Then for any minimal idempotent e, $0 = e(ex)$ and $x \in S^\perp$. Thus (3) implies (4).

Assume (4). If j is a minimal idempotent and $jj^* = 0$ then $j \in S^\perp$. But $j \in S$ and $S \cap S^\perp = (0)$. This shows that (4) implies (3).

Assume (3). Let j be a minimal idempotent, $I = jR$. Since $jj^* \neq 0$, $jj^*R = I$. There exists $u \in R, jj^*u = j$. As noted above j^* is a minimal idempotent. By (3), $0 \neq j^*j$. Then $0 \neq (u^*jj^*)(jj^*u) = u^*(jj^*)^2u$. Therefore $(jj^*)^2 \neq 0$. Set $h = jj^*$. Since I is minimal, $I = hI$. As in the proof of [17, Lemma 2.1] there exists $u \in I$ such that $h = hu$. Set $e = uu^*$. As in that proof, e is a s.a. idempotent and it remains only to check that $e \neq 0$ to obtain (2) from (3). If $e = 0$ then $0 = uu^* = huu^*h = h^2$ which is impossible.

5. Normed algebras with minimal ideals.

We are concerned here with*-representations of semi-simple normed algebras B with an involution where B has minimal one-sided ideals. B may be incomplete.

5.1. LEMMA. *Let B be a complex semi-simple normed algebra with minimal one-sided ideals. Let e_1, e_2 be minimal idempotents of B. Then the following statements are equivalent.*

(1) $e_1Be_2 \neq (0)$.

(2) $e_2Be_1 \neq (0)$,

(3) e_1Be_2 *is one-dimensional.*

(4) e_2Be_1 *is one-dimensional.*

Suppose (1). There exists $u \in B$, $e_1ue_2 \neq 0$. Since $e_1ue_2B = e_1B$, there exists $v \in B$ where $e_1ue_2v = e_1$. Then $e_2ve_1 \neq 0$ and (1) implies (2). Let $E = \{\lambda e_2ve_1 \mid \lambda \text{ complex}\}$. Clearly $e_2Be_1 \supset E$. Let $e_2xe_1 \in e_2Be_1$. Then $e_2xe_1 = e_2x(e_1ue_2ve_1) = (e_2xe_1ue_2)e_2ve_2$, a scalar multiple of e_2 by the Gelfand-Mazur Theorem. Thus (1) implies (4). The remainder of the argument is trivial.

For the remainder of § 5, B denotes a semi-simple complex normed algebra with an involution and with minimal one-sided ideals.

5.2. THEOREM. *The following statements concerning B are equivalent.*

(1) *Every minimal one-sided ideal is generated by a* s.a. *idempotent.*

(2) *There exists a*-representation with kernel $S^\perp$.*

(3) *There exists a*representation with kernel contained in $S^\perp$.*

(4) $j - j^*$ *is quasi-regular for every minimal idempotent j.*

(5) $jBj^* \neq (0)$ *for every minimal idempotent j and $xx^* = 0$ implies $x^*x \in S^\perp$, $x \in B$.*

Suppose that (1) holds. Let Q be the set of all s.a. minimal idempotents of B and let $j \in Q$. By the Gelfand-Mazur Theorem, $jBj = \{\lambda j \mid \lambda \text{ complex}\}$. Suppose $jx^*xj = \lambda j$. Taking adjoints, $\lambda = \bar{\lambda}$ so λ is real. We show that $jx^*xj = -j$ is impossible. For suppose $jx^*xj = -j$. Now $jxj = \alpha j$ for some scalar $\alpha = a + bi$, where a, b are real. Set $c = a + (a^2 + 1)^{1/2}$. By the use of $jx^*xj = -j$ one obtains $(jx^* - cj)(jx^* - cj)^* = 0$. From Lemma 4.3 we have $jx^* - cj = 0$. Then $(a - bi)j = jx^*j = cj$. It follows that $c = a$ and $b = 0$. This is impossible.

For $j \in Q$ we define the functional $f_j(x)$ on B by the rule $f_j(x)j = jxj$. By the above, $f_j(x^*x) \geqq 0$, $x \in B$, $x \in B$ and $f_j(x^*) = \overline{f_j(x)}$. The functional f_j is a positive linear functional on B and is continuous on B.

The following inequality of Kaplansky [9, p. 55] is then available.

$$f_j(y^*x^*xy) \leqq \nu(x^*x)f_j(y^*y),\ x, y \in B\ , \tag{5.1}$$

where $\nu(x^*x) = \lim \| (x^*x)^n \|^{1/n}$. Let $I_j = \{x \mid f_j(x^*x) = 0\}$. Let π be the natural homomorphism of B onto B/I_j. The definition $(\pi(x), \pi(y)) = f_j(y^*x)$ makes B/I_j a pre-Hilbert space. Let $\mathfrak{H}_j$ be its completion. See the discussion of the Gelfand-Neumark procedure in § 3. To each $y \in B$ we correspond the operator A_y^j defined by $A_y^j[\pi(x)] = \pi(yx)$. Then

$$\| A_y^j[\pi(x)] \|^2 = f_j(x^*y^*yx) \leqq \nu(y^*y) \| \pi(x) \|^2$$

by (5.1). Thus A_y^j can be extended to a bounded linear operator T_y^j on $\mathfrak{H}_j$, and the mapping $y \to T_y^j$ is a*-representation of B.

Since $\| T_y^j \| \leqq \nu(y^*y)^{1/2}$ and the estimate is independent of $j \in Q$ we can take the direct sum $\mathfrak{H}$ of the Hilbert spaces $\mathfrak{H}_j$, $j \in Q$ and the direct sum $x \to T_x$ of the representations $x \to T_x^j$. This gives a*-representation of B with kernel K where

$$K = \{x \in B \mid xy \in \bigcap_{j \in Q} I_j, \text{ for all } y \in B\} .$$

We show that $K = S^\perp$.

It is clear that $S^* = S$ and therefore $(S^\perp)^* = S^\perp$. Using this and Lemma 4.3 we obtain the following chain of equivalences: $x \varepsilon \cap I_j \leftrightarrow jx^*xj = 0$, all $j \in Q \leftrightarrow jx^* \in S^\perp$, all $j \in Q \leftrightarrow jx^* = 0$, all $j \in Q \leftrightarrow x^* \in S^\perp \leftrightarrow x \in S^\perp$. Therefore $\cap I_j = S^\perp$. Thus $K = \{x \mid xy \in S^\perp, \text{ all } y \in B\}$. If $x \in K$ then $xj \in S^\perp \cap S = (0)$ for all $j \in Q$ and $x \in S^\perp$. Clearly $S^\perp \subset K$. Therefore $K = S^\perp$. Hence (1) implies (2). Clearly (2) implies (3).

Assume (3) and let φ be a*-representation whose kernel $\subset S^\perp$. Let j be a minimal idempotent of B. Let A be the subalgebra of B generated by j and j^*. By the Gelfand-Mazur Theorem, $jj^*j = \lambda j$ for some scalar λ. Thus A is the linear space spanned by j, j^*, jj^* and j^*j. A is finite-dimensional and $A \subset S$. Since $S \cap S^\perp = (0)$, φ is one-to-one on A. Note that $A = A^*$. Let E be the B^*-algebra obtained by taking the closure in the operator algebra on the appropriate Hilbert space of $\varphi(B)$. Clearly $\varphi(A)$ is a closed*-subalgebra of E. The element $\varphi(j - j^*)$ is a skew element of E and therefore quasi-regular in E. By [8, Theorem 4.2] its quasi-inverse in E already lies in $\varphi(A)$. As φ is one-to-one on A, $j - j^*$ has a quasi-inverse in A. Thus (3) implies (4).

Assume (4). Let j be a minimal idempotent of B. There exists $u \in B$ such that $j - j^* + u - (j - j^*)u = 0$. If $jj^* = 0$ then left multiplication by j gives $j = 0$ which is impossible. Therefore $jj^* \neq 0$. By Lemma 4.3, we see that (4) implies (1). Clearly (1) implies (5) by Lemma 4.3. Assume (5). Let j be a minimal idempotent of B. If $j^*j = 0$ then $0 = x^*j^*jx = (jx)^*(jx)$. Also $jxx^*j^* \in S^\perp \cap S = (0)$ for all $x \in B$. Since $jBj^* \neq (0)$, jBj^* is one-dimensional by Lemma 5.1. Hence there exists $u \neq 0$ in B and a linear functional $f(x)$ on B such that $jxj^* = f(x)u$. Then $f(xx^*) = 0$ for all $x \in B$. Expanding $0 = f[(x + y)(x + y)^*] = f[(x + iy)(x + iy)^*]$ we see that $f(xy^*) = 0$ for all $x, y \in B$. Hence f vanishes on B^2. Take any $z \in B$. We have $f(jz)=0$ or $jzj^*=0$. Thus $jBj^*=(0)$ which is impossible. Therefore $j^*j \neq 0$. By Lemma 4.3, (5) implies (1).

Algebras to which Theorem 5.2 can be applied most easily are those for which $S^{\perp} = (0)$. Examples are semi-simple annihilator algebras studied by Bonsall and Goldie [3] and primitive algebras (Corollary 5.4).

5.3. COROLLARY. *If B is an Arens*-algebra with non-zero socle then* $N \subset S^{\perp}$.

Let $x_0 \in N$, $sp\,(x_0x_0^*) \subset (-\infty, 0]$. Then we can write $x_0x_0^* = -h^2$ where h is s.a. The ideal $S^{\perp}$ is closed and self-adjoint. Let π be the natural homomorphism of B onto $B/S^{\perp}$. An involution can be defined in $B/S^{\perp}$ by the rule $[\pi(x)]^* = \pi(x^*)$. Since B is semi-simple, $B/S^{\perp}$ has non-zero socle. Let $\pi(x)$ be a minimal idempotent of $B/S^{\perp}$. Then $[\pi(x)]^* - \pi(x) = \pi(x^* - x)$ is quasi-regular in $B/S^{\perp}$ since $x^* - x$ is quasi-regular in B. By Theorem 5.2 and Lemma 4.1, $B/S^{\perp}$ has a faithful*-representation. Then, by Theorem 3.4, $\pi(x_0x_0^*) = 0 = \pi(h^2)$. Therefore $x_0x_0^* \in S^{\perp}$ and $(jx_0)(jx_0)^* = 0$ for each minimal idempotent j of B. Therefore $jx_0 = 0$ for all such j and $x_0 \in S^{\perp}$.

We call the involution $x \to x^*$ *proper* if $xx^* = 0$ implies $x = 0$. We call the involution *quasi-proper* if $xx^* = 0$ implies $x^*x = 0$. Not every involution is quasi-proper. For example let B be all 2×2 matrices with the involution defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} \bar{a} & -\bar{c} \\ -\bar{b} & \bar{d} \end{pmatrix}.$$

To see that this is not quasi-proper choose x as

$$\begin{pmatrix} 1 & i \\ 0 & 0 \end{pmatrix}.$$

Every proper involution is quasi-proper but the converse is false. Consider, for example $B = C([0, 1])$ and set $x^*(t) = \overline{x(1-t)}$.

5.4. COROLLARY. *Let B be primitive with non-zero socle. Then the following statements are equivalent.*

(1) *The involution* is proper.*

(2) *The involution* is quasi-proper.*

(3) *There exists a faithful*-representation of B.*

Suppose that $S^{\perp} \neq (0)$. Then by [5, p. 75], $S \subset S^{\perp}$. Since $S \cap S^{\perp} = (0)$ this is impossible. Therefore $S^{\perp} = (0)$. Assume (2). Let j be a minimal idempotent of B. Then $jBj^* \neq (0)$ (see the prooof of [16, Theorem 4.4]) and, consequently (5) of Theorem 5.2 is satisfied. Then by Theorem 5.2, (2) implies (3); the remainder of the proof is obvious.

The equivalence of (1) and (3) was noted by Rickart [17, Theorem 3.5]. By Lemma 4.3 and Theorem 5.2 this equivalence of (1) and (3) holds for any B for which $S^{\perp} = (0)$.

If B is complete the following statements hold. (1) Any*-representation of B is continuous [16, Theorem 6.2]. (2) If B has a faithful*-representation then the involution is continuous [16, Lemma 5.3]. We show that both these statements can be false for B incomplete. Our discussion is based on work of Kakutani and Mackey [6, p. 56] (see also [7] for the complex case). Let $\mathfrak{X}$ be an infinite-dimensional complex Hilbert space, $(x, x)^{1/2} = \| x \|$. Let $||| x |||$ be any other norm on $\mathfrak{X}$ such that $|||x||| \leqq \| x \|$, $x \in \mathfrak{X}$. Let $\mathfrak{X}_1 = \{y \in \mathfrak{X} \mid (x, y)$ is continuous on $\mathfrak{X}$ in the norm $|||x|||\}$ and endow $\mathfrak{X}_1$ with the norm $||| x |||$. Then [6, p. 56] a linear functional $f(x)$ on $\mathfrak{X}_1$ has the form $f(x) = (x, y)$. Moreover $\mathfrak{X}_1$ is dense in $\mathfrak{X}$ in both norms. If there exists $c > 0$ such that $\| x \| \leqq c \, ||| x |||$, $x \in \mathfrak{X}_1$ then $\mathfrak{X} = \mathfrak{X}_1$ and $\mathfrak{X}_1$ is complete.

Let $\mathfrak{E}(\mathfrak{X}_1)$ be the normed algebra of all bounded linear operators on $\mathfrak{X}_1$. As shown in [6, p. 56], $\mathfrak{E}(\mathfrak{X}_1)$ has an involution $T \to T^*$ where $(T(x), y) = (x, T^*(y))$, $x, y \in \mathfrak{X}_1$. In these terms we show the following.

5.5. THEOREM. *The following statements are equivalent.*

(1) *$\mathfrak{X}_1$ is complete.*

(2) *The involution in $\mathfrak{E}(\mathfrak{X}_1)$ is continuous.*

(3) *The faithful*-representation of Theorem* 5.2 *for $\mathfrak{E}(\mathfrak{X}_1)$ is continuous.*

As already noted (1) implies (2) and (3). Assume (2) and let M be the norm of the involution. By [2] any minimal idempotent of $\mathfrak{E}(\mathfrak{X}_1)$ is one-dimensional and the operator J defined by the rule $J(x) = (x, u)u$ where $(u, u) = 1$ is a minimal idempotent. Since $(J(x), y) = (x, u)(u, y) = (x, J(y))$ we have $J = J^*$. The functional f defined by $f(U)J = JUJ$ is a continuous positive linear functional on $\mathfrak{E}(\mathfrak{X}_1)$. For $z \in \mathfrak{X}_1$ define the operator W_z by the rule $W_z(x) = (x, u)z$. Then we can write the norm of W_z as $C \, ||| z |||$ where C is independent of z. A simple computation gives $JW_z^* W_z J = (z, z)J$. By formula (5.1), where $\| U \|$ denotes the norm in $\mathfrak{E}(\mathfrak{X}_1)$,

$$\| z \|^2 = (z, z) \leqq \nu(W_z^* W_z) \leqq \| W_z^* W_z \| \leqq C^2 M \, ||| z |||^2 .$$

This shows that $\mathfrak{X}_1$ is complete.

Assume (3) and let N be the norm of the faithful*-representation. Let $I_f = \{U \in \mathfrak{E}(\mathfrak{X}_1) \mid f(U^*U) = 0\}$, π be the natural homomorphism of $\mathfrak{E}(\mathfrak{X}_1)$ onto $\mathfrak{E}(\mathfrak{X}_1)/I_f$ and $(\xi, \eta)_f$ be the inner product for the pre-Hilbert space $\mathfrak{E}(\mathfrak{X}_1)/I_f$. Let $V \to T_V^f$ be the partial*-representation induced by f. Its norm cannot exceed N. Now $(\pi(J), \pi(J))_f = 1$ and

$$N^2 \| U \|^2 \geqq \| T_U^f[\pi(J)] \|^2 = (UJ, UJ)_f = f(JU^*UJ) = f(U^*U) .$$

Applying this formula to $U = W_z$ we obtain $N^2 C^2 \, ||| z |||^2 \geqq (z, z)$ and again $\mathfrak{X}_1$ is complete.

A specific example is suggested in [6, p. 57]. Let $\mathfrak{X} = l^2$, $||| \{x_n\} ||| = \sup | x_n |$. An easy computation gives $\mathfrak{X}_1 = l^2 \cap l^1$ in the sup norm. Here the involution and*-representation are therefore not continuous.

6. Involutions on $\mathfrak{E}(\mathfrak{H})$. Let $\mathfrak{H}$ be a Hilbert space and $\mathfrak{E}(\mathfrak{H})$ the B^*-

algebra of all bounded linear operators on $\mathfrak{H}$. We determine in Theorem 6.2 all the involutions on $\mathfrak{E}(\mathfrak{H})$ for which there are faithful adjoint-preserving representations.

6.1. LEMMA. *Let*$^\#$ *be any involution on* $\mathfrak{E}(\mathfrak{H})$. *Then there exists an invertible* s.a. *element* U *in* $\mathfrak{E}(\mathfrak{H})$ *such that* $T^\# = U^{-1}T^*U$ *for all* $T \in \mathfrak{E}(\mathfrak{H})$. *Conversely any such mapping is an involution.*

The mapping $T \to T^{\#*}$, $T \in \mathfrak{E}(\mathfrak{H})$, is an automorphism of $\mathfrak{E}(\mathfrak{H})$. Thus there exists $V \in \mathfrak{E}(\mathfrak{H})$ where $T^{\#*} = VTV^{-1}$, $T \in \mathfrak{E}(\mathfrak{H})$. Set $U = V^*$. Then $T^\# = U^{-1}T^*U$. Since $T^{\#\#} = T$, $T = (U^{-1}T^*U)^\# = U^{-1}U^*T(U^*)^{-1}U$. Thus $U^{-1}U^*$ lies in the center of $\mathfrak{E}(\mathfrak{H})$. Consequently $U = \lambda U^*$ for some scalar λ. Since $U^*U = |\lambda|^2 U^*U$ we see that $|\lambda| = 1$. Set $\lambda = \exp(i\theta)$ and $W = \exp(-i\theta/2)U$. Then $W^* = W$ and $T^\# = W^{-1}T^*W$, $T \in \mathfrak{E}(\mathfrak{H})$. The remaining statement is easily verified.

6.2. THEOREM. *An involution* $T \to T^\#$ *on* $\mathfrak{E}(\mathfrak{H})$ *is proper if and only if it can be expressed in the form* $T^\# = U^{-1}T^*U$, $U \in \mathfrak{E}(\mathfrak{H})$ where U is s.a. and $sp(U) \subset (0, \infty)$.

If $T \to T^\#$ is a proper involution then (see [7]) an inner product can be defined in $\mathfrak{H}$ in terms of which $T^\#$ is the adjoint of T. Hence the proper involutions are those for which there is an adjoint preserving faithful representation.

Let W be a one-dimensional operator, $W(x) = (x, z)w$ with $w \neq 0$, $z \neq 0$. Then $W^*(x) = (x, w)z$. By Lemma 6.1 we can write $T^\# = U^{-1}T^*U$, $T \in \mathfrak{E}(\mathfrak{H})$, where U is s.a. Then $0 \neq W^\# W = U^{-1}W^*UW$. Hence $0 \neq W^*UW$. But $W^*UW(x) = (x, z)W^*U(w) = (x, z)(U(w), w)z$. Therefore $(U(w), w) \neq 0$ for an arbitrary non-zero $w \in \mathfrak{H}$. Hence $(U(w), w) \neq 0$ for an arbitrary non-zero $w \in H$. Hence $(U(w), w)$ has a constant sign and, by changing to $-U$ if necessary, we may suppose that $(U, w), w) \geqq 0$, $w \in \mathfrak{H}$. Then we can write $U = V^2$ where V is s.a. in $\mathfrak{E}(\mathfrak{H})$.

Suppose conversely that $T^\# = V^{-2}T^*V^2$, $T \in \mathfrak{E}(\mathfrak{H})$ where V is s.a. Then $TT^\# = (TV^{-1})(TV^{-1})^*V^2$. Thus $TT^\# = 0$ implies that $TV^{-1} = 0$ and that $T = 0$.

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