POLARITY AND DUALITY

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One of the most frequently encountered situations in mathematics is the existence of a Galois correspondence between two partially ordered systems. An abstract formulation of this concept has been given by Garrett Birkhoff [1, pp. 54-57] and Ore [5], in the following terms.

DEFINITION. Let A and B be two partially ordered sets, and let $\# : A \rightarrow B$ and $+ : A \rightarrow B$ be two mappings such that:

(i) if $p_1 \leq p_2$ in A, then $p_2^\# \leq p_1^\#$ in B;

(ii) if $q_1 \leq q_2$ in B, then $q_2^+ \leq q_1^+$ in A; and

(iii) for any $p \in A$ and any $q \in B$, $p \leq q^{\#+}$ and $q \leq q^{+\#}$.

Then the mappings $\#$ and $+$ are said to define a Galois correspondence between A and B.

The number of ways in which a Galois correspondence can arise is quite large, and most of them are very well known instances of what is usually called "duality theory". Perhaps the commonest source is the existence of a relation between the elements of two sets. Birkhoff has described this procedure as follows. Let S and T be two sets, and let ρ be a relation from S to T. That is, ρ is a subset of the cartesian product $S \times T$; we write $s\rho t$ to denote $(s, t) \in \rho$, as is customary. For any subset $S_1 \subset S$, define $S_1^\#$ to be the set of all those elements $t \in T$ such that $s_1 \rho t$ for all $s_1 \in S_1$. Similarly, for any subset $T_1 \subset T$, denote by T_1^+ the set of all those $s \in S$ such that $s \rho t_1$ for all $t_1 \in T_1$. Then the mapping $\# : A \rightarrow B$ and $+ : B \rightarrow A$ define a Galois correspondence between the Boolean algebra A of all subsets of S and the Boolean algebra B of all subsets of T.

This example has some special features which are not available for general partially ordered systems. If ϕ denotes the empty set of S, then $\phi^\# = T$, and if S_1 and S_2 are any two subsets of S, then $(S_1 \cup S_2)^\# = S_1^\# \cup S_2^\#$. A similar result holds for the other mapping $+$. This is due to the fact that Boolean algebras are special cases of lattices which satisfy the conditions of the following result.

LEMMA. *Let A and B be lattices, each having a greatest element* 1 *and a least element* 0, *and let $\# : A \rightarrow B$ and $+ : B \rightarrow A$ define a Galois correspondence between A and B. Then $0^\# = 1$, $0^+ = 1$, and $(p_1 \vee p_2)^\# = p_1^\# \wedge p_2^\#$, $(q_1 \vee q_2)^+ = q_1^+ \wedge q_2^+$, for any $p_1, p_2 \in A$ and $q_1, q_2 \in B$.*

This result is well known and, in any event, easily proved. (The

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symbols 0, 1 denote ambiguously the least and greatest elements of both A and B.)

This suggests that these two conditons might perhaps be taken as more primitive embodiments of general duality concepts. In so doing, of course, one loses the full generality of partially ordered systems. The purpose of this note is to consider mappings of Boolean algebras which have these two properties. It will be shown that, in this case, the method of Birkhoff described above is not only sufficient for constructing a Galois corresponcence but is also necessary.

To be precise, we introduce the following terminolgy.

DEFINITION. Let A and B be two Boolean algebras. By a polarity of A into B, we shall mean a mapping $\sharp$ of A into B satisfying the two requirements: (i) $0^\sharp = 1$, and (ii) for any $p, q \in A$, $(p \vee q)^\sharp = p^\sharp \wedge q^\sharp$.

Some recent developments in the duality theory of Boolean algebras may be used to characterize completely such mappings. It may be well to summarize these developments.

If A is any Boolean algebra, its dual space X is a Boolean space—that is, a compact, totally disconnected, Hausdorff space. The algebra A is isomorphic with the Boolean algebra $\Phi(X)$ of all continuous functions from the space X to the (discrete) two-element Boolean algebra Φ. The algebra A will, in fact, be identified with the algebra $\Phi(X)$, so that each element $p \in A$ is a continuous function from X to Φ, and each such function is an element of A. This relationship between Boolean algebras A and Boolean spaces X is the basis of the duality theory of M. H. Stone [6, 7].

Let A and B be two Boolean algebras, with dual spaces X and Y respectively, so that $A = \Phi(X)$ and $B = \Phi(Y)$. By a hemimorphism α of A into B is meant a mapping $\alpha : AB$ such that (i) $\alpha 0 = 0$, and (ii) $\alpha(p \vee q) = \alpha p \vee \alpha q$, for any $p, q \in A$. Every hemimorphism α of A into B defines a relation, denoted by α^*, of Y into X, as follows: $y\alpha^*x$ if and only if $p(x) \leq \alpha p(y)$ for every $p \in A$. The relation α^* so defined has two special topological properties. If E is any subset of X, let $\alpha^{*-1}E$ denote the set of all those $y \in Y$ such that $y\alpha^*x$ for some $x \in E$. Then α^* has the property that $\alpha^{*-1}P$ is a clopen set in Y whenever P is a clopen set in X. (A clopen set in a topological space is one which is both closed and open.) Another way of expressing this property is to say that $\alpha^{*-1}\{x \in X\colon\ p(x) = 1\} = \{y \in Y\colon\ \alpha p(y) = 1\}$, for each $p \in A$. Moreover, if $y \in Y$ is any point, then the set of all those $x \in X$ such that $y\alpha^*x$ is compact.

Conversely, let ρ be any relation from the space Y to the space X. For any element $p \in A$, we define a function ρ^*p from Y to Φ by setting $\rho^*p(y) = \text{lub}\,\{p(x)\colon\ y\rho x\}$. It is easily seen that $\rho^*0(y) = 0$ for every $y \in Y$ and that $\rho^*(p \vee q)(y) = \rho^*p(y) \vee \rho^*q(y)$ for any $y \in Y$ and any two elements $p, q \in A$. If ρ has the property that $\rho^{-1}P$ is a clopen set in Y whenever P is a clopen set in X, then ρ^*p will be a continuous function for each element $p \in A$, and hence ρ^* will be a hemimorphism of A into B.

If α is a hemimorphism of A into B, and if α^* is the relation from Y to X described above, then α^{**} is also a hemimorphism of A into B. One easily shows that $\alpha^{**} = \alpha$. (Any mapping α has a dual relation α^*, defined as before; in order that $\alpha = \alpha^{**}$, it is necessary and sufficient that α be a hemimorphism.) On the other hand, suppose that ρ is a relation from Y to X such that $\rho^{-1}P$ is clopen in Y whenever P is clopen in X; then ρ^* is a hemimorphism of A into B, and hence ρ^{**} is a relation from Y to X. A necessary and sufficient condition that $\rho = \rho^{**}$ is that for each $y \in Y$, the set $\{x \in X : y\rho x\}$ is compact. Such a relation is called a Boolean relation. The correspondence between hemimorphisms and Boolean relations just described is one-to-one. This extension of Stone's duality theory is due to Halmos [2]. See also Jonsson and Tarski [4] and Wright [8].

Cognizance should be taken of the fact that topological considerations may be ignored when the algebras A and B are the algebras of *all* subsets of two sets, say of S and T, respectively. If $A = \Phi(X)$ and $B = \Phi(Y)$, then the Boolean spaces X and Y are the Stone-Čech compactifications of the discrete spaces S and T respectively. Then a Boolean relation from Y to X defines a relation from T to S, and any relation from T to S may be extended to a Boolean relation from Y to X.

The duality between hemimorphisms and Boolean relations is sufficient to describe completely the structure of polarities, because the theory of polarities is coextensive with the theory of hemimorphisms. (If p is an element of a Boolean algebra, we denote the complement of p by the symbol p'.)

THEOREM 1. *If $\#$ is a polarity of a Boolean algebra A into a Boolean algebra B, and if, for each $p \in A$, we set $\alpha p = (p^\#)'$, then α is a hemimorphism of A into B. Conversely, if α is a hemimorphism of A into B, and if, for each $p \in A$, we set $p^\# = (\alpha p)'$, then $\#$ is a polarity of A into B.*

Proof. This is quite trivial: let α and $\#$ be two mappings of A into B such that $(\alpha p)' = p^\#$ for each $p \in A$. Then $\alpha 0 = 0$ if and only if $0^\# = 1$, and $\alpha(p \vee q) = \alpha p \vee \alpha q$ if and only if $(p \vee q)^\# = p^\# \wedge q^\#$.

This means that every special property of a polarity can be translated into a corresponding special property of a hemimorphism, and consequently into a special property of a Boolean relation. It is, however, sometimes more revealing to use the complementary relation. If ρ is a relation from Y to X, the complementary relation ρ' from Y to X is the complement of ρ in the cartesian product $Y \times X$; that is, the settheoretic complement of ρ considered as a subset of $Y \times X$. Since it will be convenient to use such complementary relations, we shall introduce the following name for them.

DEFINITION. A relation ρ from one Boolean space Y to another Boolean space X will be called a polarity relation if it is the complementary relation σ' of a Boolean relation σ of Y into X. If $\#$ is a polarity of one Boolean algebra A into another Boolean algebra B, and if α is the hemimorphism of A into B defined by $\alpha p = (p^{\#})'$, then α and $\#$ will be said to be associated. If α^* is the dual Boolean relation for the hemimorphism α, the polarity relation $\alpha^{*\prime}$ will be called the conjugate relation of the polarity $\#$ associated with α.

Suppose, in the notation of this definition, that $\#$ is a polarity from A to B. For any clopen set P in X, there is an element $p \in A$ such that $p = \{x \in X : p(x) = 1\}$. We may, temporarily, denote by $p^{\#}$ the set $\{y \in Y : p^{\#}(y) = 1\}$. The comprement $(p^{\#})'$ of $p^{\#}$ in Y is given by the formulas

$$(p^{\#})' = \{y \in Y : (p^{\#})'(y) = 1\} = \{y \in Y : \alpha p(y) = 1\}$$
$$= \alpha^{*-1}\{x \in X : p(x) = 1\} = \alpha^{*-1}p \ .$$

Thus $y \in (P^{\#})'$ if and only if there is an element $x \in P$ such that $y\alpha^* x$, and hence $y \in P^{\#}$ if and only if $y\alpha^{*\prime}x$ for all $x \in P$. If ρ denotes the polarity relation $\rho = \alpha^{*\prime}$, then $y \in P^{\#}$ if and only if $y\rho x$ for all $x \in P$. In other words, every polarity has the form given by Birkhoff, if consideration is given to the topological structure of the dual spaces. Then Theorem 1 may be restated in the following (somewhat telegraphic) form.

THEOREM 2. *There is a one-to-one correspondence between polarities of Boolean algebras and polarity relations of Boolean spaces.*

Special properties of hemimorphisms have been investigated in terms of their dual Boolean relations [8]. It is thus quite easy to obtain the corresponding facts about polarities.

DEFINITION. A polarity $\#$ of A into B is called a DeMorgan polarity if $(p \wedge q)^{\#} = p^{\#} \vee q^{\#}$, for each $p, q \in A$, and if $1^{\#} = 0$.

THEOREM 3. *Let $\#$ be a polarity of A into B. The following are then equivalent:*

(i) *$\#$ is a DeMorgan polarity;*

(ii) *the associated hemimorphism α is a homomorphism:*

(iii) *the Boolean relation α^* is a function;*

(iv) *the polarity relation $\rho = \alpha^{*\prime}$, has the property that for any $y \in Y$, if x_1 and x_2 are distinct elements of X, then either $y\rho x_1$ or $y\rho x_2$; and*

(v) $(p^{\#})' = (p')^{\#}$, *for any* $p \in A$.

Proof. A hemimorphism α is a homomorphism if and only if $\alpha(p \wedge q) = \alpha p \wedge \alpha q$ for all $p, q \in A$, and $\alpha 1 = 1$. The theorem follows from this and from the fact that α is a homomorphism if and only if α^* is a function [2].

Let A, B, C be Boolean algebras, with dual spaces X, Y, Z respectively. If $\alpha : A \to B$ and $\beta : B \to C$ are hemimorphisms, then the product $\beta\alpha : A \to C$ is also a hemimorphism. Correspondingly if α is a Boolean from Z to Y and if ρ is a Boolean relation from Y to X, then the prodnct $\rho\sigma$ is a Boolean relation from Z to X [3]. Recall then $z(\rho\sigma)x$ if and only if there is an element $y \in Y$ such that $z\sigma y$ and $y\rho x$. If $\alpha^* = \rho$ and $\beta^* = \sigma$, then $(\beta\alpha)^* = \rho\sigma = \alpha^*\beta^*$ [2].

The iteration of two polarities is not, is general, a polarity. However, it may have other important properties; in particular, we are interested in the properties of a Galois correspondence. Note that if $\# : A \to B$ is a polarity, then $p_1 \leq p_2$ in A implies $p_2^{\#} \leq p_1^{\#}$ in B. This means that it is only the third condition in the definition of a Galois correspondence which needs investigation.

Recall that if ρ is a relation from Y to X, the inverse relation ρ^{-1} is a relation from X to Y, defined by declaring $x\rho^{-1}y$ if and only if $y\rho x$.

THEOREM 4. *Let* $\#$ *be a polarity from a Boolean algebra* A, *with dual space* X, *to a Boolean algebra* B, *with dual space* Y. *Let* α *be the associated hemimorphism, let* α^* *be the dual Boolean relation of* α, *and let* ρ *be the conjugate polarity relation of* $\#$. *Let* $+$ *be a polarity from* B *into* A, *let* β *be the associated hemimorphism, let* β^* *be the dual Boolean relation of* β, *and let* σ *be the conjugate polarity relation of* $\#$. *Then the following are equivalent*:

(i) $p \leq p^{\#+}$ *for each* $p \in A$;

(ii) $\beta(\alpha p)' \leq p'$ *for each* $p \in A$;

(iii) $x\beta^* y$ *implies* $y\alpha^* x$ *for each* $x \in X, y \in Y$;

(iv) $\beta^* \subset \alpha^{*-1}$:

(v) $y\rho x$ *implies* $x\sigma y$ *for each* $x \in X, y \in Y$; *and*

(vi) $\rho \subset \sigma^{-1}$.

Proof. The only problem is to show that (ii) and (iii) are equivalent. This will follow from the slightly more general result: for any two elements x_1 and $x_2 \in X$, we have $\beta(\alpha p)'(x_1) \leq p'(x_2)$ for all $p \in A$ if and only if, for any $y \in Y$, $x_1\beta^* y$ implies $y\alpha^* x_2$. For we have $\beta(\alpha p)'(x_1) = \text{lub}\{(\alpha p)'(y) : x_1\beta^* y\}$, so that $\beta(\alpha p)'(x_1) \leq p'(x_2)$ if and only if $x_1\beta^* y$ implies $p(x_2) \leq \alpha p(y)$. This last inequality holds for each $p \in A$ if and only if $y\alpha^* x_2$.

This result has a number of immediate corollaries which clarify the nature of Galois correspondences between Boolean algebras.

THEOREM 5. *In the notation of Theorem 3, the polarities $\sharp$ and $+$ define a Galois correspondence of A and B if and only if $\beta^* = \alpha^{*-1}$.*

This means that a given polarity $\sharp$ can have at most one other polarity $+$ which may be paired with it to yield a Galois correspondence. Theorem 2 showed that the method given by Birkhoff is the only way to obtain a polarity; this result shows that the same method is the only way to obtain a Galois correspondence. These facts also given an answer to an important question in connection with Boolean relations themselves: when is the inverse of a Boolean relation again a Boolean relation ?

THEOREM 6. *Let θ be a Boolean relation from a Boolean space Y to another Boolean space X. A necessary and sufficient condition that the inverse relation θ^{-1} be a Boolean relation is that the dual hemimorphism θ^* of $\phi(X)$ into $\phi(Y)$ be the associated hemimorphism of a polarity of $\phi(X)$ to $\phi(Y)$ which is part of a Galois correspondence.*

In the special case of most importance, when $X = Y$, this condition becomes very simple.

THEOREM 7. *Let $\sharp$ be a polarity of a Boolean algebra A into itself, let α be its associated hemimorphism, let α^* be the dual relation of α, and let ρ be the conjugate polarity relation of $\sharp$. Then the following are equivalent:*

(i) $p \leq p^{\sharp\sharp}$ *for each* $p \in A$;
(ii) $\alpha(\alpha p)' \leq p'$ *for each* $p \in A$;
(iii) α^* *is symmetric*; *and*
(iv) ρ *is symmetric*.

A polarity has some to the properties of the complementation mapping $p \rightarrow p'$. We may ask what other properties of complementation it can have, and in particular, we may seek a characterization of complementation. Since we are given a Boolean algebra at the outset, there is already available one characterization of complementation: In any distributive lattice with 0 and 1, if every element p has an element p' such that $p \vee p' = 1$ and $p \wedge p' = 0$, then the element p' is unique. Furthermore, the mapping $p \rightarrow p'$ is a DeMorgan polarity satisfying $p = p''$. When we ask for a characterization of complementation, we ask for additional assumptions about a polarity $\sharp$ which imply that $p^\sharp = p'$ for each element p.

Let $\sharp$ be a polarity of a Boolean algebra A into itself. Theorem 3 gives precise conditions that $\sharp$ satisfy DeMorgan's laws, and Theorem 7 gives equally precise conditions that $p \leq p^{\sharp\sharp}$ for each $p \in A$. From the above list of properties of complementation, this leaves three attributes to be investigated; (1) $p \vee p^{\#} = 1$ for each $p \in A$; (2) $p \wedge p^{\#} = 0$ for each $p \in A$; and (3) $p^{\#\#} \leq p$ for each $p \in A$.

In the next two theorems, let α be the associated hemimorphism of the polarity $\#$ of A into itself, let α^* be the dual Boolean relation of α, and let ρ be the conjugate polarity ralation of $\#$.

THEOREM 8. *The following are equivalent*:

(i) $p \vee p^{\#} = 1$ *for each* $p \in A$;

(ii) $p^{\#} = p' \vee b$ *for each* $p \in A$, *where* $b \in A$ *is some fixed element*;

(iii) *if* $x_1 \alpha^* x_2$, *then* $x_1 = x_2$; *and*

(iv) *if* $x_1 \neq x_2$, *then* $x_1 \rho x_2$.

In particular, the following are equivalent:

(I) $p \vee p^{\#} = 1$ *for each* $p \in A$, *and* $1^{\#} = 0$;

(II) $p^{\#} = p'$ *for each* $p \in A$;

(III) $x_1 \alpha^* x_2$ *if and only if* $x_1 = x_2$; *and*

(IV) $x_1 \rho x_2$ *if and only if* $x_1 \neq x_2$.

Proof. It is easily seen that $p \vee p^{\#} = 1$ if and only if $\alpha p \leq p$. It is known [8] that a hemimorphism α satisfies this condition for each $p \in A$ if and only if $\alpha p = p \wedge a$, for some fixed $a \in A$. The theorem follows from this fact.

THEOREM 9. *The following are equivalent*:

(i) $p \wedge p^{\#} = 0$ *for each* $p \in A$;

(ii) $p \leq \alpha p$ *for each* $p \in A$;

(iii) α^* *is reflexive*; *and*

(iv) ρ *is irreflexive*.

Proof. The equivalence of (ii) with (iii) is proved in [8]; the equivalence of the others is then trivial. (Note that an irreflexive relation ρ is one such that either $x \rho' x$ or else $x \rho x$ implies $x \rho y$ for all y.)

The problem of condition (3) can be treated in more generality. Return for a moment to the definition of a Galois correspondence. If we retain (i) and (ii) of this definition, but alter (iii) to read (iii′) $p^{\#+} \leq p$ and $q^{+\#} \leq q$, then the lemma stated at the beginning must also be altered. In fact, the conclusion becomes $1^{\#} = 0$, $1^{+} = 0$, $(p_1 \wedge p_2)^{\#} = p_1^{\#} \vee p_2^{\#}$, $(q_1 \wedge q_2)^{+} = q_1^{+} \vee q_2^{+}$. These properties might also be considered in the manner in which we have treated of polarities. There is obviously no need to do this.

However, if we have two *polarities* $\#$ and $+$ having the property given by (iii′) above, then the altered lemma shows that we have two DeMorgan polarities. If α and β are the associated hemimorphisms, then both α and β are homomorphisms. Furthermore, $\beta \alpha p = p^{\#'+'} = p^{\#+} \leq p$, for each $p \in A$. Then [8], $\beta \alpha p = p \wedge a$, and since $\beta \alpha 1 = 1$, we have $\beta\alpha p = p$ for each $p \in A$. This gives the following result.

THEOREM 10. *Let* $\sharp$ *be a polarity of* A *into* B *and let* $+$ *be a polarity of* B *into* A. *In the notation of Theorem* 4, *the following are equivalent*:

(i) $p^{\sharp +} \leq p$ *and* $q^{+\sharp} \leq q$ *for each* $p \in A$ *and* $q \in B$;

(ii) $p^{\sharp +} = p$ *and* $q^{+\sharp} = q$ *for each* $p \in A$ *and* $q \in B$;

(iii) α *and* β *are reciprocal isomorphisms of* A *onto* B *and of* B *onto* A *respectively*; *and*

(iv) α^* *and* β^* *are reciprocal homeomorphisms of* Y *onto* X *and of* X *onto* Y *respectively.*

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