COMPUTATIONS OF THE MULTIPLICITY FUNCTION

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1. Introduction. Let H be a separable Hilbert space. The following two problems will be studied:

1. Given a bounded normal operator A, of multiplicity m, what are the conditions, on the bounded measurable function f, so that the multiplicity of $S = f(A)$ is $n, n < \infty$?

2. How to compute the multiplicity of a normal operator that commutes with a given normal operator, of finite multiplicity?

NOTATION. Let S be a normal operator of multiplicity n, $n < \infty$. There exist a Borel measure μ and n Borel sets in the complex plane $e_1 \supset e_2 \supset \cdots \supset e_n$, such that, up to unitary equivalence,

$$H = \sum_{i=1}^{n} L_2(\mu, e_i) \tag{1.1}$$

$$S\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \begin{pmatrix} \lambda f_1(\lambda) \\ \vdots \\ \lambda f_n(\lambda) \end{pmatrix}$$

This is the Multiplicity Theorem. (See Theorem X. 5.10) of [1]. The operator S has uniform multiplicity if $e_1 = e_2 = \cdots = e_n$.

The resolution of the identity, of a normal operator A, will be denoted by $E(A; \alpha)$. The Boolean algebra of projections, generated by $E(A; \alpha)$ will be denoted by $\mathfrak{E}_A$. Let $E(\alpha)$ stand for $E(S; \alpha)$ and $\mathfrak{E}$ for $\mathfrak{E}_S$. Throughout this note all operators are assumed to be bounded.

We shall use the following results from [2]:

Let S be a normal operator of multiplicity n, and B a normal operator that commutes with S. Let H and S be represented by 1.1.

THEOREM A. *There exist k Borel measurable bounded complex functions $y_1(\lambda), \cdots, y_k(\lambda)$ and k matrices of Borel measurable bounded complex functions $\varepsilon_1(\lambda), \cdots, \varepsilon_k(\lambda)$ such that:*

For a fixed λ the matrices $\varepsilon_i(\lambda)$ are disjoint self adjoint projections whose sum is the identity and

$$B\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \left(\sum_{i=1}^{k} y_i \varepsilon^i(\lambda)\right)\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix}. \tag{1.2}$$

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Equivalently, if the self adjoint projections E_i, are defined by

$$E_i\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix} = \varepsilon_i(\lambda)\begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_n(\lambda) \end{pmatrix}$$

then

$$(1.3) \qquad \begin{cases} B = \sum_{i=1}^{k} y_i(S)E_i \\ E(B;\alpha) = \sum_{i=1}^{k} E(y_i^{-1}(\alpha))E_i \; . \end{cases}$$

REMARK. In the above decomposition the numbers $y_i(\lambda)$ for a fixed λ are different eigenvalues of a certain matrix. Thus for each λ there is an integer $k' \leq k$ such that

$$y_i(\lambda) \neq y_j(\lambda) \qquad i \neq j \qquad i, j \leq k', \qquad \varepsilon_i(\lambda) \neq 0 \qquad i \leq k' ,$$

and

$$y_{k'+1}(\lambda) = \cdots = y_k(\lambda) = 0 \; ,$$
$$\varepsilon_{k+1}(\lambda) = \cdots = \varepsilon_{k+1}(\lambda) = 0 \; .$$

This is essential for the proof of Lemma 2.1. Also the matrices $\varepsilon_i(\lambda)$ are $n \times n$ matrices.

THEOREM B. *The number n is the largest integer such that there exists a nilpotent operator, commuting with S, of order n. See* [2] *Theorem* 3.1 *and its corollary.*

2. The multiplicity of a function of an operator. The main result in this section is:

THEOREM 2.1. *Let A be a normal operator of multiplicity m, $m < \infty$, and f a bounded measurable function. The operator $S = f(A)$ has finite multiplicity, if and only if, there exist k disjoint Borel sets $\beta_1, \cdots, \beta_k$ and k bounded measurable functions $z_1(\lambda), \cdots, z_k(\lambda)$ such that:*

a. $\sigma(A) = \bigcup_{i=1}^{k} \beta_i$.

b. *if $\lambda \in \beta_i$ then $z_i(f(\lambda)) = \lambda$ almost*

everywhere, with respect to $E(A;\alpha)$.

Proof of sufficiency of conditions a and b. Let S_i and A_i be the restrictions of S and A to $E(A;\beta_i)H$. Then

$$S_i = \int_{\beta i} f(\lambda) E(A; d\lambda)$$

hence

$$z_i(S_i) = A_i \; .$$

Now, it follows from Theorem B that

$$muA_i \geq muS_i \qquad (muT = \text{multiplicity of } T)$$

But the multiplicity function is subadditive:

$$muS \leq \sum_{i=1}^{k} muS_i \; .$$

To see this we have to observe that muS is the smallest number n such that there exists a set of n elements, $\{x_1, \cdots x_n\}$, $x_i \in H$ and span $\{E(\alpha)x_i, \alpha \text{ a Borel set}\} = H$. ($n$ generating elements.)

Thus

$$muA \leq \sum_{i=1}^{k} muS_i \leq \sum_{i=1}^{k} muA_i \leq mk < \infty \; .$$

In order to prove necessity we need the following:

LEMMA 2.1. *Let $S = f(A)$ have finite multiplicity n and let*

$$A = \sum_{i=1}^{k} z_i(S) E_i$$

be the representation 1.3 *then $E_i \in \mathfrak{E}_A$.*

Proof. For every Borel set α $E(\alpha) \in \mathfrak{E}_A$ because $S = f(A)$. Let $E(\alpha)$ be maximal with respect to the property that $E(\alpha)E_1 \in \mathfrak{E}_A$. Such a maximal projection exists by Zorn's Lemma. Now if $E(\sigma(S) - \alpha) \neq 0$ there exists, by the proof of 3.2 in [2] a set β such that:

$$\beta \subseteq \sigma(S) - \alpha \qquad\qquad E(\beta) \neq 0$$

and for some Borel set γ

$$E(\beta)E_1 = E(\beta)E(A; \gamma) \in \mathfrak{E}_A \; .$$

This contradicts the maximality of α, hence $E(\alpha) = I$.

Proof of necessity of conditions a and b. Let S hsve finite multiplicity n. By Lemma 2.1 there exist n sets β_i such that $E(A; \beta_i) = E_i$. Thus

$$E(A;\beta_i)E(A;\beta_j)=0 \text{ if } i\neq j$$

and

$$\sum_{i=1}^{k} E(A;\beta_i)=I\ .$$

Therefore the sets β_i can be chosen to be disjoint and satisfy condition a. Also

$$A=\sum_{i=1}^{k} z_i(S)E_i=\sum_{i=1}^{k} z_i(f(A))E(A;\beta_i)=\sum_{i=1}^{k}\int_{\beta} z_i(f(\lambda)E(A;d\lambda)\ .$$

Hence, if $\beta\subset\beta_i$ then

$$E(A;\beta)A=\int_{\beta}\lambda E(A;d\lambda)=\int_{\beta} z_i(f(\lambda))E(A;d\lambda)$$

or: on the set $\beta_i\lambda=z_i(f(\lambda))$ almost everywhere with respect to the measure $E(A;\alpha)$.

DEFINITION. The function f will be said to have k repetitions, with respect to the measure $E(A;\alpha)$, if conditions a and b of Theorem 2.1 are satisfied.

In the rest of this section we compute muS. It is enough to consider the case where the operator A has uniform multiplicity m: otherwise A can be written as direct sum of operators of uniform multiplicity and one has to study each component of A separately.

The following Theorem is needed:

THEOREM 2.2 *Let H be the direct sum of the orthogonal subspaces $H_1,\cdots,H_k$. Let S_i be a normal operator, on H_i, of uniform multiplicity m_i and S be the direct sum of S_i.*

If

$$E(S;\alpha)=0 \textit{ whenever } E(S_i;\alpha)=0 \textit{ for some } i$$

then

$$muS=\sum_{i=1}^{k} m_i\ .$$

Proof. It is enough to prove that $muS\geq\sum_{i=1}^{k} m_i$. Let $\sigma=\sigma(S_1)=\cdots=\sigma(S_k)=\sigma(S)$. By the Spectral Multiplicity Theorem each operator S_i can be described as follows: There exists a measure μ_i on σ and H_i is the direct sum of m_i spaces $L_2(\mu_i)$. The operator S_i is given by

$$S_i\begin{pmatrix} f_1(\lambda)\\ \vdots\\ f_{m_i}(\lambda)\end{pmatrix}=\begin{pmatrix}\lambda f_1(\lambda)\\ \vdots\\ \lambda f_{m_i}(\lambda)\end{pmatrix}.$$

Now, the measures μ_i are equivalent, by the condition of the Theorem. Thus there exist functions φ_i, $\varphi_i \in L(\mu_{i+1})$ $1 \le i \le k-1$ such that

$$\mu_i(e) = \int_e \varphi_i(\lambda) d\mu_{i+1}$$

for every Borel set e. (Radon Nikodym Theorem, see [3], p. 128). Let us define an operator on H:

If $x \in H_i$,

$$x = \begin{pmatrix} f_1(\lambda) \\ \vdots \\ f_{m_i-1}(\lambda) \\ 0 \end{pmatrix}$$

then

$$Mx \in H_i, \quad Mx = \begin{pmatrix} 0 \\ f_1(\lambda) \\ \vdots \\ f_{m_i-1}(\lambda) \end{pmatrix}.$$

If

$$x \in H_i, \quad x = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ f_{m_i}(\lambda) \end{pmatrix}$$

then

$$Mx \in H_{i+1}, \quad Mx = \begin{pmatrix} \sqrt{\varphi_i(\lambda)}\, f_{m_i}(\lambda) \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Where H_{k+1} is the zero space.

It is easy to see that M is a bounded operator and

$$M^{\sum_{i=1}^{k} m_i} = 0$$

but

$$M^{\sum_{i=1}^{k} m_i - 1} \neq 0 .$$

Also $MS = SM$, hence $muS \geq \sum_{i=1}^{k} m_i$.

REMARK. It was proved in Theorem 2.1 that if a function f has k repetitions then

$$muf(A) \leq kmuA .$$

However the number of repetitions of a function is not uniquely defined. In order to compute $muf(A)$ we have to find the minimal number of repetitions. This is what the next Theorem does.

THEOREM 2.3. *Let A be a normal operator of uniform multiplicity m. Let f be a bounded measurable function which has k repetitions with respect to the measure $E(A;\alpha)$. A necessary and sufficient condition that $muS = mk$, where $S = f(A)$, is:*

There exists a Borel set α_0

$$(2.1) \qquad E(A; f^{-1}(\alpha_0)) \neq 0$$

and

$E(A; f^{-1}(\alpha)) = 0$ whenever $E(A; f^{-1}(\alpha) \cap \beta_i) = 0$ for some i and $\alpha \subset \alpha_0$.

Proof. Assume condition 2.1. We may restrict A and S to $E(A; f^{-1}(\alpha_0))H$. Let

$$H_i = E(A; f^{-1}(\alpha_0) \cap \beta_i)H ,$$

and A_i, S_i the restriction of A, S to H_i. Now

$$f(A_i) = S_i \qquad z_i(S_i) = A_i$$

(See Theorem 2.1.). Thus the operators S_i have uniform multiplicity m because the operators A_i do. It follows from Theorem 2.2 that the multiplicity of S restricted to $E(A; f^{-1}(\alpha_0))H$ is mk. But $muS \leq mk$, hence $muS = mk$.

(Note that on α_0 the operator S has uniform multiplicity mk). Conversely, let us assume that for each Borel set α_0 with $E(A; f^{-1}(\alpha_0)) \neq 0$, there exists a subset α such that $E(A; f^{-1}(\alpha)) \neq 0$ but $E(A; f^{-1}(\alpha) \cap \beta_i) = 0$ for some i. Let $E(A; f^{-1}(\alpha_1))$ be maximal with respect to the property

$$E(A; f^{-1}(\alpha_1))E(A; \beta_1) = 0$$

Let $E(A; f^{-1}(\alpha_2))$ be maximal, with respect to the property

$$\alpha_2 \cap \alpha_1 = \varphi \text{ and } E(A; f^{-1}(\alpha_2))E(A; \beta_2) = 0$$

and choose inductively $\alpha_3 \cdots \alpha_n$, $\alpha_i \cap \alpha_j = \varphi$

$$E(A; f^{-1}(\alpha_j))E(A: \beta_j) = 0$$

There exist such maximal projections by Zorn's Lemma. Now if $E(A; \bigcup_{i=1}^{k} f^{-1}(\alpha_i)) \neq I$ there will be a set α and an integer j such that

$$\alpha \cap \left(\bigcup_{i=1}^{k} \alpha_i\right) = 0; \qquad E(A; f^{-1}(\alpha) \cap \beta_j) = 0$$

Thus α_j will not be maximal. Let

$$\bar{\beta}_j = \beta_j \cup (f^{-1}(\alpha_j) \cap \beta_1), \quad j \geq 2 .$$

Then $\bigcup_{j=2}^{u} \bar{\beta}_j = \sigma(A)$ and on $\bar{\beta}_j$ the function f possesses a bounded measurable inverse. Thus f has $k-1$ repetitions and $muS \leq m(k-1)$.

3. The multiplicity of a matrix of functions. Let S be a normal operator of uniform multiplicity n. Let B be a normal operator and $BS = SB$. The operator B is represented as the matrix of functions $\sum_{i=1}^{k} y_i(\lambda)\varepsilon_i(\lambda)$ and also $B = \sum_{i=1}^{k} y_i(S)E_i$ (Equation 1.2 and 1.3). Let us denote by B_i and S_i the restrictions of B and S, respectively, to $E_iH = H_i$.

THEOREM 3.1. *The operator B has finite multiplicity, if and only if, the functions y_i have $j_i (j_i < \infty)$ repetitions with respect to the spectral measure of S_i.*
Also

$$\max_i muB_i \leq \sum_{i=1}^{k} mu\, B_i \leq \sum_{i=1}^{k} j_i muS_i .$$

Proof. From the definition of multiplicity, as the smallest number of generating elements, it follows that

$$\max_i muB_i \leq muB \leq \sum_{i=1}^{k} muB_i .$$

Now, $B_i = y_i(S_i)$, hence the rest of the Theorem follows from Theorem 2.1. The problem of this section is reduced to the following

$$H = \sum_{i=1}^{k} E_iH \text{ where } E_iE_j = 0 \text{ if } i \neq j$$

and B_i = restriction B to E_iH, where the multiplicity of B_i is known. Now by decomposing each operator B_i into sum of operators of uniform multiplicity we will have $H = \sum_{i=1}^{m} H_i$, where the spaces H_i are mutually orthogonal, and C_i = restriction of B to H_i is an operator of uniform multiplicity. We shall show how to compute muB from muC_i by reducing this case to the one studied in Theorem 2.2.

Denote the projection on H_i by F_i. Let $E(B;\alpha_i)$ be the maximal projection such that

$$E(C_i;\alpha_i) = E(B;\alpha_i)F_i = 0\ .$$

Such a projection exists by Zorn's Lemma. Finally let $\beta_i = \sigma(B) - \alpha_i$. On β_i the spectral measure of C_i can vanish only when the spectral measure of B vanishes. Now $E(B;\bigcup_{i=1}^m \beta_i) = I$ because $\sum_{i=1}^m F_i = I$.

The set $\sigma(B)$ can be decomposed into disjoint sets γ_j such that

a. Each γ_j is a subset of one of the sets β_{j_0}.

b. If $\gamma_j \cap \beta_i \neq \varphi$ then $\gamma_j \subset \beta_i$.

Assuming, for a moment, that this decomposition is given then

$$muB = \max_j mu(B \text{ restricted to } E(B;\gamma_j)H)\ .$$

But the multiplicity of B restricted to $E(B;\gamma_j)H$ is

$$\sum_{i\mid\gamma_j\subset\beta_i} mu(C_i \text{ restricted to } E(B;\gamma_j)H_i)$$

by Theorem 2.2.

We shall show how to choose the sets γ_i by an induction argument on the number m. Let $\gamma_1 = \beta_1 - \bigcup_{i\geq 2}\beta_1\beta_i$. This set (which might be void) satisfies conditions a and b. The rest of $\sigma(B)$ is

$$\left(\bigcup_{i\geq 2}\beta_1\beta_i\right)\cup\left(\bigcup_{i\geq 2}(\beta_i - \beta_1)\right)$$

In both sets there are only $m-1$ subsets and by induction there exists a decomposition.

Bibliography

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