MAXIMAL MEANS AND TAUBERIAN THEOREMS

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Introduction. In this paper, we study two well-known mathematical ideas that have hitherto been regarded as unconnected. It is our purpose to show that they are closely related. The first idea is one developed by Pólya in his theory of maximal density [3]. The second is the idea of repeated differentiation introduced by Littlewood in Tauberian arguments [2], [1].

Our account of the Pólya theory is virtually a direct translation of certain sections of [3]. To apply his methods to bounded Lebesgue-measurable functions in general requires only simple changes. Consequently, we state only the results, leaving the proofs to the reader.

Then the idea of Littlewood is developed in a theory of Littlewood means that compares closely, theorem by theorem, with the theory of Pólya means. Thus, Theorems 1.1 and 2.1, Theorems 1.2 and 2.2, etc., should be compared. The theory of Littlewood means may be regarded as the theory of the p-fold application of l'Hospital's rule to a certain class of indeterminate forms. The order p is not restricted to positive integral values; indeed, p may be any real number between -1 and $+\infty$.

But the connection between the two theories goes deeper than mere analogy. Our principle result, Theorem 3.1, asserts that the Pólya maximal upper mean, $\mathscr{L}(1)$, is equal to the Littlewood maximal upper mean, $\Lambda(\infty)$, and that the minimal lower means are also equal to each other. An immediate corollary of this theorem is the celebrated Tauberian theorem of Littlewood that a bounded and Lebesgue-measurable function has a Cèsaro average if and only if it has an Abel average.

Finally, in §4, we give an intrinsic characterization of the mean $\mathscr{L}(1)$ as the infimum of the averages of all Cèsaro-averageable functions f^* with $f^*(x) \geq f(x)$ for all x. This might be compared with the characterization of the outer measure of a set as the infimum of the measures of all measurable sets that cover it.

1. The Pólya means. Let $f(x)$ be a given bounded and Lebesgue-measurable function on $(0, \infty)$. For $0 < \xi < 1$ define

$$\mathscr{L}(\xi) = \limsup_{x \to \infty} \frac{1}{x - \xi x} \int_{\xi x}^{x} f(t) dt ,$$

$$l(\xi) = \liminf \text{(same)} .$$

The quantities $\mathscr{L}(0)$ and $l(0)$ are the ordinary Cèsaro lim sup and

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lim inf, respectively, of f. We define $\mathscr{L}(1) = \lim_{\xi\to 1} \mathscr{L}(\xi)$ and $l(1) = \lim_{\xi\to 1} l(\xi)$, where the existence of the limits is guaranteed by Theorem 1.3. We call $\mathscr{L}(1)$ the "maximal Pólya upper mean" and $l(1)$ the "minimal Pólya lower mean".

THEOREM 1.1. *$\mathscr{L}(\xi)$ and $l(\xi)$ are continuous functions of ξ for $0 \le \xi \le 1$.*

THEOREM 1.2. *$\mathscr{L}(\xi) \ge \mathscr{L}(\xi^n)$ and $l(\xi) \le l(\xi^n)$ for each $\xi, 0 \le \xi \le 1$ and $n = 1, 2, 3, \cdots$.*

THEOREM 1.3. *$\mathscr{L}(1) = \lim_{\xi\to 1-} \mathscr{L}(\xi)$ and $l(1) = \lim_{\xi\to 1-} l(\xi)$ exist.*

REMARK. It may happen that $\mathscr{L}(\xi)$ and $l(\xi)$ fail to be monotone functions. Nevertheless, we have the next theorem.

THEOREM 1.4. *For all ξ, $0 \le \xi \le 1$, $\mathscr{L}(1) \ge \mathscr{L}(\xi) \ge \mathscr{L}(0)$ and $l(1) \le l(\xi) \le l(0)$.*

THEOREM 1.5. *If there are numbers ξ_0 and ξ_1 with $0 \le \xi_0 \le 1$, $0 \le \xi_1 \le 1$ for which $l(\xi_1) \ge \mathscr{L}(\xi_0)$, then there is a constant, L, such that $\mathscr{L}(\xi) = l(\xi) = L$ for all $\xi, 0 \le \xi \le 1$.*

2. The Littlewood means. Let $f(x)$ be a bounded, Lebesgue-measurable function on $(0, \infty)$. The Abel averaging method studies the behaviour of $F(x)/I(x)$ as $x \to 0+$, where $F(x) = \int_0^\infty f(t)e^{-xt}dt$ and $I(x) = \int_0^\infty e^{-xt}dt = x^{-1}$. We regard $F(x)/I(x)$ as an "indeterminate form ∞/∞" at $x = 0$. The Littlewood means arise by applying l'Hospital's rule p times to the fraction $F(x)/I(x)$. Thus, we study also the ratios $F^{(p)}(x)/I^{(p)}(x)$ where

$$F^{(p)}(x) = (-)^p\int_0^\infty t^p e^{-xt} f(t)dt$$

and

$$I^{(p)}(x) = (-)^p\int_0^\infty t^p e^{-xt}dt = (-)^p x^{-(p+1)}\Gamma(p+1)\ .$$

Here, p may be any real number in the range $-1 < p < \infty$ and we put $(-)^p = e^{i\pi p}$. When p is a positive integer, $F^{(p)}(x)$ is the pth derivative of $F(x)$ and $I^{(p)}(x)$ is the pth derivative of $I(x)$. We define

$$\Lambda(p) = \limsup_{x\to 0+} F^{(p)}(x)/I^{(p)}(x)$$
$$\lambda(p) = \liminf_{x\to 0+} F^{(p)}(x)/I^{(p)}(x)\ .$$

The quantities $\Lambda(0)$ and $\lambda(0)$ are the ordinary Abel lim sup and Abel lim inf, respectively, of f. We define $\Lambda(\infty) = \lim_{p\to\infty} \Lambda(p)$, $\lambda(\infty) = \lim_{p\to\infty} \lambda(p)$, $\Lambda(-1) = \lim_{p\to -1+} \Lambda(p)$, and $\lambda(-1) = \lim_{p\to -1+} \lambda(p)$, where the existence of the limits is guaranteed by Theorem 2.3. We call $\Lambda(\infty)$ the "maximal Littlewood mean" and $\lambda(\infty)$ the "minimal Littlewood mean."

THEOREM 2.1. *$\Lambda(p)$ and $\lambda(p)$ are continuous functions of p for $-1 \le p \le \infty$.*

Proof. We have

$$(2.1) \qquad \begin{aligned} \Lambda(q) - \Lambda(p) &= \limsup_{x\to 0+} F^{(q)}(x)/I^{(q)}(x) - \limsup_{x\to 0+} F^{(p)}(x)/I^{(p)}(x) \\ &\le \limsup_{x\to 0+} \{F^{(q)}(x)/I^{(q)}(x) - F^{(p)}(x)/I^{(p)}(x)\} \\ &= \limsup_{x\to 0+} \int_0^\infty \{u^q/\Gamma(q+1) - u^p/\Gamma(p+1)\} f(u/x)e^{-u}du \ , \end{aligned}$$

with a similar inequality in the opposite direction. Using now the fact that $|f(u/x)|$ is uniformly bounded, standard techniques of estimation may be applied to the integral in (2.1) to show that $\lim_{q\to p} \Lambda(q) = \Lambda(p)$. The case $p = -1$ and $p = \infty$ follow by definition.

THEOREM 2.2. *$\Lambda(p)$ is a non-decreasing function of p and $\lambda(p)$ is a non-increasing function of p.*

REMARK. Theorem 2.2 is clearly a stronger kind of statement than Theorem 1.2. A direct analogue of Theorem 1.2 would state merely that $\Lambda(p+1) \ge \Lambda(p)$ and $\lambda(p+1) \le \lambda(p)$. It is interesting to note that there is a special proof of this direct analogue, based on the fact that $F^{(p+1)}(x)$ and $I^{(p+1)}(x)$ are the respective derivatives of $F^{(p)}(x)$ and $I^{(p)}(x)$. We need only put $(-)^p g(x) = F^{(p)}(x)$ and $(-)^p h(x) = I^{(p)}(x)$ in the following statement of one form of l'Hospital's rule, after $f(x)$ is suitably normalized.

L'Hospital's rule. If $g(x)$ and $h(x)$ are differentiable functions of x for $x > 0$ and $g(0+) = \infty$ and $h(0+) = \infty$ then

$$\limsup_{x\to 0+} g(x)/h(x) \le \limsup_{x\to 0+} g'(x)/h'(x) \ .$$

Proof of Theorem 2.2. The notation of $*$ for Mellin convolution will simplify our proof considerably. If $A(x)$, $B(x)$, $C(x)$ and $D(x)$ are suitably restricted functions on $0 < x < \infty$, we define

$$(A * B)(x) = \int_0^\infty A(t)B(x/t)\frac{dt}{t} \ ,$$

and see that

$$A * (B * C) = (A * B) * C, \quad \text{and} \quad A * B = B * A .$$

Further, if $C(x) \geq 0$ for all x and $(C * 1)(x) \equiv 1$, where $1(x) = 1$ for all x, then $\limsup_{x \to \infty} (C * D)(x) \leq \limsup_{x \to \infty} D(x)$.

If we now suppose $p < q$ and let

$$K_p(x) = \frac{x^{p+1} e^{-x}}{\Gamma(p+1)}, \quad K_q(x) = \frac{x^{q+1} e^{-x}}{\Gamma(q+1)}, \quad g(x) = f\left(\frac{1}{x}\right),$$

and

$$K_{p,q}(x) = \begin{cases} \dfrac{\Gamma(q+1)}{\Gamma(p+1)\Gamma(q-p)} x^{p+1}(1-x)^{q-p-1} & \text{for } x < 1 \\ 0 & \text{for } x \geq 1 \end{cases}$$

then we have

$$\begin{aligned} \Lambda(p) &= \limsup_{x \to \infty} (K_p * g)(x) \\ \Lambda(q) &= \limsup_{x \to \infty} (K_q * g)(x) \\ K_p &= K_q * K_{p,q} \\ K_{p,q} &\geq 0 \\ (K_{p,q} &* 1)(x) \equiv 1 . \end{aligned}$$

In accordance with the preceding remarks, it follows that $\Lambda(p) \leq \Lambda(q)$, and we are done.

THEOREM 2.3. *$\Lambda(\infty) = \lim_{p \to \infty} \Lambda(p)$, $\lambda(\infty) = \lim_{p \to \infty} \lambda(p)$, $\Lambda(-1) = \lim_{p \to -1+} \Lambda(p)$, and $\lambda(-1) = \lim_{p \to -1+} \lambda(p)$ exist.*

Proof. This is an obvious consequence of the monotonicity of the bounded functions $\Lambda(p)$ and $\lambda(p)$.

THEOREM 2.4. *For all p, $-1 \leq p \leq \infty$, $\Lambda(\infty) \geq \Lambda(p) \geq \Lambda(-1)$ and $\lambda(\infty) \leq \lambda(p) \leq \lambda(-1)$.*

Proof. Same as that of Theorem 2.3.

THEOREM 2.5. *If there are numbers p_0 and p_1 with $-1 < p_0 \leq \infty$, $-1 < p_1 \leq \infty$ for which $\lambda(p_1) \geq \Lambda(p_0)$, then there is a constant, L, such that $\Lambda(p) = \lambda(p) = L$ for all p, $-1 \leq p \leq \infty$.*

REMARK. As is shown in the example following the proof of this theorem, the values $p_0 = -1, p_1 = -1$ must be excluded from the hypotheses. This is in contrast to Theorem 1.5 where the values $\xi_0 = 0$, $\xi_1 = 0$ are not only included in the hypotheses, but play the most important role in the proof. In the example, $f(x) = \sin \log x$, $\lambda(-1) = \Lambda(-1) = 0$, but $\Lambda(\infty) = 1$.

Proof of Theorem 2.5. Suppose $p_1 \geqq p_0$. Then $\lambda(p_0) \geqq \lambda(p_1)$ and hence $\lambda(p_0) = \Lambda(p_0)$. If, on the other hand, $p_1 \leq p_0$, then $\Lambda(p_0) \geq \Lambda(p_1)$ and hence $\lambda(p_1) = \Lambda(p_1)$. It is therefore enough to show that if there exists a $p_0 > -1$ such that $\lambda(p_0) = \Lambda(p_0)$, then $\lambda(p) = \Lambda(p) = \text{constant}$.

Now the following familiar lemma will be enough to show that if $\lambda(p_0) = \Lambda(p_0) = L$, say, then $\lambda(p_0 + 1) = \Lambda(p_0 + 1) = L$. By repeating this argument, we deduce that $\lambda(\infty) = \Lambda(\infty) = L$. But since, for all p, $\lambda(\infty) \leq \lambda(p) \leq \Lambda(p) \leq \Lambda(\infty)$, we have the desired result.

LEMMA. *Let $g(x)$ be a differentiable function for $x > 0$ and let $g'(x)$ be a non-decreasing function (or a non-increasing function) there. If, for some $\alpha < 0$, $L = \lim_{x\to 0+} g(x)/x^\alpha$ exists, then $\lim_{x\to 0+} g'(x)/\alpha x^{\alpha-1}$ also exists and equals L.*

Proof. Fix $\theta > 0$. Then $g(x + \theta x) - g(x) \geq \theta x g'(x)$. But

$$\lim_{x\to 0+} x^{-\alpha}\{g(x + \theta x) - g(x)\} = L\{(1 + \theta)^\alpha - 1\} .$$

Hence $\limsup_{x\to 0+} g'(x)/x^{\alpha-1} \leq L\theta^{-1}\{(1 + \theta)^\alpha - 1\}$ and we may now let $\theta \to 0$ to get $\limsup_{x\to 0+} g'(x)/x^{\alpha-1} \leq \alpha L$. A similar argument shows that $\liminf_{x\to 0+} g'(x)/x^{\alpha-1} \geq \alpha L$ and we are done.

EXAMPLE. Let $f(t) = \sin \log t$. Now

$$\begin{aligned} \Gamma(p + 1)\Lambda(p) &= \limsup_{x\to 0+} \int_0^\infty t^p e^{-t} \sin(\log(t/x))dt \\ &= \limsup_{x\to 0+} \Big\{(\cos \log x)\int_0^\infty t^p e^{-t} \sin \log t\,dt \\ &\qquad - (\sin \log x)\int_0^\infty t^p e^{-t} \cos \log t\,dt\Big\} \\ &= \left|\int_0^\infty t^p e^{-t} \exp(i \log t)dt\right| = |\Gamma(p + 1 + i)| . \end{aligned}$$

Hence

$$\Lambda(p) = \left|\frac{\Gamma(p + 1 + i)}{\Gamma(p + 1)}\right| ,$$

and similary

$$\lambda(p) = -\left|\frac{\Gamma(p + 1 + i)}{\Gamma(p + 1)}\right| .$$

In particular, since, as $p \to -1, \Gamma(p+1) \to \infty$ and $\Gamma(p+1+i) \to \Gamma(i)$, we have $\lambda(-1) = \Lambda(-1) = 0$. On the other hand, $\Lambda(\infty) = 1$, and $\lambda(\infty) = -1$, by Stirling's formula.

3. The main theorem.

THEOREM 3.1. *$\mathscr{L}(1) = \Lambda(\infty)$ and $l(1) = \lambda(\infty)$.*

The following Tauberian theorem of Littlewood is an immediate corollary of this result.

COROLLARY. *The bounded Lebesgue-measurable function f is Abel-averageable, if and only if it is Cèsaro-averageable, and then $av(f: C) = av(f: A)$.*

Proof. We must prove that $l(0) = \mathscr{L}(0)$ if and only if $\lambda(0) = \Lambda(0)$. Suppose $\lambda(0) = \Lambda(0)$. Then $\lambda(\infty) = \Lambda(\infty)$ (Theorem 2.5). But $\lambda(\infty) = l(1)$ and $\Lambda(\infty) = \mathscr{L}(1)$ (Theorem 3.1) so $l(1) = \mathscr{L}(1)$, and hence $l(0) = \mathscr{L}(0)$ (Theorem 1.5). This is the proof of the hard part of the result. The other part follows directly from the inequalities

$$l(0) \leq \lambda(0) \leq \Lambda(0) \leq \mathscr{L}(0) \tag{3.1}$$

that are derived by a familiar integration by parts in the Laplace transform of f; $x\int_0^\infty f(t)e^{-xt}dt = x^2\int_0^\infty \left\{t^{-1}\int_0^t f(x)dy\right\}te^{-xt}dt$.

Proof of Theorem 3.1. We prove here that $\mathscr{L}(1) = \Lambda(\infty)$, and first that $\mathscr{L}(1) \geq \Lambda(\infty)$. An easy roundabout proof would be by way of Theorem 4.1 in which we construct a bounded measurable function $f^*(x), f^*(x) \geq f(x)$, and $av(f^*: C) = \mathscr{L}(1)$. But the same sort of integration by parts that yields inequalities (3.1) tells us that $\Lambda(\infty: f^*) = \mathscr{L}(1)$. Since $f^*(x) \geq f(x)$, $\Lambda(\infty: f^*) \geq \Lambda(\infty)$, and thus $\mathscr{L}(1) \geq \Lambda(\infty)$.

It is important to give a direct proof, because, in certain generalizations of this theory, the analogue of Theorem 4.1 may be false, but the analogue of the present theorem is always true.

To prove directly that $\mathscr{L}(1) \geq \Lambda(\infty)$ it is enough to show that $\mathscr{L}(1) \geq \Lambda(p)$ for each finite positive p. We make normalization $0 \leqq f_{(x)} \leqq 1$. Keeping p fixed, for any $\varepsilon > 0$ we choose R so large that if $0 < A < R^{-1}$, then for all x

$$\frac{F^{(p)}(x)}{I^{(p)}(x)} = \frac{1}{\Gamma(p+1)}\int_0^\infty t^p e^{-t} f(t/x)dt \leq \varepsilon + \frac{1}{\Gamma(p+1)}\int_A^R t^p e^{-t} f(t/x)dt\ .$$

Now, for $0 < \xi < 1$, we put $\sigma_n = R\xi^n$, $n = 1, 2, \cdots$, and choose $N = N(R, \xi)$ so that $\sigma_N < R^{-1}$. Let

$$M_n = \max t^p e^{-t} \text{ for } \sigma_{n+1} \le t \le \sigma_n .$$

We then have

$$\int_{\sigma_N}^{R} t^p e^{-t} f(t/x) dt = \sum_{n=0}^{N-1} \int_{\sigma_{n+1}}^{\sigma_n} t^p e^{-t} f(t/x) dt \le \sum_{n=0}^{N-1} M_n \int_{\sigma_{n+1}}^{\sigma_n} f(t/x) dt .$$

We may then conclude that

$$\limsup_{x \to 0+} \int_{\sigma_N}^{R} t^p e^{-t} f(t/x) dt \le \mathscr{L}(\xi) \sum_{n=0}^{N-1} M_n(\sigma_n - \sigma_{n+1}) .$$

Since $\mathscr{L}(\xi) \le \mathscr{L}(1)$ we may write

$$\Lambda(p) \le \varepsilon + \mathscr{L}(1) \frac{1}{\Gamma(p+1)} \sum_{n=0}^{\infty} M_n(\sigma_n - \sigma_{n+1}) .$$

We now let $\xi \to 1-$ so that $\sum M_n(\sigma_n - \sigma_{n+1}) \to \int_0^R t^p e^{-t} dt,$ then let $R \to \infty$ and then $\varepsilon \to 0$ to get $\Lambda(p) \le \mathscr{L}(1)$.

We now prove that $\Lambda(\infty) \ge \mathscr{L}(1)$. Our proof depends on the fact that when p is large, the function $t^p e^{-t}$ has a single and very sharp maximum. We fix $R > 0$, then a positive integer N, and consider, for $p > R^2$, the dissection of the interval $p - Rp^{1/2} \le t \le p + Rp^{1/2}$ into the $2N$ subintervals of equal length, $\lambda_k \le t \le \lambda_{k+1}$, where $-N \le k < N$. For convenience of notation, we put $\mu_k = \lambda_{k+1}$. We see that $\lambda_k = p + k\delta p^{1/2}$ where $\delta = R/N$. Finally, we put $\xi = \lambda_{-N}/\lambda_N$, and $\xi_k = \lambda_k/\mu_k$.

We choose $0 < \gamma < 1$ and then (for reasons that appear in the proof of Lemma 3.1) choose τ so that $\gamma < \tau < 1$ and $\gamma < \tau^{-1} - 2N(\tau^{-1} - \tau^2)$. We again make the convenient normalization $0 \le f(x) \le 1$.

Since, for fixed R and N and each k, $\lim_{p\to\infty} \xi_k = \lim_{p\to\infty} \xi = 1$, we choose p_0 so that for $p > p_0$ we have $\mathscr{L}(\xi_k) > \tau\mathscr{L}(1)$ and $\mathscr{L}(\xi) > \tau\mathscr{L}(1)$.

We now choose a sequence of x tending to ∞ for which

$$\frac{1}{\lambda_N x - \lambda_{-N} x} \int_{\lambda_{-N} x}^{\lambda_N x} f(t) dt \ge \tau \mathscr{L}(\xi) > \tau^2 \mathscr{L}(1) \tag{3.2}$$

and for which

$$\frac{1}{\mu_k x - \lambda_k x} \int_{\lambda_k x}^{\mu_k x} f(t) dt \leqq \frac{1}{\tau} \mathscr{L}(\xi) \le \frac{1}{\tau} \mathscr{L}(1) . \tag{3.3}$$

LEMMA 3.1. *For x in the sequence described above*

$$\frac{1}{\mu_k x - \lambda_k x} \int_{\lambda_k x}^{\mu_k x} f(t) dt \ge \gamma \mathscr{L}(1) .$$

Proof.

$$\int_{\lambda_k x}^{\mu_k x} f(t) dt = \int_{\lambda_{-N} x}^{\lambda_N x} f(t) dt - \sum_{m \ne k} \int_{\lambda_m x}^{\mu_m x} f(t) dt .$$

Applying 3.2 and 3.3, we have

$$\int_{y_k x}^{\mu_k x} f(t)dt \geq \tau^2(\lambda_N x - \lambda_{-N}x)\mathscr{L}(1) - \sum_{m \neq k} \frac{\mathscr{L}(1)}{\tau}(\mu_m x - \lambda_m x)$$
$$= (\mu_k x - \lambda_k x)\mathscr{L}(1)\left\{\frac{1}{\tau} - 2N\left(\frac{1}{\tau} - \tau^2\right)\right\} ,$$

since

$$\mu_k - \lambda_k = \mu_m - \lambda_m = \frac{1}{2N}(\lambda_N - \lambda_{-N}) .$$

But τ was chosen to make the expression in brackets exceed γ, and the proof of the lemma is complete.

Continuing with our proof of the theorem, we put $y = 1/x$ for x in the sequence described above, so that $y \to 0+$ through some set of values. Now

$$\int_0^\infty t^p e^{-ty} f(t)dt \geq \int_{\lambda_{-N}x}^{\lambda_N x} t^p e^{-ty} f(t)dt = \sum_{k=-N}^{-1} \int_{\lambda_k x}^{\mu_k x} + \sum_{k=0}^{N-1} \int_{\lambda_k x}^{\mu_k x} = \sum\nolimits_1 + \sum\nolimits_2 .$$

In each of the integrals in $\sum_1$, the variable t of integration lies in the range where $t^p e^{-ty}$ is increasing, so that for $-N \leq k < 0$

$$\int_{\lambda_k x}^{\mu_k x} t^p e^{-ty} f(t)dt \geq (\lambda_k x)^p e^{-\lambda_k} \int_{\lambda_k x}^{\mu_k x} f(t)dt ,$$

and applying Lemma 3.1, we have

$$\int_{\lambda_k x}^{\mu_k x} t^p e^{-ty} f(t)dt \geq x^{p+1} \gamma \mathscr{L}(1) \lambda_k^p e^{-\lambda_k}(\mu_k - \lambda_k) .$$

Similarly, for $0 \leq k < N$, we get

$$\int_{\lambda_k x}^{\mu_k x} t^p e^{-ty} f(t)dt \geq x^{p+1} \gamma \mathscr{L}(1) \mu_k^p e^{-\mu_k}(\mu_k - \lambda_k) .$$

Thus, for $p > p_0$,

$$\Lambda(p) \geq \frac{\gamma \mathscr{L}(1)}{\Gamma(p+1)}\left\{\sum_{k=-N}^{-1} \lambda_k^p e^{-\lambda_k}(\mu_k - \lambda_k) + \sum_{k=0}^{N-1} \mu_k^p e^{-\mu_k}(\mu_k - \mu_k)\right\} .$$

We apply Stirling's formula, taking p_0 also so large that for $p > p_0$, $\Gamma(p+1) \leq \gamma^{-1} p^p e^{-p}(2\pi p)^{1/2}$. Hence

$$\Lambda(p) \geq \gamma^2 \mathscr{L}(1)\frac{\delta}{\sqrt{2\pi}}\left\{\sum_{k=-N}^{-1}\left(1 + \frac{k\delta}{\sqrt{p}}\right)^p \exp(-k\delta\sqrt{p})\right.$$
$$\left. + \sum_{k=0}^{N-1}\left(1 + \frac{(k+1)\delta}{\sqrt{p}}\right)^p \exp(-(k+1)\delta\sqrt{p})\right\} .$$

Since

$$\lim_{p\to\infty}\left(1+\frac{k\delta}{\sqrt{p}}\right)^p \exp(-k\delta\sqrt{p}) = \exp\left(-\frac{(k\delta)^2}{2}\right),$$

we let $p \to \infty$ above to get

$$\Lambda(\infty) \geq \gamma^2 \mathscr{L}(1)\frac{1}{\sqrt{2\pi}}\left\{\sum_{k=-N}^{-1} \delta \exp\left(-\frac{(k\delta)^2}{2}\right) + \sum_{k=0}^{N-1} \delta \exp\left(-\frac{((k+1)\delta)^2}{2}\right)\right\}.$$

But the expression in brackets is simply an approximating Riemann sum for $\int_{-R}^{R} \exp(-x^2/2)dx$. Letting first $N \to \infty$ and then $R \to \infty$, we have

$$\Lambda(\infty) \geq \gamma^2 \mathscr{L}(1)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \exp(-x^2/2)dx = \gamma^2 \mathscr{L}(1).$$

Since $\gamma < 1$ was arbitrary, we let $\gamma \to 1$ to complete our proof.

4. The Cèsaro outer and inner means. Let C be the class of Cèsaro-averageable functions, i.e. those for which $\mathscr{L}(0) = l(0)$, and denote the common value by $av(f\colon C)$. Let $C^*(f)$ be the class of all functions $f^* \in C$ for which

$$f^*(x) \geq f(x) \text{ for all } x$$

and

$$f^*(x) \leq \sup_{0<y<\infty} f(y).$$

We define

$$\mathscr{L}^* = \inf av(f^*\colon C), f^* \in C^*(f)$$

and call $\mathscr{L}^*$ the "Cèsaro outer mean" of f. The inner mean l_* is similarly defined.

THEOREM 4.1. *$\mathscr{L}^* = \mathscr{L}(1)$ and $l_* = l(1)$. Moreover, there exists an $f^* \in C^*(f)$ with $av(f^*\colon C) = \mathscr{L}(1)$ and there exists an $f_* \in C_*(f)$ with $av(f_*\colon C) = l(1)$.*

Proof. That $\mathscr{L}^* \geq \mathscr{L}(1)$ is obvious, since, if $f^*(x) \geq f(x)$ then $\mathscr{L}(1\colon f^*) \geq \mathscr{L}(1)$, $\mathscr{L}(1\colon f^*) = av(f^*\colon C)$ by Theorem 1.5, and therefore $av(f^*\colon C) \geq \mathscr{L}(1)$. We must now construct our minimizing $f^* \in C^*(f)$.

Let $\{\varepsilon_k\}$ be a sequence tending to 0, and $\{\lambda_k\}$ a sequence decreasing to 1. For each λ_k there is an $x_k > k$ such that

$$\frac{1}{\lambda_k x - x}\int_x^{\lambda_k x} f(t)dt \leq \mathscr{L}\left(\frac{1}{\lambda_k}\right) + \varepsilon_k \leq \mathscr{L}(1) + \varepsilon_k$$

for all $x \geq x_k$. We shall define a sequence $S_1, S_2, S_3, \cdots$ of finite geometric progressions, and denote by $S = s_1, s_2, s_3, \cdots$ the sequence of numbers we get when we write first all the terms of S_1, then those of S_2, those of S_3, and so on.

Let $S_1 = x_1\lambda_1, x_1\lambda_1^2, \cdots, x_1\lambda_1^{n_1}$, where, if we write the last term as $M_2 = x_1\lambda_1^{n_1}$, we choose n_1 large enough to make $M_2 > x_2$. We also write $M_1 = x_1$. In general, $S_k = M_k\lambda_k, M_k\lambda_k^2, \cdots, M_k\lambda_k^{n_k}$, where $M_{k+1} = M_k\lambda_k^{n_k}$, and n_k is chosen to make $M_{k+1} > x_{k+1}$.

The important properties of S are

$$s_n \longrightarrow \infty \tag{4.1}$$

$$\frac{s_{n+1}}{s_n} \longrightarrow 1 \tag{4.2}$$

$$\frac{1}{s_{n+1} - s_n}\int_{s_n}^{s_{n+1}} f(t)dt < \mathscr{L}(1) + \delta_n \,, \tag{4.3}$$

as $n \to \infty$. Here $\{\delta_n\}$ is the sequence whose first n_1 terms are ε_1, whose next n_2 terms are ε_2, and so on, so that $\delta_n \to 0$.

We now define $f^*(x)$. For $0 < x < \lambda_1 x_1$ put $f^*(x) = f(x)$ and then define $f^*(x)$ in each of the intervals $s_n \leq x < s_{n+1}$ by

$$f^*(x) = \begin{cases} \sup\limits_{0<y<\infty} f(y) \text{ for } s_n \leq x < s_n + \mu_n \\ f(x) \text{ for } s_n + \mu_n \leq x < s_{n+1}\,, \end{cases}$$

where we choose $\mu = \mu_n$ in the interval $0 \leq \mu < s_{n+1} - s_n$ so that

$$\mathscr{L}(1) - \delta_n \leq h(\mu) \leq \mathscr{L}(1) + \delta_n \,,$$

where

$$h(\mu) = \frac{1}{s_{n+1} - s_n}\int_{s_n}^{s_{n+1}} f^*(t)dt \,.$$

To see that such a choice of μ is possible, we first note that if $h(0) \geq \mathscr{L}(1) - \delta_n$ then we may choose $\mu = 0$, since by (4.3), $h(0) \leq \mathscr{L}(1) + \delta_n$. But if $h(0) < \mathscr{L}(1) - \delta_n$, we observe that $h(\mu)$ is a continuous function of μ with

$$h(s_{n+1} - s_n) = \sup_{0<y<\infty} f(y) \geq \mathscr{L}(1) > \mathscr{L}(1) - \delta_n$$

and we may therefore choose μ to make $h(\mu) = \mathscr{L}(1) - \delta_n$.

Our construction of f^* is now complete and it remains only to show that $\lim x^{-1}\int_0^x f^*(t)dt = \mathscr{L}(1)$. But it is easily verified that because of (4.1) and (4.2) we need only show that

$$\frac{1}{s_n}\int_{s_1}^{s_n} f^*(t)dt \longrightarrow \mathscr{L}(1) \,.$$

Now

$$\begin{aligned}\frac{1}{s_n}\int_{s_1}^{s_n} f^*(t)dt &= \frac{1}{s_n}\sum_{k=1}^{n-1}\int_{s_k}^{s_{k+1}} f^*(t)dt \\ &\leq \frac{1}{s_n}\sum_{k=1}^{n-1}(\mathscr{L}(1)+\delta_k)(s_{k+1}-s_k) \\ &= \mathscr{L}(1)\left(1-\frac{s_1}{s_n}\right)+\frac{1}{s_n}\sum_{k=1}^{n-1}\delta_k(s_{k+1}-s_k)\ ,\end{aligned}$$

with a similar inequality in the opposite direction. Now $s_1/s_n \to 0$, and an easy estimate shows that

$$\frac{1}{s_n}\sum_{k=1}^{n-1}\delta_k(s_{k+1}-s_k) \longrightarrow 0\ .$$

Hence $av(f^*: C) = \mathscr{L}(1)$ and we are done.

REMARK. We could similarly define Λ^*, the outer Abel mean, and λ_*, the inner Abel mean, and prove the analogue of Theorem 4.1, namely that $\Lambda^* = \Lambda(\infty)$ and $\lambda_* = \lambda(\infty)$. The proof would use Theorem 4.1, Theorem 3.1, and its corollary. It would be interesting to find a direct proof that $\Lambda^* = \Lambda(\infty)$ without either using these results or essentially reproving them.

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