A MULTIMOVE INFINITE GAME WITH LINEAR PAYOFF

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1. Introduction. Games can be classified in terms of the number of moves by each player—unimove or multimove—and in terms of the number of choices—finite or infinite—available at each move. The original work of von Neumann [2] on the existence and structure of solutions of games was, in effect, restricted to unimove finite games. Later, Ville [3] proved the existence of optimal strategies for unimove infinite games with continuous payoff function.

Except for games with perfect information, multimove finite games have been analyzed only very recently; and multimove infinite games with an arbitrary number of moves have hardly been touched upon.

In this paper, we analyze a multimove infinite game with a linear payoff function. The game is symmetric in every respect except that the initial conditions of the two players are different. We prove that one player has an optimal pure strategy and that the other player must randomize on the strategies. The optimal strategies and game value are derived.

Although this game had its origin in a military problem concerning allocation of resources among several tasks, it is presented here solely for its mathematical interest. A complete discussion of the military problem and its solution is given in [1].

2. Description of game. We shall analyze the following multimove zero-sum two-person game. At the nth move, or stage of the game, Blue has resources given by the state variable p_n and assigns a value to each of two tactical variables under his control, x_n and u_n, subject to the constraints

$$x_n \geq 0, \quad u_n \geq 0, \quad x_n + u_n \leq p_n . \tag{2.1}$$

At the same time, Red has resources given by the state variable q_n and controls the values of the tactical variables y_n and w_n, subject to the constraints

$$y_n \geq 0, \quad w_n \geq 0, \quad y_n + w_n \leq q_n . \tag{2.2}$$

Let us number the moves from the end of the game; i.e., the nth move means n moves to the end of the game. The state variables at the $(n-1)$-st move are defined by

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$$p_{n-1} = \max [0, p_n - \max (0, y_n - u_n)] ,$$
$$q_{n-1} = \max [0, q_n - \max (0, x_n - w_n)] . \tag{2.3}$$

The payoff to Blue is given by

$$\sum_{n=1}^{N} [(p_n - x_n - u_n) - (q_n - y_n - w_n)] , \tag{2.4}$$

where N is the number of moves in the game.

The play of the game proceeds by first making the Nth move, then the $(N-1)$-st move, $\cdots$, the second move, and the first move. An nth move of the game consists of a choice by Blue of x_n and u_n satisfying (2.1) and simultaneously a choice by Red of y_n and w_n satisfying (2.2). We assume that each player knows the manner in which the game proceeds from stage to stage; namely, each player has the information expressed by equations (2.3). We also assume that at each stage of the game both players know the state variables and the entire past history of the play; that is, at the nth move, both players know N, p_N, q_N, and also know x_i, u_i, y_i, w_i for $i = N, N-1, \cdots, n+2, n+1$. It follows that p_i, q_i, for $i = N, N-1, \cdots, n+1, n$, are known at the nth move.

The strategies of the game in normal form will be defined inductively on the number of moves. First, a strategy for Blue in a one-move game is a point $X_1 = (x_1, u_1)$, where $x_1 \geq 0, u_1 \geq 0$, and $x_1 + u_1 \leq p_1$. Similarly a strategy for Red in a one move game is a point $Y_1 = (y_1, w_1)$ where $y_1 \geq 0, w_1 \geq 0$, and $y_1 + w_1 \leq q_1$. Now let σ_N be a strategy for Blue in an N-move game. Of course, σ_N is a function of p_N and q_N. Then, in a game of $N+1$ moves, at the $(N+1)$-st move Blue chooses a point $X_N = (x_{N+1}, u_{N+1})$ in the triangle Δ_{N+1} defined by

$$x_{N+1} \geq 0, \quad u_{N+1} \geq 0, \quad x_{N+1} + u_{N+1} \leq p_{N+1} , \tag{2.5}$$

and simultaneously Red chooses a point $Y_{N+1} = (y_{N+1}, w_{N+1})$ in the triangle D_{N+1} defined by

$$y_{N+1} \geq 0, \quad w_{N+1} \geq 0, \quad y_{N+1} + w_{N+1} \leq q_{N+1} . \tag{2.6}$$

These choices yield the state variables p_N and q_N, by equations (2.3). A strategy σ_{N+1} for Blue in the $(N+1)$ move game is then defined as a choice X_{N+1} in Δ_{N+1} and a function Φ_N that associates, with each point $(x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1}) \equiv (X_{N+1}, Y_{N+1})$ in the product space $\Delta_{N+1} D_{N+1}$ a strategy σ_N in the N-move game. Thus σ_{N+1} can be written as

$$\sigma_{N+1} = (X_{N+1}; \Phi_N) = (x_{N+1}, u_{N+1}; \Phi_N) ,$$

where Φ_N assigns the strategy σ_N to the point $(x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1})$.

In a like manner, a strategy τ_{N+1} for Red in the $(N+1)$-move game is defined as a choice Y_{N+1} and a function $\bar{\psi}_N$ that associates, with each

(X_{N+1}, Y_{N+1}), a strategy τ_N in the N-move game. Thus we have

$$\tau_{N+1} = (Y_{N+1}; \bar{\psi}_N) = (y_{N+1}, w_{N+1}; \psi_N) .$$

3. Solution of game. The main result of this paper is the following:

THEOREM 1. *If $N = 1$ or 2, the value of the game is given by*

$$V_N(p_N, q_N) = N(p_N - q_N) .$$

Blue has an optimal pure strategy:

$$\bar{x}_m = \bar{u}_m = 0 \text{ for } m \leq N .$$

Red has an optimal pure strategy:

$$\bar{y}_m = \bar{w}_m = 0 \text{ for } m \leq N .$$

If $N \geq 3$, the value of the game is given by the $(N-2)$-piecewise-linear function.

$$V_N(p_N, q_N) = a_N^i p_N - b_N^i q_N, \qquad i = 1, 2, \cdots, N-2 ,$$

where the constants a_N^i and b_N^i are positive and monotone decreasing in i for fixed N; the value of the superscript i is determined by the ratio p_N/q_N. The optimal strategies for the two players are as follows:

(i) *At move $m = 1, 2$ (counting from the end) the players choose*

$$\bar{x}_m = \bar{u}_m = \bar{y}_m = \bar{w}_m = 0 .$$

(ii) *At move $m = 3$, if, $p_3 \geq q_3$, then Blue chooses $\bar{x}_3, \bar{u}_3$ such that*

$$q_3 \leq \bar{x}_3 \min\left(\frac{p_3 + q_3}{2}, \frac{3q_3}{2}\right),$$

$$\bar{u}_3 = \bar{x}_3 - q_3.$$

Red chooses either $\bar{y}_3 = q_3$ or $\bar{w}_3 = q_3$, each with probability 1/2.

(iii) *At the $(m+1)$-st move, where $3 \leq m \leq N-1$, if $p_{m+1} \geq q_{m+1}$, then the ratio p_{m+1}/q_{m+1} determines an integer $i, 1 \leq i \leq m-1$, and Blue chooses*

$$\bar{x}_{m+1} = \frac{(2m - a_m^i)p_{m+1} - (m - 2b_m^i)q_{m+1}}{m + b_m^i}$$

$$\bar{u}_{m+1} = p_{m+1} - \bar{x}_{m+1}, \qquad \text{for } i = 1, 2, \cdots, m-2 ,$$

and

$$\bar{x}_{m+1} = \left(2 - \frac{1}{b_m^{m-2}}\right)q_{m+1} ,$$

Table 1

Solution of game of N Moves, for $N \leq 8$

Move m	Ratio of State Variables p/q	Associated Superscript i	Value of Game $V(p, q)$	Blue's Optimal Choice		Red's Optimal Choice		
				$\bar{x}_m^i$	$\bar{u}_m^i$	α_m^i	β_m^i	γ_m^i
1	1.00 to ∞	1	$p-q$	0	0	0	0	1
2	1.00 to ∞	1	$2(p-q)$	0	0	0	0	1
3	1.00 to 2.00	1	$3(p-q)$	q	0	.50	.50	0
	2.00 to ∞	2	$3(p-q)$	$1.5q$	$.5q$	.50	.50	0
4	1.00 to 2.33	1	$4.5(p-q)$	$.5p+.5q$	$.5p-.5q$	.50	.50	0
	2.33 to ∞	2	$4.00p-3.33q$	$1.67q$	$.67q$	.33	.33	.33
5	1.00 to 1.70	1	$6.35p-6.35q$	$.41p+.59q$	$.59p-.59q$	.53	.47	0
	1.70 to 2.45	2	$5.82p-5.45q$	$.55p+.36q$	$.45p-.36q$	.45	.55	0
	2.45 to ∞	3	$5.00p-3.45q$	$1.70q$	$.75q$	.25	.30	.45
6	1.00 to 1.44	1	$8.39p-8.39q$	$.32p+.68q$	$.68p-.68q$	.56	.44	0
	1.44 to 1.78	2	$8.00p-7.83q$	$.40p+.56q$	$.60p-.56q$	.52	.48	0
	1.78 to 2.51	3	$7.04p-6.12q$	$.59p+.22q$	$.41p-.22q$	.41	.59	0
	2.51 to ∞	4	$6.00p-3.51q$	$1.71q$	$.80q$	.20	.29	.51
7	1.00 to 1.29	1	$10.50p-10.50q$	$.25p+.75q$	$.75p-.75q$	.58	.42	0
	1.29 to 1.53	2	$10.26p-10.19q$	$.29p+.70q$	$.71p-.70q$	.57	.43	0
	1.53 to 1.84	3	$9.54p-9.09q$	$.41p+.51q$	$.59p-.51q$	.50	.50	0
	1.84 to 2.55	4	$8.21p-6.64q$	$.63p+.11q$	$.37p-.11q$	.37	.63	0
	2.55 to ∞	5	$7.00p-3.55q$	$1.72q$	$.84q$	.17	.28	.55
8	1.00 to 1.25	1	$12.60p-12.60q$	$.20p+.80q$	$.80p-.80q$	.60	.40	0
	1.25 to 1.40	2	$12.48p-12.45q$	$.22p+.78q$	$.78p-.78q$	.59	.41	0
	1.40 to 1.59	3	$12.06p-11.86q$	$.28p+.69q$	$.72p-.69q$	.56	.44	0
	1.59 to 1.88	4	$11.03p-10.22q$	$.42p+.46q$	$.58p-.46q$	.49	.51	0
	1.88 to 2.58	5	$9.36p-7.07q$	$.66p+.01q$	$.34p-.01q$	.34	.66	0
	2.58 to ∞	6	$8.00p-3.58q$	$1.72q$	$.86q$	.14	.28	.58

$$\bar{u}_{m+1} = \left(1 - \frac{1}{m}\right) q_{m+1}, \qquad \text{for } i = m - 1 ,$$

where the constants a_m^i *and* b_m^i *are those associated with a game of length* m *and initial condition* p_m, q_m. *Red chooses either* y_{m+1} *or* $w_{m+1} = q_{m+1}$ *with probabilities* $\alpha_m^i = b_m^i/(m + b_m^i)$ *and* $\beta_m^i = m/(m + b_m^i)$, *respectively, for* $i = 1, 2, \cdots, m - 2$; *however, if* $i = m - 1$, *Red chooses* $y_{m+1} = q_{m+1}$ *with probability* $\alpha_m^i = 1/m$, *or* $w_{m+1} = q_{m+1}$ *with probability* $\beta_m^i = 1/b_m^{m-2}$, *or* $y_{m+1} = w_{m+1} = 0$ *with probability* $\gamma_m^i = 1 - 1/m - 1/(b_m^{m-2})$.

The proof of Theorem 1 will be carried out by induction on N, the number of moves of the game. In the course of this argument, recursive definitions will be given for the constants a_N^i and b_N^i. As an illustration of the theorem, Table 1 shows the solutions for games with eight or less moves.

4. A three-part sufficiency condition with mixed strategies. From the statement of the theorem, it is seen that mixed strategies will have to be introduced, at least for Red. However, it is sufficient to introduce a restricted class of mixed strategies in order to prove the theorem.

For a game of one move, a mixed strategy for Red is a probability distribution G_1 over D_1. Now suppose G_N is a mixed strategy for Red in a game of N moves and state variables p_N and q_N. Then a probability distribution g_{N+1} over D_{N+1} and a function ψ_{N+1} that associates $(x_{N+1}, u_{N+1}, y_{N+1}, w_{N+1})$ with G_N is a mixed strategy G_{N+1} in the $(N + 1)$-move game. Thus we may write the mixed strategy as

$$G_{N+1} = (g_{N+1}, \psi_N) .$$

Mixed strategies F_{N+1} for Blue are defined similarly by a distribution function f_{N+1} and a function ϕ_N, and can be written as

$$F_{N+1} = (f_{N+1}, \phi_N) .$$

Let $\hat{F}_{N+1}$ denote a mixed strategy for Blue in the $(N + 1)$-move game in which he selects $X_{N+1} = (x_{N+1}, u_{N+1})$ with probability 1 at the $(N + 1)$-st move. Let $\hat{G}_{N+1}$ denote a mixed strategy for Red in which he selects $Y_{N+1} = (y_{N+1}, w_{N+1})$ with probability 1 at the $(N + 1)$-st move.

Suppose that Theorem 1 is valid for games of length $N = n$. Let F_n^* and G_n^* be optimal strategies for Blue and Red, respectively. Let ϕ_n^*, ψ_n^* denote the functions that associate $(x_{n+1}, u_{n+1}, y_{n+1}, w_{n+1})$ with F_n^*, G_n^*, respectively. Suppose, further, that $p_{n+1} \geq q_{n+1}$ (from symmetry, it suffices to consider this case only).

The theorem asserts that at the $(n + 1)$-st move Blue's optimal choice is a point (x_{n+1}, u_{n+1}) that is determined by the ratio p_{n+1}/q_{n+1}. Denote this point by

$$X^*_{n+1} = (x^*_{n+1}, u^*_{n+1}) = (x^*_{n+1}(p_{n+1}, q_{n+1}), u^*_{n+1}(p_{n+1}, q_{n+1})) ,$$

and let $F^*_{n+1} = (X^*_{n+1}, \phi^*_n)$.

The theorem further asserts that Red chooses (y_{n+1}, w_{n+1}) to be

$$Y^{(1)}_{n+1} \equiv (q, 0),\ Y^{(2)}_{n+1} \equiv (0, q),\ Y^{(3)}_{n+1} \equiv (0, 0) ,$$

with probabilities $\alpha_{n+1}, \beta_{n+1}$, and $\gamma_{n+1} = (1 - \alpha_{n+1} - \beta_{n+1})$, respectively, the values of α_{n+1} and β_{n+1} being determined by the ratio p_{n+1}/q_{n+1}. Denote this distribution in Δ_{n+1} by $g^*_{n+1}(p_{n+1}, q_{n+1})$, and set $G^*_{n+1} = (g^*_{n+1}(p_{n+1}, q_{n+1}), \psi^*_n)$.

Define

$$L_{n+1}(X_{n+1}, Y_{n+1}) \equiv p_{n+1} - x_{n+1} - u_{n+1} - q_{n+1} + y_{n+1} + w_{n+1}$$

and

$$M_{n+1}(X_{n+1}, Y_{n+1}) \equiv L_{n+1}(X_{n+1}, Y_{n+1}) + V_n(p_n, q_n) ,$$

where p_n, q_n are obtained from p_{n+1}, q_{n+1} by means of (2.3) and the choices $x_{n+1}, u_{n+1}, y_{n+1}, w_{n+1}$. Let $E_N(F_N, G_N)$ denote the expected payoff of the game of length N if Blue chooses a strategy F_N and Red chooses a strategy G_N. Then

$$\begin{aligned} E_{n+1}(\hat{F}^*_{n+1}, \hat{G}_{n+1}) &= L_{n+1}(X^*_{n+1}, Y_{n+1}) + E_n(F^*_n, G_n) \\ &\geq M_{n+1}(X^*_{n+1}, Y_{n+1}), \text{ for all } Y_{n+1} , \end{aligned}$$

where $\psi_n(X^*_{n+1}, Y_{n+1}) = G_n$. Furthermore, we have

$$\begin{aligned} E_{n+1}(\hat{F}_{n+1}, G^*_{n+1}) &= \alpha_{n+1}[L_{n+1}(X_{n+1}, Y^{(1)}_{n+1}) + E_n(F_n, G^*_n)] \\ &\quad + \beta_{n+1}[L_{n+1}(X_{n+1}, Y^{(2)}_{n+1}) + E_n(F_n, G^*_n)] \\ &\quad + (1 - \alpha_{n+1} - \beta_{n+1})[L_{n+1}(X_{n+1}, Y^{(3)}_{n+1}) + E_n(F_n, G^*_n)] \\ &\leq \alpha_{n+1}M_{n+1}(X_{n+1}, Y^{(1)}_{n+1}) + \beta_{n+1}M_{n+1}(X_{n+1}, Y^{(2)}_{n+1}) \\ &\quad + (1 - \alpha_{n+1} - \beta_{n+1})M_{n+1}(X_{n+1}, Y^{(3)}_{n+1}) , \end{aligned}$$

for all X_{n+1}, where $\phi_n(X_{n+1}, Y_{n+1}) = F_n$.

The validity of the following lemma is now apparent.

LEMMA 1. *Given that Theorem* 1 *is true for* $N = n$, *to prove the theorem for* $N = n + 1$ *with initial conditions* $p_{n+1} \geq q_{n+1}$, *it suffices to exhibit the* X^*_{n+1}, α_{n+1}, *and* β_{n+1} *for which*

$$E_{n+1}(\hat{F}^*_{n+1}, G^*_{n+1}) = V_{n+1}(p_{n+1}, q_{n+1}) , \tag{4.1}$$

$$M_{n+1}(X^*_{n+1}, Y_{n+1}) \geq V_{n+1}(p_{n+1}, q_{n+1}) \tag{4.2}$$

for all Y_{n+1}, *and*

$$\begin{aligned} &\alpha_{n+1}M_{n+1}(X_{n+1}, Y^{(1)}_{n+1}) + \beta_{n+1}M_{n+1}(X_{n+1}, Y^{(2)}_{n+1}) \\ &\quad + (1 - \alpha_{n+1} - \beta_{n+1})M_{n+1}(X_{n+1}, Y^{(3)}_{n+1}) \leq V_{n+1}(p_{n+1}, q_{n+1}) \end{aligned} \tag{4.3}$$

for all X_{n+1}.

5. Some special cases. It will also be useful to tabulate the information given by equations (2.3). We may assume that $p_n \geq q_n$, whence $y_n - u_n > p_n$ is impossible and the equations (2.3) can be tabulated as follows, where *the subscript n is suppressed*:

Table 2
Determination of Values of State Variables p_{n-1} and q_{n-1}

Region in (X, Y) Space	Region Number	p_{-1}	q_{-1}
$y - u < 0, \quad x - w < 0$	I	p	q
$y - u < 0, \ 0 \leq x - w \leq q$	II	p	$q - x + w$
$y - u < 0, \quad x - w > q$	III	p	0
$0 \leq y - u \leq p, \quad x - w \leq 0$	IV	$p - y + u$	q
$0 \leq y - u \leq p, \ 0 \leq x - w \leq q$	V	$p - y + u$	$q - x + w$
$0 \leq y - u \leq p, \quad x - w > q$	VI	$p - y + u$	0

Games of length $N = 1, 2, 3$ will now be discussed. From the statement of the theorem, it is clear that separate arguments are needed for $N = 1, 2$, and for $N \geq 3$.

For $N = 1$, an examination of the payoff (2.4) shows that optimal play for Blue is to choose $x_1 = u_1 = 0$, and that optimal play for Red is to choose $y_1 = w_1 = 0$.

As a consequence of Lemma 1, for $N = 2$ it suffices to consider

$$M_2(X_2, Y_2) = p_2 - x_2 - u_2 - q_2 + y_2 + w_2 + (p_1 - q_1) ,$$

with $p_2 \geq q_2$. Using Table 2 and dropping the subscript 2, we may write this

$$M(X, Y) = \begin{cases} 2(p - q) - (x + u) + (y + w) & \text{in region I,} \\ 2(p - q) - u + w & \text{in region II,} \\ 2p - q - (x + u) + (y + w) & \text{in region III,} \\ 2(p - q) - x + w & \text{in region IV,} \\ 2(p - q) & \text{in region V,} \\ 2p - q - x + w, & \text{in region VI,} \end{cases}$$

where the region in the (X, Y) space for which each expression on the right is valid is that given in Table 2. It now follows that the optimal choices at the second move are

$$(\bar{x}, \bar{u}) = (0, 0) \text{ for Blue; } (\bar{y}, \bar{w}) = (0, 0) \text{ for Red;}$$

and that $V_2 = 2(p_2 - q_2)$. Thus the theorem is proved for $N = 2$.

For $N = 3$, it suffices to consider

$$M_3(X_3, Y_3) = p_3 - x_3 - u_3 - q_3 + y_3 + w_3 + 2(p_2 - q_2) ,$$

where $p_3 \geq q_3$. It follows from Table 2 that, dropping the subscript 3, we may write $M_3(X_3, Y_3)$ as

$$Ap - Bq + \varepsilon_1 x + \varepsilon_2 u + \varepsilon_3 y + \varepsilon_4 V, \tag{5.1}$$

where the values of $A, B, \varepsilon_i = \pm 1, i = 1, \cdots, 4$ depend on the region of the (X, Y) space. To prove the theorem for $N = 3$, we use (5.1) and Lemma 1 to verify that the choices given for the third move are actually optimal.

Games of length $N = 3$, and $N = 4$, as can be seen from Table 1, are somewhat transitional in character, and do not exhibit all the characteristics of games of arbitrary length. For $N = 5$, however, all of the characteristics of the game's structure become apparent. For the reader who wishes to gain further insight into the structure of the game, as well as to obtain a motivation for the general induction step, the computation of the functions $m_i(x, u) \equiv \min_Y M_i(X_i, Y_i)$, $i = 3, 4, 5$, is recommended. By way of illustration we give $m_5(x, u)$, valid for $1 \leq p/q \leq 7/3$. The authors found the construction of a diagram quite instructive, and also indicative of the form of $m_5(x, u)$ for $p/q \geq 7/3$. We have that $m_5(x, u)$ is given as follows, (where the subscript 5 is omitted):

(i) $5p - q - x - u$, if $x \geq 17q/10, u \geq 3q/4$;

(ii) $5p - 20q/3 + 7x/3 - u$, if $2q - 3p/7 \leq x \leq 17q/10, 5x - 6u \leq 4q$;

(iii) $11p/2 - 9q + 7x/2 - u$, if $q \leq x \leq 2q - 3p/7$, $9x - 8u \leq 10q - p$;

(iv) $11(p - q)/2 - u$, if $0 \leq x \leq q, 9u + 7x \geq 7q$, $14x + 24u \geq 3p + 11q$;

(v) $11p/2 - 9q + 7(x + u)/2$, if $x \geq 0, u \geq 0$, $9u + 7x \leq 7q, 7x + 3u \leq 10q - 3p$;

(vi) $5p + 22q/3 + 7x/3 + 3u$, if $x \leq q, u \geq 0$, $7x + 3u \geq 10q - 3p, 14x + 24u \leq 3p + 11q$;

(vii) $5p - 4q - x + 3u$ if $x \geq q, u \geq 0, 5x - 6u \geq 4q$, $9x - 8u \geq 10q - p$.

6. Definitions and properties of constants. The first step of the proof of Theorem 1 is to define the sequences $\{a_N^i\}$, $\{b_N^i\}$, $\{\lambda_N^i\}$. To this end, consider the following sequences defined in the manner and order indicated:

$$a_3^1 = 3, a_{n+1}^{n-1} = a_n^{n-2} + 1; n \geq 3; \tag{6.1}$$

$$b_3^1 = 3, b_{n+1}^{n-1} = 4 - \frac{1}{a_n^{n-2}} - \frac{1}{b_n^{n-2}}, n \geq 3; \tag{6.2}$$

$$b_n^0 = a_n^0 = 0, n \geq 3; \tag{6.3}$$

$$\left.\begin{aligned} a_{n+1}^i &= \frac{a_n^{n-2}(2b_n^i + a_n^i)}{b_n^i + a_n^{n-2}} \qquad (6.4)\\ b_{n+1}^i &= \frac{3a_n^{n-2}b_n^i}{b_n^i + a_n^{n-2}} \qquad (6.5)\end{aligned}\right\} \quad i \geq 1;\ n = i+2, i+3, \cdots;$$

$$\lambda_n^{n-1} = +\infty,\ n = 3, 4, 5, \cdots; \qquad (6.6)$$

$$\lambda_3^1 = 1, \qquad (6.7)$$

$$\lambda_{n+1}^{n-1} = 3 - \frac{1}{a_n^{n-2}} - \frac{1}{b_n^{n-2}} = b_{n+1}^{n-1} - 1,\ n = 3, 4, 5, \cdots;$$

$$\lambda_n^i = \frac{b_n^{i-1} - b_n^i}{a_n^{i-1} - a_n^i},\ n \geq 4; \quad i = 1, 2, 3, \cdots, n-3. \qquad (6.8)$$

The following properties of the foregoing sequences will be useful in the proof of the theorem; indications of the derivations of the properties are given after the listing:

$$a_n^{n-2} = n, \qquad n \geq 3; \qquad (6.9)$$

$$a_n^1 = b_n^1, \qquad n \geq 3; \qquad (6.10)$$

$$4 > b_{n+1}^{n-1} > b_n^{n-2} \geq 3, \qquad n \geq 3; \qquad (6.11)$$

$$a_n^i \geq b_n^i > 0, \qquad i = 1, 2, \cdots, n-2;\ n = 3, 4, 5, \cdots; \qquad (6.12)$$

$$n \leq a_n^i < 2n, \qquad i = 1, \cdots, n-2;\ n = 3, 4, 5, \cdots, ; \qquad (6.13)$$

$$\left.\begin{aligned} a_n^i &< a_n^{i-1}, \\ b_n^i &< b_n^{i-1}, \end{aligned}\right\} \quad n \geq 3;\ i = 2, 3, \cdots, n-2; \qquad (6.14)$$

$$\lambda_{n+1}^i = \frac{3n\lambda_n^i}{(2n - a_n^i)\lambda_n^i + (n + b_n^i)}, \quad n \geq 3;\ i = 1, 2, \cdots, n-2; \qquad (6.15)$$

$$3 > \lambda_{n+1}^{n-1} > \lambda_n^{n-2} \geq 2, \qquad n \geq 3; \qquad (6.16)$$

$$\lambda_n^{i+1} > \lambda_n^i, \qquad n \geq 3;\ i = 1, 2, \cdots, n-2; \qquad (6.17)$$

$$\lambda_{n+1}^i \leq \lambda_n^i, \qquad n \geq 3;\ i = 1, \cdots, n-2. \qquad (6.18)$$

Statements (6.9)–(6.11) follow from the definitions and from trivial inductive arguments.

Inequalities (6.12) and (6.13) are proved by induction on n, $n \geq i+2$, for each fixed i.

The monotonicity properties in (6.14) are established as follows. The monotonicity of $\{b_{n+1}^i\}$, $i = 1, \cdots, n-2$, follows, by induction on n, from the monotonicity of $\{b_n^i\}$, $i = 1, \cdots, n-2$. To show that $b_{n+1}^{n-2} > b_{n+1}^{n-1}$, it suffices to show that $b_{n+1}^{n-2} \geq 4$. This inequality, however, is obvious

from (6.5) and (6.11). When we compute $a_{n+1}^{i-1} - a_{n+1}^{i}$, $i = 2, 3, \cdots, n-2$, we obtain

$$\frac{n}{(n + b^{i-1})(n + b^{i})}[b^{i-1}(2n - a^{i}) - b^{i}(2n - a^{i-1}) + n(a^{i-1} - a^{i})] ,$$

where the subscript n is omitted. From the inductive hypothesis that $\{a_n^i\}$ is monotone decreasing in i, and from (6.13), the bracketed expression in turn is seen to be larger than the positive quantity

$$b^{i-1}(2n - a^{i-1}) - b^{i}(2n - a^{i-1}) = (b^{i-1} - b^{i})(2n - a^{i-1}) .$$

Furthermore, for $n \geq 3$, by (6.1), (6.4), and (6.11) we have

$$a_{n+1}^{n-2} - a_{n+1}^{n-1} = \frac{b_n^{n-2}(n-1) - 1}{b^{n-2} + n} \geq \frac{3(n-1) - n}{b^{n-2} + n} > 0 .$$

Thus the fact that $\{a_{n+1}^i\}$ is monotone decreasing in i is established.

To prove (6.15), we use (6.4) and (6.5) in the definition (6.8) of λ_{n+1}^i.

Inequality (6.16) is obvious, and (6.17) is established inductively as follows. Suppose $\lambda_n^{i+1} > \lambda_n^i$ for $i = 1, \cdots, n-1$. Consider

$$\frac{\lambda_{n+1}^{i+1}}{\lambda_{n+1}^{i}} = \frac{\lambda^{i+1}[(2n - a^{i})\lambda^{i} + (n + b^{i})]}{\lambda^{i}[(2n - a^{i+1})\lambda^{i+1} + (n + b^{i+1})]}, \qquad i = 1, 2, \cdots, n-3 ,$$

where the subscript n is omitted. To prove that this ratio exceeds 1, it therefore suffices to show that

$$n(\lambda^{i+1} - \lambda^{i}) + \lambda^{i+1}b^{i} - \lambda^{i}b^{i+1} > \lambda^{i}\lambda^{i+1}(a^{i} - a^{i+1}) .$$

Upon replacing λ^{i+1} on the right by its definition (6.8), we see that this last inequality is equivalent to $(n + b^i)(\lambda^{i+1} - \lambda^i) > 0$, the validity of which follows from the inductive hypothesis. The chain

$$\lambda_{n+1}^{n-2} = \frac{3n\lambda^{n-2}}{(\lambda^{n-2} + 1)n + b^{n-2}} = \frac{3n\lambda^{n-2}}{b^{n-2}(n+1)} \leq \frac{3\lambda^{n-2}}{b^{n-2}} < \lambda^{n-2} < \lambda_{n+1}^{n-1}$$

completes the proof of (6.17).

It is seen from (6.15) that to verify (6.18) it suffices to show that

$$\lambda_n^i \geq \frac{2n - b_n^i}{2n - a_n^i}, \; n \geq 3; \qquad i = 1, 2, \cdots, n-2 . \tag{6.19}$$

This inequality is shown to hold by induction on i, as follows. For $i = 1$, equality is obvious. Suppose that the inequality holds for $i = k$. It is then seen to hold for $i = k + 1$, $1 \leq i \leq n - 3$, by writing (omitting the subscript n)

$$\begin{aligned} \lambda^{k+1}(2n - a^{k+1}) - (2n - b^{k+1}) &= \lambda^{k+1}(2n - a^{k}) - (2n - b^{k}) \\ &\quad + \lambda^{k+1}(a^{k} - a^{k+1}) - (b^{k} - b^{k+1}) \\ &> \lambda^{k}(2n - a^{k}) - (2n - b^{k}) \geq 0 . \end{aligned}$$

7. Miscellaneous preparations. For $N \leq 3$, Theorem 1 has been proved in §5. The theorem will now be proved inductively for arbitrary $N > 3$. Suppose then that it has been established for $N = n \geq 3$. It is required to show that it holds for $N = n + 1$.

In order to simplify notation, for the remainder of the proof we shall omit the subscript n. Thus the symbol a_n^i, say, will be written merely as a^i, the symbol λ_{n+1}^{i-1} as λ_1^{i-1}, b_{n-1}^{i+1} as b_{-1}^{i+1}, etc.

From the symmetry of Theorem 1, it is clear that it suffices to consider the case $p_1 \geq q_1$. Define $X_1^* = X_1^*(p_1/q_1)$ as follows:

$$X_1^* = \bar{X}_1^i \equiv (\bar{x}_1^i, \bar{u}_1^i) \text{ if } \lambda_1^i \leq \frac{p_1}{q_1} \leq \lambda_1^{i+1}, \qquad i = 1, \cdots, n-1 ,$$

where

$$\begin{aligned} &\bar{x}_1^i = \frac{(2n - a^i)p_1 - (n - 2b^i)q_1}{b^i + n}, \ \bar{u}_1^i = p_1 - \bar{x}_1^i, \qquad i = 1, \cdots, n-2 , \\ &\bar{x}_1^{n-1} = \left(2 - \frac{1}{b^{n-2}}\right)q_1, \qquad \bar{u}_1^{n-1} = \left(1 - \frac{1}{n}\right)q_1 . \end{aligned} \tag{7.1}$$

Define $\alpha_1 = \alpha_1(p_1/q_1)$ and $\beta_1 = \beta_1(p_1/q_1)$ thus:

(a) if $\lambda_1^i \leq \dfrac{p_1}{q_1} \leq \lambda_1^{i+1}$, $\qquad 1 \leq i \leq n-2$,

then

$$\alpha_1 = \alpha_1^i = \frac{b^i}{n + b^i}, \qquad \beta_1 = \beta_1^i = \frac{n}{n + b^i} ; \tag{7.2}$$

(b) if $\dfrac{p_1}{q_1} \geq \lambda_1^{n-1}$,

then

$$\alpha_1^{n-1} = \frac{1}{n}, \qquad \beta_1^{n-1} = \frac{1}{b^{n-2}} .$$

Clearly, $\alpha_1^i > 0, \beta_1^i > 0$ for all i satisfying $i \leq 1 \leq n-1$; $\alpha^i + \beta^i = 1$ for all i satisfying $1 \leq i \leq n-2$; and $\alpha^i + \beta^i < 1$ for $i = n-1$. Thus α_1 and β_1 are probabilities. Lemma 2 will show that X_1^* is an admissible choice for Blue and will furnish some useful bounds for $\bar{x}_1^i$ and $\bar{u}_1^i$.

LEMMA 2. *The point X_1^* is an admissible choice of strategic variable for Blue. Furthermore, for all i satisfying $1 \leq i \leq n-1$, we have*

$$\begin{aligned} &q_1 \leq \bar{x}_1^i \leq 2q_1 , \\ &0 \leq \bar{u}_1^i < q_1 . \end{aligned} \tag{7.3}$$

Since X_1^* is defined piecewise, the first step in showing that X_1^* is admissible is to show that the pieces come together—i.e., that X_1^* is well defined. Substitution of $p_1 = \lambda_1^{n-1}q_1$ into the definition (7.1) of $\bar{x}_1^{n-2}$, and application of (6.7), show that if $p_1 = \lambda_1^{n-1}q_1$, then $\bar{x}_1^{n-2} = \bar{x}_1^{n-1}$. Similarly, it is seen that $\bar{u}_1^{n-2} = \bar{u}_1^{n-1}$ for $p_1 = \lambda_1^{n-1}q_1$. Substitution of $p_1 = \lambda_1^i q_1$, $i = 1, \cdots, n-2$, into the definition (7.1) of $\bar{x}_1^i$, and use of (6.15), show that, for $p_1 = \lambda_1^i q_1$,

$$\bar{x}_1^i = \left(2 - \frac{\lambda_1^i}{\lambda^i}\right)q_1 .$$

Substitution of $p_1 = \lambda_1^{i+1}q_1$, and use of (6.15), show that, for $p_1 = \lambda_1^{i+1}q_1$, $i = 1, \cdots, n-3$,

$$\bar{x}_1^i = \left(2 - \frac{\lambda_1^{i+1}}{\lambda^{i+1}}\right)q_1 .$$

Thus X_1^* is well defined, and for $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $i = 1, 2, \cdots, n-3$,

$$\left(2 - \frac{\lambda_1^i}{\lambda^i}\right)q_1 \leq \bar{x}_1^i \leq \left(2 - \frac{\lambda_1^{i+1}}{\lambda^{i+1}}\right)q_1 , \tag{7.4}$$

with equality on the left occurring for $p_1 = \lambda_1^i q_1$, and on the right for $p_1 = \lambda_1^{i+1}q_1$. Similarly, we obtain

$$\left(2 - \frac{\lambda_1^{n-2}}{\lambda^{n-2}}\right)q_1 \leq \bar{x}_1^{n-2} \leq \bar{x}_1^{n-1} . \tag{7.5}$$

Clearly, (7.3) implies $\bar{x}_1^i \geq 0$, $\bar{u}_1^i \geq 0$ for all i satisfying $1 \leq i \leq n-1$. By definition, $\bar{x}_1^i + \bar{u}_1^i = p_1$ for $1 \leq i \leq n-2$; and for $i = n-1$, we obtain $\bar{x}_1^{n-1} + \bar{u}_1^{n-1} = \lambda_1^{n-1}q_1 \leq p_1$. Thus the proof of the lemma depends on the proof of (7.3). For $i = n-1$, (7.3) is obvious. The inequality (6.18) implies $\lambda_1^i \leq \lambda^i$, whence it follows that $(2 - \lambda_1^i/\lambda^i)q_1 \geq q_1$ for $i = 1, 2, \cdots, n-2$. Clearly, the inequality $(2 - \lambda_1^i/\lambda^i)q_1 < 2q_1$ holds for $i = 1, \cdots, n-2$. Hence, we obtain $q_1 \leq \bar{x}_1^i < 2q_1$ for $i = 1, \cdots, n-2$.

To verify $0 \leq \bar{u}_1^i < q_1$ for $1 \leq i \leq n-2$, we substitute from the definition of $\bar{x}_1^i$ into the definition $\bar{u}_1^i = p_1 - \bar{x}_1^i$ and obtain

$$\bar{u}_1^i = \frac{(b^i + a^i - n)p_1 + (n - 2b^i)q_1}{(b^i + n)} , \qquad 1 \leq i \leq n-2 .$$

Hence, showing that $\bar{u}_1^i < q_1$ is equivalent to showing that $p_1/q_1 < 3b^i/(b^i + a^i - n)$. Since $p_1/q_1 \leq \lambda_1^{i+1}$, it clearly suffices to show that $\lambda_1^{i+1} < 3b^i/(b^i + a^i - n)$. If $i = n-2$ this inequality is obvious. If $i \leqq n-3$, then by using (6.15) and (6.8) this inequality becomes

$$b^i - \lambda^{i+1}(a^i - n) > 0 , \qquad i = 1, 2, \cdots, n-3 .$$

Since $a^{n-2} = n$, it follows from (6.8) that for $i = n-3$ the expression on

the left of this inequality is positive. Further, the left hand member is a monotone decreasing function of i, as can be seen by forming the difference of the left-hand side for superscripts $i-1$ and i and getting $(\lambda^{i+1}-\lambda^i)(a^i-n)>0$. This inequality follows from (6.13) and (6.17). Thus it follows that $\bar{u}_1^i < q_1$. The inequality $\bar{u}_1^i \geq 0$, follows from

$$(b^i + a^i - n)p_1 + (n - 2b^i)q_1 \geq (a^i - b^i)q_1 \geq 0\ ; \qquad 1 \leq i \leq n-2\ ,$$

and the lemma is proved.

It follows from Lemma 1 and the definitions of X_1^*, α_1, β_1, and from the inductive hypothesis to the effect that Theorem 1 is valid for $N=n$, that the validity of Theorem 1 will be established if (4.1), (4.2), and (4.3) of Lemma 1 are shown to hold for this X_1^*, α_1, β_1. The next sections of the proof will be devoted to the verification of these three statements.

In the course of this verification, it will be necessary to compute

$$\begin{aligned}(7.6)\qquad M_1(X_1, Y_1) &= L_1(X_1, Y_1) + V(p, q)\\ &= L_1(X_1, Y_1) + a^j p - b^j q\ , \qquad j = 1, \cdots, n-2\ ,\end{aligned}$$

explicitly in terms of p_1, q_1, X_1, Y_1, for certain choices of X_1, Y_1. For any given fixed initial condition (p_1, q_1), an integer $1 \leq i \leq n-1$ is determined by the inequality $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$. Each choice (X_1, Y_1) by the players falls into one of the six regions enumerated in Table 2[1] and determines p and q and hence an integer $1 \leq j \leq n-2$ via the inequality $\lambda^j \leq p/q \leq \lambda^{j+1}$. It is this integer j that appears in (7.6). Clearly, j is a function of p_1, q_1, X_1, and Y_1. In computing $M_1(X_1, Y_1)$ explicitly in terms of the initial conditions and choices X_1, Y_1, it will thus be necessary to take into account the region of the (X_1, Y_1) space and the superscript j. The statement "(X_1, Y_1) leads to case III^{j_0}" will mean that, for the initial condition being considered, the pair (X_1, Y_1) falls into region III of the (X_1, Y_1) space and the ratio p/q is such that $j = j_0$. At first glance, it appears that there are $6(n-2)$ cases. Actually, not all of these cases are possible; and since some specialization of X_1, Y_1 will occur, not all of the possible cases will be encountered.

8. Verification, first part of sufficiency condition. We divide the discussion into two cases.

Case 1. $p_1/q_1 \geq \lambda_1^{n-1}$.

For this initial condition, it is readily seen from Table 2 that $(X_1^*, Y_1^{(1)})$ leads to case VI^{n-2}, $(X_1^*, Y_1^{(2)})$ leads to case II^{n-2}, and $(X^*, Y_1^{(3)})$ leads to case III^{n-2}. It then follows by straightforward computation

[1] The table is given for passage from n to $n-1$, whereas the present situation is for passage from $n+1$ to n. The adjustment of subscripts is left to the reader.

and the definition of §6 that

$$M_1(X_1^*, Y_1^{(1)}) = M_1(X_1^*, Y_1^{(2)}) = M_1(X_1^*, Y_1^{(3)}) = V_1(p_1, q_1) . \tag{8.1}$$

The equality $E_1(\hat{F}^*, G_1^*) = V_1(p_1, q_1)$ now follows from

$$\begin{aligned} E_1(\hat{F}_1^*, G_1^*) &= \alpha_1[L_1(X_1^*, Y_1^{(1)}) + E(F^*, G^*)] \\ &\quad + \beta_1[L_1(X_1^*, Y_1^{(2)}) + E(F^*, G^*)] \\ &\quad + (1 - \alpha_1 - \beta_1)[L_1(X_1^*, Y_1^{(3)}) + E(F^*, G^*)] \\ &= \alpha_1 M_1(X_1^*, Y_1^{(1)}) + \beta_1 M_1(X_1^*, Y_1^{(2)}) \\ &\quad + (1 - \alpha_1 - \beta_1) M_1(X_1^*, Y_1^{(3)}) . \end{aligned}$$

Case 2. $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $\quad i = 1, 2, \cdots, n-2$.

For this initial condition, it is immediately seen from Table 2 that $(X_1^*, Y_1^{(1)})$ leads to case VI^{n-2}, while $(X_1^*, Y_1^{(2)})$ leads to case II^j for appropriate j. To determine the value of j, we first observe that $p/q = p_1/(2q_1 - \bar{x}_1^i)$. It follows from (7.4) and (7.5) that $j = i$. Straightforward computation and use of the definitions in §6 now show that

$$M_1(X_1^*, Y_1^{(1)}) = M_1(X_1^*, Y_1^{(2)}) = V_1(p_1, q_1) , \tag{8.2}$$

and hence (4.1) follows as before.

9. Verification, second part of sufficiency condition. Again, as in §8, we divide the discussion into two cases.

Case 1. $p_1/q_1 \geq \lambda_1^{n-1}$.

Since $\bar{x}_1^{n-2} \geq q_1$, (X_1^*, Y_1) can never fall into region I or IV of the (X_1, Y_1) space. Since (X_1^*, Y_1) lying in region V implies that

$$y_1 + w_1 \geq \bar{x}_1^{n-2} + \bar{u}_1^{n-2} - q_1 = \left(2 - \frac{1}{b^{n-2}} + \frac{1}{n}\right)q_1 > q_1 ,$$

this event is also impossible.

For Y_1 such that (X_1^*, Y_1) falls in region II, we have

$$\frac{p}{q} = \frac{p_1}{q_1 - \bar{x}_1^{n-2} + w_1} \geq \frac{p_1}{2q_1 - \bar{x}_1^{n-2}} = b^{n-2}\frac{p_1}{q_1} \geq \lambda^{n-2} ,$$

the last inequality following from (6.11) and (6.16). Hence, the superscript asociated with region II is $n - 2$, and

$$\begin{aligned} M_1(X_1^*, Y_1) &= (n+1)p_1 - (b^{n-2} - 1)\bar{x}_1^{n-2} - \bar{u}_1^{n-2} - (1 + b^{n-2})q_1 + y_1 \\ &\quad - (b^{n-2} - 1)w_1 \geq M_1(X_1^*, Y_1^{(2)}) = V_1(p_1, q_1) , \end{aligned}$$

the last inequality following from (8.1).

The only superscript that can be associated with region III or VI

is $n-2$. Having noted this, we easily see that for (X_1^*, Y_1) in region III, $\min_{Y_1} M_1(X_1^*, Y_1)$ occurs for $Y_1 = Y_1^{(3)}$, and so (4.2) follows from (8.1). Similarly, for Y_1 such that (X_1^*, Y_1) is in VI,

$$\min_{Y_1} M_1(X_1, Y_1) = M_1(X_1^*, Y_1^{(1)}) = V_1(p_1, q_1) .$$

Case 2. $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $i = 1, \cdots, n-2$.

Since $\bar{x}_1^i \geq q_1$, (X_1^*, Y_1) cannot fall in region I or IV. If (X_1^*, Y_1) is in region II, then we have

$$M_1(X_1^*, Y_1) = (a^j + 1)p_1 - (b^j + 1)q_1 + (b^j - 1)\bar{x}_1^i - \bar{u}_1^i + y_1 - (b^j - 1)w_1 ,$$

where j is determined by the ratio $p/q = p_1/(q_1 - \bar{x}_1^i + w_1)$. Clearly, j is a nonincreasing function of w_1 alone, $j = j(w_1)$. From (7.4), it follows that $j(q_1) = i$. For each j, the minimum of $M_1(X_1^*, Y_1)$ is achieved at a point $Y_1 = (y_1, w_1)$, where $y_1 = 0$ and where w_1 is the largest value of w such that (X_1^*, Y_1), $Y_1 = (0, w)$, leads to case II^j. Hence, by the continuity of $M_1(X_1^*, Y_1)$, it follows that the minimum of $M_1(X_1^*, Y_1)$, over all Y_1 such that (X_1^*, Y_1) is in region II, occurs at $Y_1^{(2)}$. Thus, using (8.2), we get

$$M_1(X_1^*, Y_1) \geq M_1(X_1^*, Y_1^{(2)}) = V_1(p_1, q_1) .$$

The only superscript possible for (X_1, Y_1) in region III is $n-2$; thus for Y_1 such that (X_1^*, Y_1) is in III, we have

$$M_1(X_1^*, Y_1) = np_1 - q_1 + (y_1 + w_1) .$$

The minimum of this expression over region III is assumed at $Y_1 = (0, 0)$ and is $np_1 - q_1$. Since we are considering the case $p_1/q_1 \leq \lambda_1^{i+1}$, the inequality

$$(b_1^i - 1) - \lambda_1^{i+1}(a_1^i - n) \geq 0 \tag{9.1}$$

implies the inequality $np_1 - q_1 \geq a_1^i p_1 - b_1^i q_1$, and so it suffices to establish (9.1) in order to verify (4.2). With the aid of (6.8) and (6.13), it is easy to see that the left-hand side of (9.1) is a decreasing function of the superscript. Furthermore, for $i = n-2$ it follows from (6.4), (6.5), and (6.7) that the left-hand side of (9.1) is zero, and so (9.1) is verified.

In the event that Y_1 is such that (X_1^*, Y_1) is in region V, we have

$$M_1(X_1^*, Y_1) = (a^j + 1)p_1 - (b^j + 1)q_1 + (b^j - 1)\bar{x}_1^i - w_1) + (a^j - 1)(\bar{u}_1^i - y_1) , \tag{9.2}$$

where j is determined by

$$\lambda^j \leq \frac{p_1 - y_1 + \bar{u}_1^i}{q_1 - \bar{x}_1^i + w_1} \leq \lambda^{j+1}, \qquad j = 1, 2, \cdots, n-2.$$

Since X_1^* is fixed, all questions concerned with determining which points in the (X_1^*, Y_1) plane lead to the different cases V^j are thus seen to devolve upon questions concerning point sets in the Y_1 plane. Clearly, the lines L^j,

$$y_1 = -\lambda^j w_1 + \lambda^j(\bar{x}_1^i - q_1) + p_1 + \bar{u}_1^i,$$

in the (y_1, w_1) plane form a finite pencil through the point $y_1 = p_1 + \bar{u}_1^i$, $w_1 = \bar{x}_1^i - q_1$. From the monotonicity properties of the sequence $\{\lambda^i\}$, it follows that for any fixed $y_1 = c$ with $c \leq p_1 + \bar{u}_1^i$ (and so particularly for $y_1 \leq q_1$), as one moves along $y_1 = c$ in the direction of increasing w_1, the lines L^j are encountered in order of decreasing j, with the line L^{n-2} being intercepted at a value of $w_1 > \bar{x}_1^i - q_1$. Thus, the sets in the

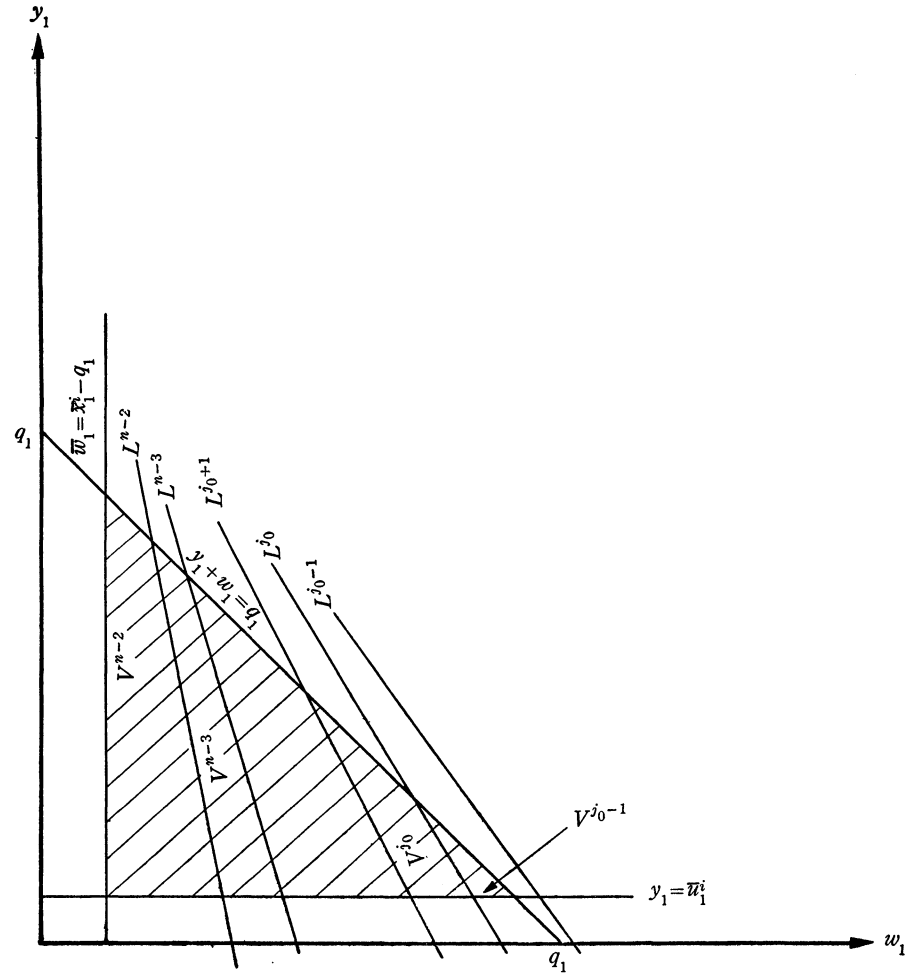

Fig. 1.

Y_1 plane giving rise to the various cases V^j are, in general, as indicated by the hatched regions in Fig. 5.

From the fact that $a^j \geq b^j$, it follows that the minimum of $M_1(X_1^*, Y_1)$, over each set V^j of Fig. 1, is assumed at the upper left-hand vertex of V^j. Hence, by the continuity of $M_1(X_1^*, Y_1)$ in Y_1, it follows that the minimum of $M_1(X_1^*, Y_1)$, over all Y_1 such that (X_1^*, Y_1) is in V, is achieved at $w_1 = \bar{x}_1^i - q_1$, $y_1 = 2q_1 - \bar{x}_1^i$. Substituting these values into (9.2) and using $\bar{x}_1^i + \bar{u}_1^i = p_1$, we see that the value of the minimum is $2n(p_1 - q_1)$. It remains to show that $2n(p_1 - q_1) \geq a_1^i p_1 - b_1^i q_1$. Since $p_1/q_1 \geq \lambda_1^i$, this inequality is implied by the inequality $\lambda_1^i \geq (2n - b_1^i)/(2n - a_1^i)$ which is established by induction in exactly the way that (6.19) was established.

Finally, the case in which (X_1^*, Y_1) lies in region VI must be considered. Examination of Table 2 shows that the only superscript possible is $j = n - 2$, and so

$$M(X_1^*, Y_1) = (n+1)p_1 - q_1 - \bar{x}_1^i + (n-1)\bar{u}_1^i - (n-1)y_1 + w_1 .$$

The minimum of this expression is assumed at $Y_1 = Y_1^{(1)}$. Since $M_1(X_1^*, Y_1^{(1)}) = V_1$, the proof of (4.2) is now concluded.

10. Verification, third part of sufficiency condition. The proof of (4.3) will clearly involve the computation of

$$\mathscr{M}(X_1) = \alpha_1 M_1(X_1, Y_1^{(1)}) + \beta_1 M_1(X_1 Y_1^{(2)}) + (1 - \alpha_1 - \beta_1) M_1(X_1, Y_1^{(3)}) . \tag{10.1}$$

Thus, for each X_1 it is necessary to know the case to which we are led by each of the points

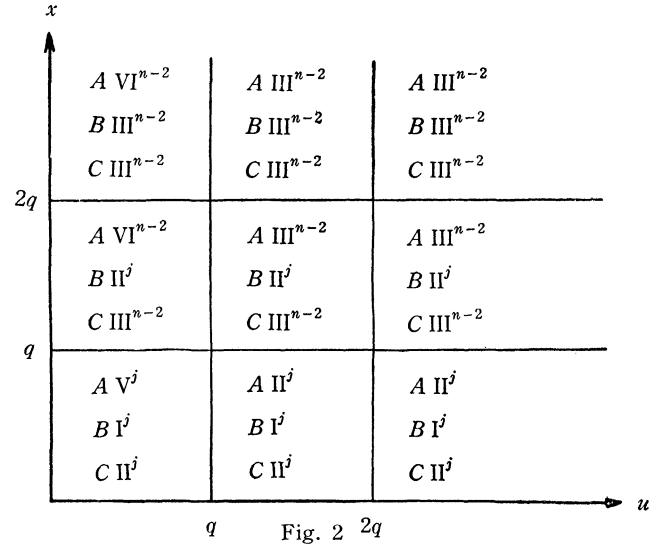

Fig. 2

$$A\colon\ (X_1, Y_1^{(1)}),\quad B\colon\ (X_1, Y_1^{(2)}),\quad C\colon\ (X_1, Y_1^{(3)})\,.$$

Part of this information is tabulated in Fig. 2. In this tabulation, a symbol such as, say, AV^j in a given region means that, for all X_1 in that region, $(X_1, Y_1^{(1)})$ leads to case V^j. In some instances, the value of j is indicated; in others, the determination of j will be made in the discussions of Cases 1 and 2 below.

Case 1. $\lambda_1^{n-1} \leq p_1/q_1$.

First, the values assumed by the superscripts j will be determined. In the case BI^j, Table 2 shows that $p_1/q_1 = p/q$; and since, by (6.16), $\lambda_1^{n-1} \geq \lambda^{n-2}$, it follows that $j = n-2$. Also, in the case BII^j, we have $j = n-2$, because

$$\frac{p}{q} = \frac{p_1}{2q_1 - x_1} \geq \frac{p_1}{q_1}\,, \qquad q_1 \leq x_1 \leq 2q_1\,.$$

Similarly, in AII^j and CII^j, we have $j = n-2$, because the relation $p/q = p_1/(q_1 - x_1) \geq p_1/q_1$ holds there. In the case AV^j, the value of j is determined by the ratio

$$\lambda^j \leq \frac{p}{q} = \frac{p_1 - q_1 + u_1}{q_1 - x_1} \leq \lambda^{j+1}\,, \qquad j = 1, \cdots, n-2\,.$$

The lines

$$l^j\colon\ \lambda^j x_1 + u_1 = \lambda^j q_1 - (p_1 - q_1)$$

form a finite pencil through the point $x_1 = q_1$, $u_1 = -(p_1 - q_1)$. It follows from the monotonicity of the λ's that if a line $u_1 = c$ with $c > -(p_1 - q_1)$ is traversed from $x_1 = q_1$ in the direction of decreasing x_1, then the lines of the pencil are encountered in order of decreasing j, with l^{n-2} being the first line encountered. Thus, the lines l^j divide the square $0 \leq x_1 \leq q_1$, $0 \leq u_1 \leq q_1$ into subregions over each of which a different superscript j

Table 3
Determination of Values of the coefficients R and S

Region of (x_1, u_1) plane		Region Number	R	S
x_1	u_1			
$2q_1 \leq x_1$	$u_1 \geq q_1$	(1)	-1	-1
$q_1 \leq x_1 \leq 2q_1$	$u_1 \geq q_1$	(2)	0	-1
$0 \leq x_1 \leq q_1$	$u_1 \geq q_1$	(3)	$b^{n-2} - 2$	-1
$2q_1 \leq x_1$	$u_1 \leq q_1$	(4)	-1	0
$q_1 \leq x_1 \leq 2q_1$	$u_1 \leq q_1$	(5)	0	0
$0 \leq x_1 \leq q_1$	$u_1 \leq q_1$	(6)	$\frac{(b^{n-2} - 1)b^j}{b^{n-2}} - 1$	$\frac{a^j}{n} - 1$

is applicable. The number of subregions depends on the ratio p_1/q_1. For sufficiently large values of this ratio, the entire square will have the superscript value $n-2$ associated with it. The important fact to be noted is that the region with superscript $n-2$ always exists and contains the line segment $x_1 = q_1$, $0 \leq u_1 \leq q_1$.

Clearly, $\mathscr{M}(X_1)$ is continuous and is of the form

$$\mathscr{M}(X_1) = F(p_1, q_1) + Rx_1 + Su_1 ,$$

where $F(p, q_1)$ is a step function on the (x_1, u_1) plane whose values are expressions involving the constants a^j, b^j and the initial conditions p_1, q_1. Its exact form is of no concern here. The coefficients R and S are also step functions on the (x_1, u_1) plane whose values involve the constants a^j, b^j. The information concerning R and S shown in Table 3 is easily obtained from Fig. 2, the preceding discussion, Table 2, and the definitions (7.2) of α_1 and β_1.

The superscript j in the entries for region (6) varies as the superscript in AV^j, and assumes the same values as the superscript in AV^j.

For each of the regions (1) through (6) of this table, the set of points at which the maximum of $\mathscr{M}(X_1)$ is achieved on that region is easily determined from the tabulated values of R and S in the region. It then follows from the continuity of $\mathscr{M}(X_1)$ that the maximum of $\mathscr{M}(X_1)$ is achieved at all points of the square $q_1 \leq x_1 \leq 2q_1$, $0 \leq u_1 \leq q_1$ In particular, it is achieved at $(\bar{x}_1^{n-2}, \bar{u}_1^{n-2})$, since by (7.3) this point is in the square. It now follows from (8.1) that

$$\mathscr{M}(X_1) \leq \mathscr{M}(X_1^*) = V_1(p_1, q_1) .$$

Case 2. $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $\qquad i = 1, 2, \cdots, n-2$

Under these initial conditions, Fig. 2 is modified from the outset as follows. Point C is eliminated since we have $\alpha_1^i + \beta_1^i = 1$ for $1 \leq i \leq n-2$; and the region $x_1 \geq 2q_1$, $u_1 \geq q_1$ need not be considered since $p_1/q_1 < 3$. In determining the superscript j and the modifications of Fig. 2, it will be convenient to distinguish two cases, namely $p_1/q_1 \geq 2$ and $p_1/q_1 \leq 2$.

Suppose that $p_1/q_1 \geq 2$. The superscript j in BII^j is determined by the ratio

$$\frac{p}{q} = \frac{p_1}{2q_1 - x_1} , \tag{10.2}$$

where $q_1 \leq x_1 \leq \min [2q_1, p_1 - u_1]$. Thus j is a nondecreasing step function of x_1 alone whose value at $x_1 = 2q_1$ is $n-q$, and whose jumps occur at

$$x_1 = 2q_1 - \frac{p_1}{\lambda^j} , \tag{10.3}$$

where j is such that $2q_1 - p_1/\lambda^j \geq q_1$. At the jump points, j is continuous from the right. Let j_0 denote the lowest value of the superscript j. This is clearly assumed at $x_1 = q_1$, and the defining relation for j_0 can be taken as

$$\lambda^{j_0} \leq \frac{p_1}{q_1} \lambda^{j_0+1} . \tag{10.4}$$

Since, by assumption and (6.18), $p_1/q_1 \leq \lambda_1^{i+1} < \lambda^{i+1}$, it follows that $j_0 \leq i$. It is also necessary to have some information concerning the superscript at $u_1 = q_1, x_1 = p_1 - q_1$. Substitution of this value of x_1 into (10.2) gives the quantity $p_1/(3q_1 - p_1)$. It can be shown that $p_1/(3q_1 - p_1) \leq \lambda^{i+1}$, and hence it follows that $j \leq i$ at the point $x_1 = p_1 - q_1, u_1 = q_1$.

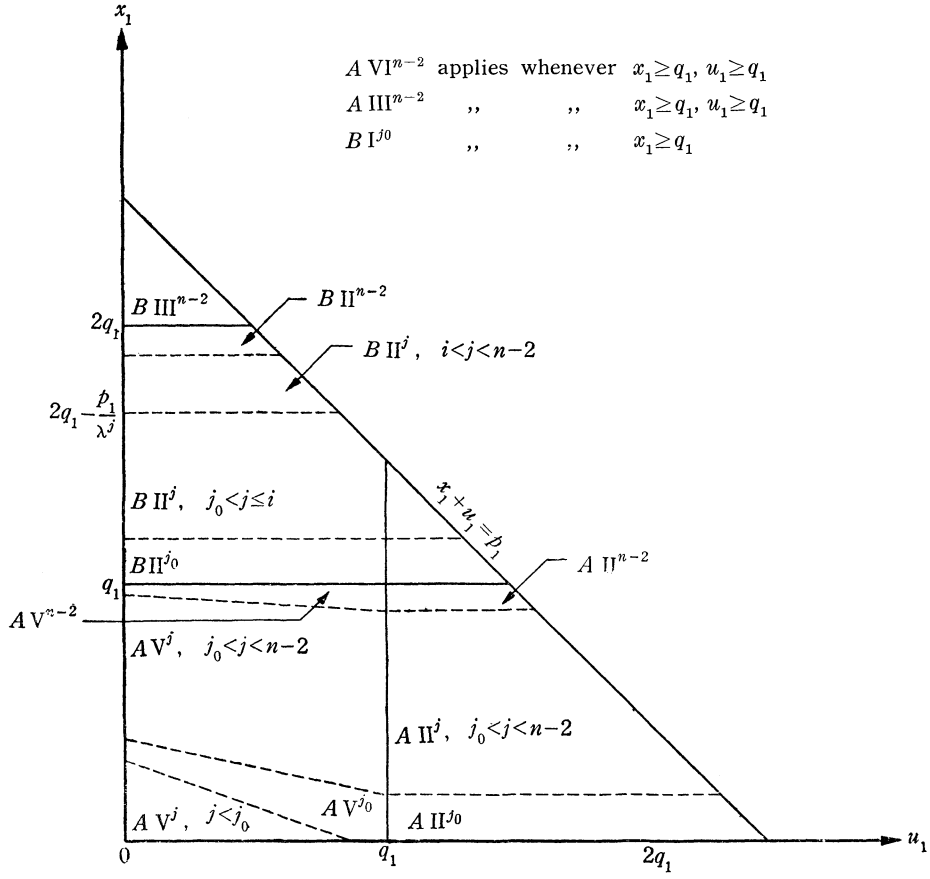

Fig. 3

The superscript j in BI^j is determined by the ratio p_1/q_1; and, in view of (10.4), this makes $j = j_0$. In AII^j the superscript is determined by $p/q = p_1/(q_1 - x_1)$. Thus j is an increasing step function of x_1 alone, having value j_0 at $x_1 = 0$, and $n - 2$ at $x_1 = q_1$. The remarks made under Case 1 concerning AV^j are applicable here, too. It is not difficult

to see that the lines l^j split up the square $0 \leq x_1 \leq q_1, 0 \leq u_1 \leq q_1$ as indicated in Fig. 3, which summarizes the foregoing discussion.

Suppose now that $p_1/q_1 \leq 2$. Most of the remarks concerning the superscript j in $B\mathrm{II}^j$ in the case $p_1/q_1 \geq 2$ are also valid here. Now, however, u_1 cannot exceed q_1 when $x_1 \geq q_1$, and so there is no need to discuss the point $u_1 = q_1, x_1 = p_1 - q_1$. If j_1 denotes the maximum value of the superscript j, it no longer need be true that $j_1 = n - 2$. However, the relation $j_1 \geqq i$ does hold. For, the maximum value of p_1/q_1 is $p_1/(2q_1 - p_1)$, and so the assertion $j_1 \geq i$ is equivalent to $p_1/(2q_1 - p_1) \geq \lambda^i$. This relation, however, is easily established.

As before, the superscript in $B\mathrm{I}^j$ takes on the value j_0. In $A\mathrm{II}^j$, it is readily seen that the superscript j is equal to j_0 at $x_1 = 0$ and increases to the maximum value of j_1. In $A\mathrm{V}^j$, the remarks made in the discussion of $p_1/q_1 \geq 2$ still hold, except that for $x_1 \geq p_1 - q_1$ the

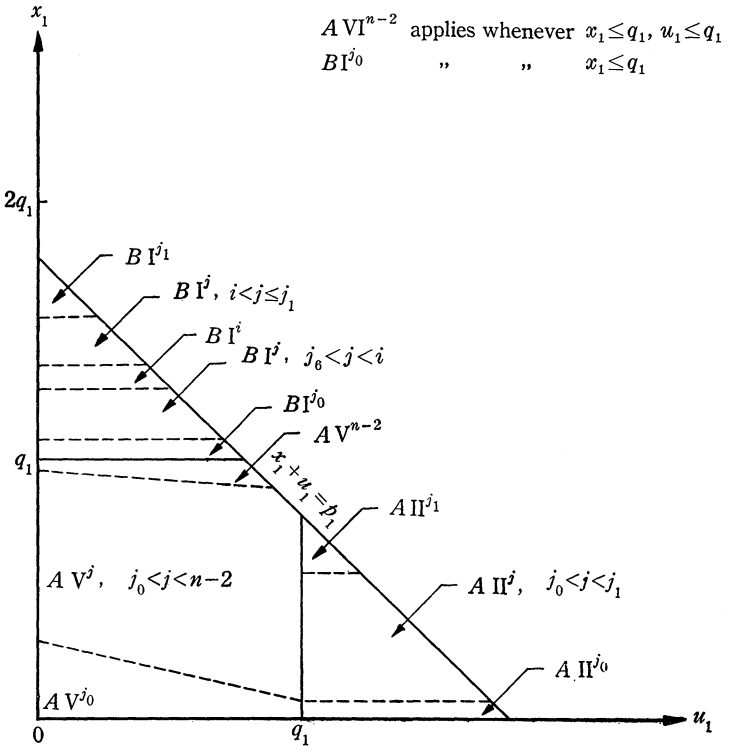

Fig. 4

regions are truncated by the line $x_1 + u_1 = p_1$. Furthermore, the smallest superscript involved in a truncated region is clearly j_1, and so is greater than i. This information is summarized in Fig. 4.

Regardless of whether $p_1 \geq 2q_1$, or $p_1 \leq 2q_1$, we may write

$$\mathscr{M}(X_1) = H(p_1, q_1) + Tx_1 + Uu_1 ,$$

where H, T, and U are step functions having values that depend on the choice of X_1, but do not involve the variables x_1 or u_1. The functional values of H, T, and U do involve the constants a^i, b^i, a^j, b^j, and those of H involve p_1 and q_1 in addition. The superscripts j, of course, are determined by X_1. The values of T and U are shown in Table 4, and the regions of constancy are indicated. Clearly, the regions of constancy of F coincide with those of T and U.

Table 4
Determination of Values of the coefficients T and U

Region of (x, u) Plane	Region Number	T	U	Remarks
$2q_1 \leq x_1$ $u_1 \leq q_1$	(1)	-1	$\frac{nb^i}{b^i + n} - 1$	Only applies if $p_1 \geq 2q_1$
$q_1 \leq x_1 \leq 2q_1$ $q_1 \leq u_1$	(2)	$\frac{nb^j}{b^i + n} - 1$	-1	Only applies if $p_1 \geq 2q_1$; exponent j varies as superscript in $B\text{II}^j$
$q_1 \leq x_1 \leq 2q_1$ $u_1 \leq q_1$	(3)	$\frac{nb^j}{b^i + n} - 1$	$\frac{nb^i}{b^i + n} - 1$	Exponent j varies as superscript in $B\text{II}^j$
$0 \leq x_1 \leq q_1$ $q_1 \leq u_1$	(4)	$\frac{b^i b^j}{b^i + n} - 1$	-1	Exponent j varies as superscript in $A\text{II}^j$
$0 \leq x_1 \leq q_1$ $u_1 \leq q_1$	(5)	$\frac{b^i b^j}{b^i + n} - 1$	$\frac{b^i a^j}{b^i + n} - 1$	Exponent j varies as superscript in $A\text{V}^j$

Suppose now that $p_1 \geq 2q_1$. Since $nb^i > n + b^i$, the maximum of $\mathscr{M}(X_1)$ over region (1) of Table 4 occurs at $x_1 = 2q_1$, $u_1 = p_1 - 2q_1$. Then the following four facts,

(a) $nb^i > b^i + n$,

(b) the b^j's are decreasing in j,

(c) the point $x_1 = p_1$, $u_1 = q_1$ lies in a set for which the superscript j in $B\text{II}^j$ does not exceed i, and

(d) the continuity of $\mathscr{M}(X_1)$, have the following implications: (a) the maximum of $\mathscr{M}(X_1)$ over region (2) and that part of region (3) lying below $x_1 = p_1 - q_1$ is attained at $x_1 = p_1 - q_1$, $u_1 = q_1$; (b) the maximum of $\mathscr{M}(X_1)$ over that part of region (2) lying above $x_1 = p_1 - q_1$ is attained at all points of the line $x_1 + u_1 = p_1$ that lie in the strip for which the superscript in $B\text{II}^j$ assumes the value i. Denote this set of points by $\mathscr{E}$. Again appealing to the continuity of $\mathscr{M}(X_1)$, we see that the maximum of $\mathscr{M}(X_1)$ over all admissible X_1 for which $x_1 \geq q_1$ is achieved on $\mathscr{E}$.

It is now asserted that X_1^* lies in $\mathscr{E}$. In view of (10.3), this is equivalent to showing that we have

$$2q_1 - \frac{p_1}{\lambda^i} \leq \bar{x}_1^i \leq 2q_1 - \frac{p_1}{\lambda^{i+1}}, \qquad i = 1, \cdots, n-2 .$$

The right-hand inequality follows immediately from (7.4) and the initial

conditions. The left-hand inequality follows from the definition (7.1) of $\bar{x}_1^i$, (6.15), and the initial conditions.

Thus, it has been proved that

$$\mathscr{M}(X_1^*) \leq \mathscr{M}(X_1) \tag{10.6}$$

for all X_1 such that $x_1 \geq q_1$. Since $\mathscr{M}(X_1^*) = V_1(p_1, q_1)$, to complete the proof of (4.3) in the case $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $p_1/q_1 \geq 2$, it is sufficient to show that (10.6) holds for all X_1 such that $x_1 \leq q_1$. From the form of S in region (5) of Table 4, it is clear that $\mathscr{M}(X_1)$ attains its maximum along the line $u_1 = q_1$ whenever $x_1 \leq q_1$. From the form of R in this region, it is clear that if $b^i b^{n-2} \geq b^i + n$, then the maximum of $\mathscr{M}(X_1)$ is attained at $(x_1, u_1) = (q_1, q_1)$. Hence, (10.6) follows for all X_1 in this event. On the other hand, if $b^i b^{n-2} < b^i + n$, then the maximum of $\mathscr{M}(X_1)$ will be attained at one of the points $x_1 = q_1 - p_1/\lambda^j \geq 0$, or $x_1 = 0$, of the line $u_1 = q_1$. In this event, it can be shown by lengthy computation that (10.6) holds for such X_1. Thus (4.3) is established for $\lambda_1^i \leq p_1/q_1 \leq \lambda_1^{i+1}$, $p_1 \geq 2q_1$. By similar methods, which will not be carried out here, (4.3) can be established for $p_1 \leq 2q_1$. Thus the validity of (4.3), and hence that of the theorem, is established.

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