NORMAL EXTENSIONS OF FORMALLY NORMAL OPERATORS

EARL A. CODDINGTON

1. Introduction. Let $\mathfrak{H}$ be a Hilbert space. If T is any operator in $\mathfrak{H}$ its domain will be denoted by $\mathfrak{D}(T)$, its null space by $\mathfrak{N}(T)$. A *formally normal* operator N in $\mathfrak{H}$ is a densely defined closed operator such that $\mathfrak{D}(N) \subset \mathfrak{D}(N^*)$, and $\|Nf\| = \|N^*f\|$ for all $f \in \mathfrak{D}(N)$. Intimately associated with such an N is the operator $\bar{N}$ which is the restriction of N^* to $\mathfrak{D}(N)$. The operator N is formally normal if and only if $\bar{N}$ is. A *normal operator* N in $\mathfrak{H}$ is a formally normal operator for which $\mathfrak{D}(N) = \mathfrak{D}(N^*)$; in this case $\bar{N} = N^*$. A densely defined closed operator N is normal if and only if $N^*N = NN^*$.[1]

Let N be formally normal in $\mathfrak{H}$. Since $\bar{N} \subset N^*$ we have $N \subset \bar{N}^*$, where $\bar{N}^* = (\bar{N})^*$. Thus we see that a closed symmetric operator is a formally normal operator such that $N = \bar{N}$, and a self-adjoint operator is a normal operator such that $N = \bar{N}$ $(= N^*)$. If a closed symmetric operator has a normal extension in $\mathfrak{H}$, this extension is self-adjoint. It is known that a closed symmetric operator may not have a self-adjoint extension in $\mathfrak{H}$. Necessary and sufficient conditions for such extensions were given by von Neumann.[2] However, until recently, conditions under which a formally normal operator N can be extended to a normal one in $\mathfrak{H}$ were known only for certain special cases.[3,4] Kilpi[5] considered the problem in terms of the real and imaginary parts of N. It is the purpose of this note to characterize the normal extensions of N in a manner similar to the von Neumann solution for the symmetric case.

If N_1 is a normal extension of a formally normal operator N in $\mathfrak{H}$, then it is easy to see that $N \subset N_1 \subset \bar{N}^*$, and $\bar{N} \subset N_1^* \subset N^*$. In Theorem 1 we describe $\mathfrak{D}(\bar{N}^*)$ and $\mathfrak{D}(N^*)$ for any two operators $N, \bar{N}$ satisfying $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. With the aid of this result a characterization of the normal extensions N_1 of a formally normal N in $\mathfrak{H}$ is given in Theorem 2. It is indicated in Theorem 3 how the domains of normal extensions

Received January 13, 1960. This work was supported in part by the National Science Foundation and the Office of Naval Research.

[1] See, e.g., B. v. Sz. Nagy, *Spektraldarstellung linearer Transformationen des Hilbertschen Raumes*, Ergeb. Math., **5** (1942), 33.

[2] Ibid; p. 39.

[3] Y. Kilpi, "Über lineare normale Transformationen im Hilbertschen Raum", Annales Academiae Scientiarum Fennicae, Series A-I, No. **154** (1953).

[4] R. H. Davis, "Singular normal differential operators", Technical Report No. 10, Department of Mathematics, University of California, Berkeley, Calif., (1955).

[5] Y. Kilpi, "Über das komplexe Momentenproblem", Annales Academiae Scientiarum Fennicae, Series A-I, No. **236** (1957).

can be described by abstract boundary conditions.

I would like to thank Ralph Phillips for instructive conversations during this work.

2. Domains.

THEOREM 1. *Let N, $\bar{N}$ be two closed densely defined operators in a Hilbert space $\mathfrak{H}$ such that $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. Then*

$$\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}\,, \qquad \mathfrak{D}(N^*) = \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}\,,$$

where $\mathfrak{M} = \mathfrak{N}(I + N^*\bar{N}^*)$, $\bar{\mathfrak{M}} = \mathfrak{N}(I + \bar{N}^*N^*)$. Here I is the identity operator, and the sums are direct sums.

Proof. Let N, $\bar{N}$ be any two closed densely defined operators in $\mathfrak{H}$ such that $N \subset \bar{N}^*$, $\bar{N} \subset N^*$. Then $(Nf, g) = (f, \bar{N}g)$ for all $f \in \mathfrak{D}(N)$, $g \in \mathfrak{D}(\bar{N})$. Define an operator $\mathscr{N}$ in the Hilbert space $\mathfrak{H}_2 = \mathfrak{H} \oplus \mathfrak{H}$ with domain $\mathfrak{D}(\mathscr{N})$ the set of all $\hat{f} = \{f_1, f_2\}$ with $f_1 \in \mathfrak{D}(N)$, $f_2 \in \mathfrak{D}(\bar{N})$, and such that $\mathscr{N}\hat{f} = \{\bar{N}f_2, Nf_1\}$. Then $\mathscr{N}$ is closed symmetric. Indeed $\mathfrak{D}(\mathscr{N})$ is dense in $\mathfrak{H} \oplus \mathfrak{H}$, and, if $\hat{f} = \{f_1, f_2\}$, $\hat{g} = \{g_1, g_2\}$ are in $\mathfrak{D}(\mathscr{N})$, we have

$$(\mathscr{N}\hat{f}, \hat{g}) = (\bar{N}f_2, g_1) + (Nf_1, g_2) = (f_1, \bar{N}g_2) + (f_2, Ng_1) = (\hat{f}, \mathscr{N}\hat{g})\,.$$

Since N and $\bar{N}$ are closed, so is $\mathscr{N}$. The adjoint $\mathscr{N}^*$ of $\mathscr{N}$ has domain $\mathfrak{D}(\mathscr{N}^*)$ the set of all $\hat{g} = \{g_1, g_2\}$ such that $g_1 \in \mathfrak{D}(\bar{N}^*)$, $g_2 \in \mathfrak{D}(N^*)$; and $\mathscr{N}^*\hat{g} = \{N^*g_2, \bar{N}^*g_1\}$.

We now show that the defect spaces of $\mathscr{N}$, namely,

$$\mathfrak{E}(+i) = \{\hat{\phi} \in \mathfrak{D}(\mathscr{N}^*) : \mathscr{N}^*\hat{\phi} = i\hat{\phi}\}\,,$$
$$\mathfrak{E}(-i) = \{\hat{\psi} \in \mathfrak{D}(\mathscr{N}^*) : \mathscr{N}^*\hat{\psi} = -i\hat{\psi}\}\,,$$

have the same dimension. We have $\hat{\phi} = \{\phi_1, \phi_2\} \in \mathfrak{E}(+i)$ if and only if $\phi_1 \in \mathfrak{D}(\bar{N}^*)$, $\phi_2 \in \mathfrak{D}(N^*)$, $N^*\phi_2 = i\phi_1$, $\bar{N}^*\phi_1 = i\phi_2$. The latter is true if and only if $N^*(-\phi_2) = -i\phi_1$, $\bar{N}^*\phi_1 = -i(-\phi_2)$. Thus we see that the unitary map $\mathscr{U}$ of $\mathfrak{H}_2$ onto itself given by $\mathscr{U}\{f_1, f_2\} = \{f_1, -f_2\}$ carries $\mathfrak{E}(-i)$ onto $\mathfrak{E}(+i)$ in an isometric way. This proves $\dim \mathfrak{E}(+i) = \dim \mathfrak{E}(-i)$.

We note that $\{\phi_1, \phi_2\} \in \mathfrak{E}(+i)$ if and only if $\phi_1 \in \mathfrak{D}(N^*\bar{N}^*)$, $(I + N^*\bar{N}^*)\phi_1 = 0$, and $\phi_2 = -i\bar{N}^*\phi_1$. Alternatively $\{\phi_1, \phi_2\} \in \mathfrak{E}(+i)$ if and only of $\phi_2 \in \mathfrak{D}(\bar{N}^*N^*)$, $(I + \bar{N}^*N^*)\phi_2 = 0$, and $\phi_1 = -iN^*\phi_2$. Thus we see that the algebraic dimensions of the spaces $\mathfrak{M} = \mathfrak{N}(I + N^*\bar{N}^*)$, $\bar{\mathfrak{M}} = \mathfrak{N}(I + \bar{N}^*N^*)$, $\mathfrak{E}(+i)$, and $\mathfrak{E}(-i)$ are all the same. Further it is easy to see that $\bar{N}^*$ maps $\mathfrak{M}$ one-to-one onto $\bar{\mathfrak{M}}$, the inverse mapping being $-N^*$ restricted to $\bar{\mathfrak{M}}$.

Since $\dim \mathfrak{E}(+i) = \dim \mathfrak{E}(-i)$ the operator $\mathscr{N}$ has self-adjoint extensions in $\mathfrak{H}_2$. They are in a one-to-one correspondence with the isometries of $\mathfrak{E}(-i)$ onto $\mathfrak{E}(+i)$. If $\mathscr{S}$ is a self-adjoint extension of $\mathscr{N}$ there is a unique isometry $\mathscr{V}$ of $\mathfrak{E}(-i)$ onto $\mathfrak{E}(+i)$ such that $\mathfrak{D}(\mathscr{S}) = \mathfrak{D}(\mathscr{N}) + (\mathscr{I} - \mathscr{V})\mathfrak{E}(-i)$, where $\mathscr{I}$ is the identity operator on $\mathfrak{H}_2$. Let us consider that self-adjoint extension $\mathscr{S}$ of $\mathscr{N}$ determined in this way by the isometry $-\mathscr{U}$ restricted to $\mathfrak{E}(-i)$. Then we have $\hat{h} \in \mathfrak{D}(\mathscr{S})$ if and only if $\hat{h} = \hat{f} + \hat{\psi} + \mathscr{U}\hat{\psi}$, for some $\hat{f} \in \mathfrak{D}(\mathscr{N})$, $\hat{\psi} \in \mathfrak{E}(-i)$. If $\hat{h} = \{h_1, h_2\}$, $\hat{f} = \{f_1, f_2\}$, $\hat{\psi} = \{\psi_1, \psi_2\}$, this means $h_1 = f_1 + 2\psi_1$, $h_2 = f_2$, where $f_1 \in \mathfrak{D}(N)$, $\psi_1 \in \mathfrak{M}$, $f_2 \in \mathfrak{D}(\bar{N})$. Thus $\mathfrak{D}(\mathscr{S})$ is the set of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N) + \mathfrak{M}$, $h_2 \in \mathfrak{D}(\bar{N})$. Now the operator $\mathscr{S}_1$ with domain all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(\bar{N}^*)$, $h_2 \in \mathfrak{D}(\bar{N})$, and such that $\mathscr{S}_1\{h_1, h_2\} = \{\bar{N}h_2, \bar{N}^*h_1\}$, is readily seen to be a self-adjoint operator in $\mathfrak{H}_2$ satisfying $\mathscr{N} \subset \mathscr{S} \subset \mathscr{S}_1 \subset N^*$. Hence $\mathscr{S} = \mathscr{S}_1$, and we see that $\mathfrak{D}(\bar{N}^*) = \mathfrak{D}(N) + \mathfrak{M}$. The sum is a direct one, for if $f \in \mathfrak{D}(N) \cap \mathfrak{M}$, $0 = (I + N^*\bar{N}^*)f = f + N^*Nf$ implying $0 = (f + N^*Nf, f) = \|f\|^2 + \|Nf\|^2$, or $f = 0$.

A similar argument shows that the self-adjoint extension $\mathscr{S}$ of $\mathscr{N}$ determined by the isometry $\mathscr{V}$ equal to $\mathscr{U}$ restricted to $\mathfrak{E}(-i)$ has domain the set of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N)$, $h_2 \in \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$. This operator is equal to the self-adjoint extension of $\mathscr{N}$ having domain the set of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N)$, $h_2 \in \mathfrak{D}(N^*)$, implying that $\mathfrak{D}(N^*) = \mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$, a direct sum. This completes the proof of Theorem 1.

Note added in proof. The results of Theorem 1 can be obtained more directly, although some of the discussion given in the proof above is required for our proof of Theorem 2. Let $\mathfrak{G}(T)$ denote the graph of an operator T. If A, B are any two closed operators with dense domain, and $A \subset B$, then it is easy to see that $\mathfrak{G}(B) \ominus \mathfrak{G}(A)$ is the set of all $\{u, Bu\} \in \mathfrak{G}(B)$ such that $u \in \mathfrak{N}(I + A^*B)$. Since

$$\mathfrak{G}(B) = \mathfrak{G}(A) \oplus [\mathfrak{G}(B) \ominus \mathfrak{G}(A)],$$

we have $\mathfrak{D}(B) = \mathfrak{D}(A) + \mathfrak{N}(I + A^*B)$, a direct sum. This implies Theorem 1.

3. Normal extensions.

THEOREM 2. *If N_1 is a normal extension of a formally normal operator N in a Hilbert space $\mathfrak{H}$, then there exists a unique linear map W of $\mathfrak{M}$ onto itself satisfying*

(i) $W^2 = I$,

(ii) $\|\phi\|^2 + \|\bar{N}^*\phi\|^2 = \|W\phi\|^2 + \|\bar{N}^*W\phi\|^2$, $(\phi \in \mathfrak{M})$,

(iii) $(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M}$,

(iv) $\|\bar{N}^*(I - W)\phi\| = \|N^*(I - W)\phi\|$, $(\phi \in \mathfrak{M})$.

In terms of W we have

$$\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}\,, \quad N_1 f = \bar{N}^* f\,, \qquad (f \in \mathfrak{D}(N_1))\,. \tag{1}$$

Conversely, if W is any linear map of $\mathfrak{M}$ onto $\mathfrak{M}$ satisfying (i)—(iv) *above, then the operator N_1 defined by* (1) *is a normal extension of N in $\mathfrak{H}$.*

REMARKS. Condition (i) implies that $P_1 = (1/2)(I + W)$ and $P_2 = (1/2)(I - W)$ are projections (not necessarily orthogonal) in $\mathfrak{M}$, and $\mathfrak{M}$ is the direct sum of $\mathfrak{M}_1 = P_1\mathfrak{M}$ and $\mathfrak{M}_2 = P_2\mathfrak{M}$. If $\phi \in \mathfrak{M}$, then $\phi \in \mathfrak{M}_1$ if and only if $W\phi = \phi$, and $\phi \in \mathfrak{M}_2$ if and only if $W\phi = -\phi$.

Condition (ii) implies that if $\phi, \phi' \in \mathfrak{M}$ then

$$(\phi, \phi') + (\bar{N}^*\phi, \bar{N}^*\phi') = (W\phi, W\phi') + (\bar{N}^* W\phi, \bar{N}^* W\phi') .$$

If $\phi \in \mathfrak{M}_1$, $\phi' \in \mathfrak{M}_2$ we see that $(\phi, \phi') + (\bar{N}^*\phi, \bar{N}^*\phi') = 0$, which means that the graph of $\bar{N}^*$ restricted to $\mathfrak{M}_1$ is orthogonal to the graph of $\bar{N}^*$ restricted to $\mathfrak{M}_2$.

Since $\bar{N}^*$ is one-to-one from $\mathfrak{M}$ onto $\bar{\mathfrak{M}}$, condition (iii) implies that $\mathfrak{M}_2 = \bar{N}^*\mathfrak{M}_1 \subset \mathfrak{M} \cap \bar{\mathfrak{M}}$, and $\mathfrak{M}_2$ has the same algebraic dimension as $\mathfrak{M}_1$. In particular the dimension of $\mathfrak{M}$ must be even.

Proof of Theorem 2. Let N_1 be a normal extension of the formally normal operator N in $\mathfrak{H}$. Then we have $N \subset N_1 \subset \bar{N}^*$, $\bar{N} \subset N_1^* \subset N^*$. Let the operator $\mathscr{N}_1$ in $\mathfrak{H}_2$ be defined with domain all $\{h_1, h_2\}$ such that $h_1 \in \mathfrak{D}(N_1)$, $h_2 \in \mathfrak{D}(N_1^*)$, and so that $\mathscr{N}_1\{h_1, h_2\} = \{N_1^*h_2, N_1h_1\}$. Then it is easily seen that $\mathscr{N}_1$ is a self-adjoint extension of the operator $\mathscr{N}$ defined in the proof of Theorem 1.

Let $\mathscr{N}_1$ be *any* self-adjoint extension of $\mathscr{N}$, and let $\mathscr{V}$ be the unique isometry of $\mathfrak{E}(-i)$ onto $\mathfrak{E}(+i)$ such that $\mathfrak{D}(\mathscr{N}_1) = \mathfrak{D}(\mathscr{N}) + (\mathscr{I} - \mathscr{V})\mathfrak{E}(-i)$. Then we may write $\mathscr{V} = \mathscr{W}\mathscr{U}$, where $\mathscr{U}$ is the isometry defined on $\mathfrak{E}(-i)$ to $\mathfrak{E}(+i)$ by $\mathscr{U}\{\psi_1, \psi_2\} = \{\psi_1, -\psi_2\}$, and $\mathscr{W}$ is a unitary map of $\mathfrak{E}(+i)$ onto itself. For $\{\phi_1, \phi_2\} \in \mathfrak{E}(+i)$ let $\mathscr{W}\{\phi_1, \phi_2\} = \{\chi_1, \chi_2\}$. Then $\phi_1, \chi_1 \in \mathfrak{M}$ and $\phi_2 = -i\bar{N}^*\phi_1$, $\chi_2 = -i\bar{N}^*\chi_1$. Define the map W of $\mathfrak{M}$ into $\mathfrak{M}$ by $W\phi_1 = \chi_1$. Then W is linear, and since $\mathscr{W}$ is unitary, W is onto, and

$$\| \{\phi, -i\bar{N}^*\phi\} \|^2 = \| \{W\phi, -i\bar{N}^* W\phi\} \|^2 , \qquad (\phi \in \mathfrak{M}) ,$$

or

$$\| \phi \|^2 + \| \bar{N}^*\phi \|^2 = \| W\phi \|^2 + \| \bar{N}^* W\phi \|^2 , \qquad (\phi \in \mathfrak{M}) . \tag{2}$$

Conversely, suppose W is a linear map of $\mathfrak{M}$ onto $\mathfrak{M}$ satisfying (2). Then for $\hat{\phi} = \{\phi, -i\bar{N}^*\phi\} \in \mathfrak{E}(+i)$ define $\mathscr{W}\hat{\phi} = \{W\phi, -i\bar{N}^* W\phi\}$. Then $\mathscr{W}$ maps $\mathfrak{E}(+i)$ onto $\mathfrak{E}(+i)$ and (2) implies that $\mathscr{W}$ is unitary. Thus we see that the self-adjoint extensions $\mathscr{N}_1$ of $\mathscr{N}$ are in a one-to-one correspondence with the linear maps W of $\mathfrak{M}$ onto $\mathfrak{M}$ satisfying (2). We have $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathscr{N}_1)$ if and only if $\hat{h}$ can be represented in the form $\hat{h} = \hat{f} + (\mathscr{I} - \mathscr{W}\mathscr{U})\hat{\psi}$, where $\hat{f} = \{f_1, f_2\} \in \mathfrak{D}(\mathscr{N})$, $\hat{\psi} = \{\phi, i\bar{N}^*\phi\} \in \mathfrak{E}(-i)$. This means $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi$, where $f_1 \in \mathfrak{D}(\mathscr{N})$, $f_2 \in \mathfrak{D}(\bar{N})$, $\phi \in \mathfrak{M}$.

The self-adjoint extension $\mathscr{N}_1$ arising from the normal extension N_1 of N has the property that if $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathscr{N}_1)$ then so does $\mathscr{P}_1\hat{h} = \{h_1, 0\}$. It will now be shown that a self-adjoint extension $\mathscr{N}_1$ of $\mathscr{N}$ has this property if and only if the W corresponding to $\mathscr{N}_1$ satisfies $W^2 = I$. First suppose $\mathscr{P}_1\hat{h} \in \mathfrak{D}(\mathscr{N}_1)$ for all $\hat{h} \in \mathfrak{D}(\mathscr{N}_1)$. Letting $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi$ as above, we see that this implies that there exist elements $f_1' \in \mathfrak{D}(N)$, $f_2' \in \mathfrak{D}(\bar{N})$, $\phi' \in \mathfrak{M}$, such that

$$\begin{aligned} f_1 + (I - W)\phi &= f_1' + (I - W)\phi' , \\ 0 &= f_2' + i\bar{N}^*(I + W)\phi' . \end{aligned}$$

Since $\mathfrak{D}(N) + \mathfrak{M}$ and $\mathfrak{D}(\bar{N}) + \bar{\mathfrak{M}}$ are direct sums these equations imply that $f_1 = f_1'$, $(I - W)\phi = (I - W)\phi'$, $f_2' = 0$, and $\bar{N}^*(I + W)\phi' = 0$. The last equation implies $(I + W)\phi' = 0$ since $\bar{N}^*$ is one-to-one from $\mathfrak{M}$ to $\bar{\mathfrak{M}}$. Thus we have

$$\begin{aligned} \phi' + W\phi' &= 0 , \\ \phi' - W\phi' &= \phi - W\phi , \end{aligned} \tag{3}$$

from which results $2\phi' = (I - W)\phi$. Returning to the first equation in (3) we obtain $(I + W)(I - W)\phi = (I - W^2)\phi = 0$ for all $\phi \in \mathfrak{M}$, showing that $W^2 = I$. Conversely, suppose $W^2 = I$ on $\mathfrak{M}$. Then if $\hat{h} = \{h_1, h_2\} \in \mathfrak{D}(\mathscr{N}_1)$, $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi$, define $\phi' = (1/2)(I - W)\phi$. Then equations (3) will be valid, implying that

$$\begin{aligned} f_1 + (I - W)\phi &= f_1 + (I - W)\phi' , \\ 0 &= 0 + i\bar{N}^*(I + W)\phi' , \end{aligned}$$

which shows that $\mathscr{P}_1\hat{h} = \{h_1, 0\} \in \mathfrak{D}(\mathscr{N}_1)$.

If $\mathscr{N}_1$ is any self-adjoint extension of $\mathscr{N}$ for which $W^2 = I$, then $\mathfrak{D}(\mathscr{N}_1)$ consists of those $\{h_1, h_2\}$ such that $h_1 = f_1 + (I - W)\phi$, $h_2 = f_2 + i\bar{N}^*(I + W)\phi'$, for some $f_1 \in \mathfrak{D}(N)$, $f_2 \in \mathfrak{D}(\bar{N})$, and $\phi, \phi' \in \mathfrak{M}$. The point is that ϕ and ϕ' need not now be the same element. Indeed, if h_1, h_2 have such representations let $\phi'' = (1/2)(I - W)\phi + (1/2)(I + W)\phi'$. Then $(I - W)\phi = (I - W)\phi''$, and $(I + W)\phi' = (I + W)\phi''$, which implies that $\{h_1, h_2\} \in \mathfrak{D}(\mathscr{N}_1)$. For such an $\mathscr{N}_1$ define N_1 to be the operator in $\mathfrak{H}$ with $\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}$, and $N_1 h_1 = \bar{N}^* h_1$ for $h_1 \in \mathfrak{D}(N_1)$. Similarly define N_2 on $\mathfrak{D}(N_2) = \mathfrak{D}(\bar{N}) + \bar{N}^*(I + W)\mathfrak{M}$ by $N_2 h_2 = N^* h_2$ for $h_2 \in \mathfrak{D}(N_2)$. In terms of N_1 and N_2 we have $\{h_1, h_2\} \in \mathfrak{D}(\mathscr{N}_1)$ if and only if $h_1 \in \mathfrak{D}(N_1)$, $h_2 \in \mathfrak{D}(N_2)$, and $\mathscr{N}_1\{h_1, h_2\} = \{N_2 h_2, N_1 h_1\}$. A short computation shows that $\mathfrak{D}(\mathscr{N}_1^*)$ is the set of all $\{g_1, g_2\}$ such that $g_1 \in \mathfrak{D}(N_2^*)$,

$g_2 \in \mathfrak{D}(N_1^*)$, and $\mathcal{N}_1^*\{g_1, g_2\} = \{N_1^* g_2, N_2^* g_1\}$. But since $\mathcal{N}_1 = \mathcal{N}_1^*$ we obtain $N_2 = N_1^*$. Hence $\mathfrak{D}(\mathcal{N}_1)$ consists of all $\{h_1, h_2\}$ with $h_1 \in \mathfrak{D}(N_1)$, $h_2 \in \mathfrak{D}(N_1^*)$, and $\mathcal{N}_1\{h_1, h_2\} = \{N_1^* h_2, N_1 h_1\}$. Here

$$\begin{aligned} \mathfrak{D}(N_1) &= \mathfrak{D}(N) + (I - W)\mathfrak{M} , \\ \mathfrak{D}(N_1^*) &= \mathfrak{D}(\bar{N}) + \bar{N}^*(I + W)\mathfrak{M} , \end{aligned} \tag{4}$$

and $N \subset N_1 \subset \bar{N}^*$, $\bar{N} \subset N_1^* \subset N^*$. Thus any self-adjoint extension $\mathcal{N}_1$ of $\mathcal{N}$ having the property that $W^2 = I$ determines a unique operator N_1 in $\mathfrak{H}$ as above, which is easily seen to be closed. In particular, if N_1 is a normal extension of N, then the equalities (4) hold.

It remains to characterize those $\mathcal{N}_1$ such that $W^2 = I$ for which N_1 is normal, that is $\mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ and $\| N_1 h \| = \| N_1^* h \|$, $h \in \mathfrak{D}(N_1)$. We claim that this is true if and only if

$$(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M} , \tag{5}$$

and

$$\| \bar{N}^*(I - W)\phi \| = \| N^*(I - W)\phi \| , \qquad (\phi \in \mathfrak{M}) . \tag{6}$$

If (5) is valid then (4) implies that $\mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$, since $\mathfrak{D}(N) = \mathfrak{D}(\bar{N})$. Let $h \in \mathfrak{D}(N_1)$, $h = f + (I - W)\phi$, $f \in \mathfrak{D}(N)$, $\phi \in \mathfrak{M}$. Then $(I - W)\phi \in \mathfrak{M} \cap \bar{\mathfrak{M}}$, and we have $N_1 h = Nf + \bar{N}^*(I - W)\phi$, $N_1^* h = \bar{N}f + N^*(I - W)\phi$. Thus

$$\begin{aligned} \| N_1 h \|^2 = \| Nf \|^2 &+ (Nf, \bar{N}^*(I - W)\phi) + (\bar{N}^*(I - W)\phi, Nf) \\ &+ \| \bar{N}^*(I - W)\phi \|^2 , \end{aligned}$$

and

$$\begin{aligned} \| N_1^* h \|^2 = \| \bar{N}f \|^2 &+ (\bar{N}f, N^*(I - W)\phi) + (N^*(I - W)\phi, \bar{N}f) \\ &+ \| N^*(I - W)\phi \|^2 . \end{aligned}$$

Since N is formally normal $\| Nf \| = \| \bar{N}f \|$. Moreover $\bar{N}^*(I - W)\phi \in \bar{\mathfrak{M}}$ implies that $(Nf, \bar{N}^*(I - W)\phi) = (f, N^*\bar{N}^*(I - W)\phi) = -(f, (I - W)\phi)$, and similarly $(\bar{N}f, N^*(I - W)\phi) = -(f, (I - W)\phi)$. Using (6) we see that $\| N_1 h \| = \| N_1^* h \|$ for all $h \in \mathfrak{D}(N_1)$, proving that N_1 is normal.

Conversely, suppose N_1 is normal. Then (6) is clearly valid, for $(I - W)\phi \in \mathfrak{D}(N_1)$ by (4). Suppose $h \in \mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ and $h = f + (I - W)\phi = f' + \bar{N}^*(I + W)\phi'$ with $f, f' \in \mathfrak{D}(N)$, $\phi, \phi' \in \mathfrak{M}$. We show that $f = f'$ and $(I - W)\phi = \bar{N}^*(I + W)\phi'$. Applying this to $f = 0$ we obtain $(I - W)\mathfrak{M} \subset \bar{N}^*(I + W)\mathfrak{M}$, and with $f' = 0$ we get $\bar{N}^*(I + W)\mathfrak{M} \subset (I - W)\mathfrak{M}$, proving (5). Now for any $g \in \mathfrak{D}(N)$ we have $(N_1 h, N_1 g) = (N_1^* h, N_1^* g)$, or

$$(Nf, Ng) + (\bar{N}^*(I - W)\phi, Ng) = (\bar{N}f', \bar{N}g) - ((I + W)\phi', \bar{N}g) .$$

Since $(\bar{N}f', \bar{N}g) = (Nf', Ng)$ and $(\bar{N}^*(I - W)\phi, Ng) = -((I - W)\phi, g)$, this yields

$$(Nf, Ng) - ((I - W)\phi, g) = (Nf', Ng) - (\bar{N}^*(I + W)\phi', g) ,$$

or

$$(N(f - f'), Ng) + (\bar{N}^*(I + W)\phi' - (I - W)\phi, g) = 0 .$$

But $\bar{N}^*(I + W)\phi' - (I - W)\phi = f - f'$, and hence

$$(N(f - f'), Ng) + (f - f', g) = 0$$

for all $g \in \mathfrak{D}(N)$. Letting $g = f - f'$ we obtain $f = f'$ as desired. This completes the proof of Theorem 2.

4. Abstract boundary conditions. For $u \in \mathfrak{D}(\bar{N}^*)$, $v \in \mathfrak{D}(N^*)$ define $\langle uv \rangle = (\bar{N}^*u, v) - (u, N^*v)$.

THEOREM 3. *If N_1 is a normal extension of the formally normal operator N such that $\mathfrak{D}(N_1) = \mathfrak{D}(N) + (I - W)\mathfrak{M}$, then $\mathfrak{D}(N_1)$ may be described as the set of all $u \in \mathfrak{D}(\bar{N}^*)$ satisfying $\langle u\alpha \rangle = 0$ for all $\alpha \in (I - W)\mathfrak{M}$.*[6]

REMARK. For differential operators the conditions $\langle u\alpha \rangle = 0$ become boundary conditions. They are self-adjoint ones, that is, $\langle \alpha\alpha' \rangle = 0$ for all $\alpha, \alpha' \in (I - W)\mathfrak{M}$. Indeed $\alpha, \alpha' \in \mathfrak{D}(N_1) = \mathfrak{D}(N_1^*)$ and for any $\alpha \in \mathfrak{D}(N_1)$, $\alpha' \in \mathfrak{D}(N_1^*)$ we have $(\bar{N}^*\alpha, \alpha') = (N_1\alpha, \alpha') = (\alpha, N_1^*\alpha') = (\alpha, N^*\alpha')$.

Proof of Theorem 3. If $u \in \mathfrak{D}(N_1)$, $\alpha \in (I - W)\mathfrak{M} \subset \mathfrak{D}(N_1^*)$, the above argument shows that $\langle u\alpha \rangle = 0$. Conversely suppose $u \in \mathfrak{D}(\bar{N}^*)$ and $\langle u\alpha \rangle = 0$ for all $\alpha \in (I - W)\mathfrak{M}$. Let $u = f + (I - W)\phi + (I + W)\phi$, where $f \in \mathfrak{D}(N)$, $\phi \in \mathfrak{M}$. We note that $\langle \ \rangle$ is linear in the first spot, and $f + (I - W)\phi \in \mathfrak{D}(N_1)$. Thus $\langle (I + W)\phi\, \alpha \rangle = 0$ for all $\alpha \in (I - W)\mathfrak{M}$. Let $\alpha = \bar{N}^*(I + W)\phi \in (I - W)\mathfrak{M}$, since $(I - W)\mathfrak{M} = \bar{N}^*(I + W)\mathfrak{M}$. Then

$$\begin{aligned}0 = \langle (I + W)\phi\, \bar{N}^*(I + W)\phi \rangle &= (\bar{N}^*(I + W)\phi, \bar{N}^*(I + W)\phi) \\ &\quad + ((I + W)\phi, (I + W)\phi) ,\end{aligned}$$

which proves that $(I + W)\phi = 0$, and hence $u \in \mathfrak{D}(N_1)$ as desired.

UNIVERSITY OF CALIFORNIA,
LOS ANGELES

[6] A result similar to Theorem 3 appears in the report by Davis (4) for the case when $\dim (\mathfrak{D}(\bar{N}^*)/\mathfrak{D}(N)) < \infty$.