# TRANSFORMATIONS WITH BOUNDED
## $m$TH DIFFERENCES

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**Introduction.** H. Whitney [8, 9] has given a thorough discussion of the problem of a function of a single real variable whose $m$th difference is bounded. Other results along the same lines have been obtained by J. C. Burkhill [1], H. Burkhill [2] and F. John [5]. Whitney distinguishes three cases, according to whether the domain of the function consists of a finite interval, a semi-infinite interval, or the whole real line. In each case, the object is to show that a function with bounded $m$th differences which also has certain regularity properties such as continuity or measurability or boundedness can be approximated by a polynomial of degree $m - 1$, and to obtain the best possible approximation in terms of the given bound on the $m$th difference.

In the present paper, an analogous problem will be considered for a transformation of a cone in a vector space into a Banach space. Our results correspond to the cases of unbounded intervals considered by Whitney. So far, we have no results for a transformation acting on a bounded domain, since our method depends essentially on the assumption that the domain be unbounded.

Our methods are completely different, and much simpler than those of Whitney for the unbounded domains considered here. By generalizing the concept of "polynomial" so as to include all transformations whose differences of a certain order are identically zero,[1] we can remove al regularity assumptions from our transformations. However, for $m > 2$, we seem to require that the $m$th difference, with $m$ *different* increments be bounded, instead of the weaker assumption with equal increments.

**1. Some definitions and known results.** Let $\mathfrak{X}$ be a vector space over the rational numbers and let $T$ be a vector space (over the reals). Let $S$ denote a convex "cone" in $\mathfrak{X}$ with vertex at the origin. This means that

( i ) if $x$ and $y \in S$, then $x + y \in S$ ;

(ii) if $x \in S$, then $\alpha x \in S$ for all rational numbers $\alpha \geqq 0$. Clearly these conditions imply that $S$ is rational-convex, in the sense that if $x$,

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[2] This is similar to the definition of a polynomial due to Fréchet [3], except that here we require no regularity assumptions.

$y \in S$ then $\alpha x + (1 - \alpha)y \in S$ for $0 \leqq \alpha \leqq 1$, where $\alpha$ is, of course, rational.

We shall make use of those results of Mazur and Orlicz [6] which are contained in the first part of their paper. These have to do with multi-additive transformations and operators of degree $m$ of one vector space into another, and we remark that all these results hold for operators defined on a vector space $\mathfrak{X}$ with rational multipliers, and *also for transformations defined on* $S$, rather than on the whole space. Paraphrasing their definitions, a transformation $V(x_1, x_2, \cdots x_k)$ defined for all $x_j \in S \, (j = 1, \cdots, k)$, with values in $T$, will be called *k-additive* if it is additive with respect to each $x_j$, for all $x_j \in S \, (j = 1, \cdots, k)$. By the usual device, it follows at once that if $V(x_1, \cdots, x_k)$ is $k$-additive on $S$ then

$$V(\tau_1 x_1, \tau_2 x_2, \cdots, \tau_k x_k) = \tau_1 \cdots \tau_k \, V(x_1, \cdots, x_k)$$

for all nonnegative rational $\tau_j$ and all $x_j \in S$.

If $V^*(x_1, \cdots, x_k)$ is $k$-additive on $S$ to $T$, define

$$V_k(x) = V^*(x, \cdots, x)$$

for all $x \in S$. Then

$$V_k(\tau x) = \tau^k V_k(x)$$

for all nonnegative rationals $\tau$. A transformation $V_k(x)$ defined in this way will be called a *rational-homogeneous form of degree k, on S,* providing that it does not vanish identically on $S$.

A transformation $V(x)$ on $S$ to $T$ is said to be *at most of the mth degree* providing that

$$(1) \qquad\qquad V(x) = V_0(x) + V_1(x) + \cdots + V_m(x) \,,$$

where each $V_k(x)$ is a rational-homogeneous form of degree $k$ on $S$, or else vanishes identically on $S$. If $V_m(x)$ does not vanish identically, we say that $V(x)$ is *of degree m*. Here $V_0(x)$ denotes a constant element of $T$.

We denote the (forward) $m$th difference of $V(x)$, with the same increment $h$ used at each step, by $\varDelta_h^m V(x)$. The $m$th difference with *different* increments $h_1, \cdots, h_m$ will be denoted by $\varDelta_{h_1 \cdots h_m}^m V(x)$. Thus, for example, $\varDelta_{h_1 h_2}^2 V(x) = V(x + h_1 + h_2) - V(x + h_1) - V(x + h_2) + V(x)$.

The following results, proved in [6], p. 56 and p. 62, also carry over without change to our present situation.

THEOREM A. *In order that the transformation $V(x)$ on $S$ to $T$ be at most of the mth degree, it is necessary and sufficient that $\varDelta_h^{m+1} V(x) = 0$ for all $x$ and $h$ in $S$.*

THEOREM B. *If $V_m(x)$ is a rational-homogeneous form of degree $m$, then there is a unique symmetric $m$-additive transformation $V(x_1, \cdots, x_m)$ such that*

$$V_m(x) = V(x, \cdots, x) .$$

*This $m$-additive transformation is given by the formula*

$$V(x_1, \cdots, x_m) = \frac{1}{m!} \, \Delta^m_{x_1 \cdots x_m} V_m(x) .$$

*Note.* This multiadditive transformation $V$ is often called the *polar* of the transformation $V_m$.

## 2. The case $m = 2$.

THEOREM I. *Let $S$ be the "cone" defined in §1 and let $B$ be a Banach space. If $\beta$ is a fixed positive number and if $F$ is a transformation on $S$ into $B$ satisfying the condition*

$$\| \Delta^2_h F(x) \| \leqq \beta ,$$

*for all $x$ and $h$ in $S$, then there exists a transformation $V$ on $S$ into $B$ with the properties*:
  (a)   $V(x + y) = V(x) + V(y)$ ,
  (b)   $\| V(x) - F(x) + F(0) \| \leqq \beta$ ,
  (c)   $V(x) = \lim_{n \to \infty} n^{-1} F(nx)$ ,
*for all $x$ and $y$ in $S$.*

*Proof.* (The proof is similar to that given by the author in [4]. However, it is quite short and will be reproduced here for convenience). Set $g(x) = F(x) - F(0)$, so that $g(0) = 0$ and $\| \Delta^2_h g(x) \| \leqq \beta$ for all $x$ and $h$ in $S$. Replacing $x$ by 0 and $h$ by $1/2 \, x$, and dividing the result by 2, we have

$$( 2 ) \qquad \left\| \frac{1}{2} g(x) - g\left(\frac{1}{2} x\right) \right\| \leqq \frac{1}{2} \beta ,$$

for all $x$ in $S$. We make the induction assumption:

$$( 3 ) \qquad \| 2^{-n} g(x) - g(2^{-n}x) \| \leqq (1 - 2^{-n})\beta ,$$

for all $x \in S$, and show that (3) holds with $n$ replaced by $n + 1$. To do this, replace $x$ by $2^{-n}x$ in (2):

$$( 4 ) \qquad \left\| \frac{1}{2} g(2^{-n}x) - g(2^{-n-1}x) \right\| \leqq \frac{1}{2} \beta ;$$

now divide (3) by 2:

$$(5) \qquad \left|\left|2^{-n-1}g(x) - \frac{1}{2}g(2^{-n}x)\right|\right| \leqq \left(\frac{1}{2} - 2^{-n-1}\right)\beta .$$

Thus, on adding (4) and (5) and using the triangular inequality we have

$$||2^{-n-1}g(x) - g(2^{-n-1}x)|| \leqq (1 - 2^{-n-1})\beta .$$

Since by (2), (3) holds for $n = 1$, the result (3) has been established by induction.

Next, we put $q_n(x) = 2^{-n}g(2^n x)$, where $n = 1, 2, 3, \cdots$, and note that

$$q_{n+p}(x) - q_n(x) = 2^{-n}\{2^{-p}g(2^{n+p}x) - g(2^{-p}(2^{n+p}x))\} ,$$

where $p$ is any positive integer. Applying (3) with $x$ replaced by $2^{n+p}x$ and with $n$ replaced by $p$, it follows that

$$||q_{n+p}(x) - q_n(x)|| \leqq 2^{-n}(1 - 2^{-p})\beta < 2^{-n}\beta .$$

Thus $q_n(x)$ is a Cauchy sequence, and hence converges to some element $V(x)$, since $B$ is complete.

Now, for any two elements $x$ and $h$ in $S$, we form the second difference at $2^n x$ with the increment $2^n h$ and divide it by $2^n$. Since the second difference of $g$ has the bound $\beta$, we get:

$$||2^{-n}g[2^n(x + 2h)] - 2\cdot2^{-n}g[2^n(x + h)] + 2^{-n}g(2^n x)|| \leqq 2^{-n}\beta .$$

Taking limits as $n \to \infty$, we see that

$$(6) \qquad \Delta_h^2 V(x) = 0 ,$$

for all $x$ and $h$ in $S$. Thus $V$ is additive on $S$ by Theorem A of § 1.

By using (3) with $x$ replaced by $2^n x$, one has $||q_n(x) - g(x)|| \leqq (1 - 2^{-n})\beta$; again taking limits as $n \to \infty$ and recalling the definitions of $g(x)$, and $V(x) = \lim_{n\to\infty}q_n(x)$, we obtain

$$(b) \qquad || V(x) - F(x) + F(0)|| \leqq \beta .$$

If we replace $x$ by $nx$ in the last inequality, divide both sides by $n$ and take the limit as $n \to \infty$, there results

$$(c) \qquad V(x) = \lim_{n\to\infty} F(nx)/n .$$

The following corollary to Theorem I is essentially just the result of [4], Theorem 1, adapted to our present case. It will be useful in the next section.

COROLLARY. *Let $F(x)$ on $S$ to $B$ satisfy the inequality*

$$\|F(x+y) - F(x) - F(y) + F(0)\| \leqq \beta$$

*for all $x$ and $y$ in $S$. Then there exists an additive transformation*

$$V(x) = \lim_{n\to\infty} n^{-1}\Delta_{nx}F(0) = \lim_{n\to\infty} F(nx)/n ,$$

*such that* $\|F(x) - F(0) - V(x)\| \leqq \beta$ ,
*for all $x$ in $S$.*
(The proof of the corollary is simillar to that of Theorem I except that the additivity of $V$ may now be proved directly.)

## 3. The general case.

LEMMA 1. *Let $F(x)$ on $S$ to $B$ satisfy the inequality*

(7)    $\|F(x+y) - F(x) - F(y) + F(0) - Q(x,y) - H(x,y)\| \leqq \beta$

*for all $x$ and $y$ in $S$, where $H(x,y)$ is either identically zero or else is a rational-homogeneous form of degree $k - 1(k > 1)$ in $x$ for each $y$, while $Q(x,y)$ is a transformation of degree at most $k - 2$ in $x$ which vanishes for $x = 0$.*

*Then $H(x,x) = kH(x)$, where $H(x)$ is either identically zero or else a homogeneous form of degree $k$ which is given by the formula*

$$H(x) = \frac{1}{k!}\lim_{n\to\infty}\frac{1}{n^k}\Delta_{nx}^k F(0) .$$

*Moreover, $H(x,y)$ is given by the formula*

$$H(x,y) = \frac{1}{(k-1)!}\lim_{n\to\infty}\frac{1}{n^{k-1}}\Delta_{nx}^{k-1}g(0,y) ,$$

*where*       $g(x,y) = F(x+y) - F(x) .$

*Proof.* By hypothesis there exists a transformation $V(x_1, \cdots, x_{k-1}, y)$ which is additive and symmetric in its first $k - 1$ arguments, such that

(8)                $H(x,y) = \frac{1}{(k-1)!} V(x, \cdots, x, y) .$

We take the $(k - 1)$th difference of the quantity inside the norm signs on the left side of (7), treating $y$ as a constant, and using the increments $x_1, \cdots, x_{k-1}$. It follows from (7) that

$$\|\Delta_{x_1\cdots x_{k-1}y}^k F(x) - \Delta_{x_1\cdots x_{k-1}}^{k-1}Q(x,y) - \Delta_{x_1\cdots x_{k-1}}^{k-1}H(x,y)\| \leqq 2^{k-1}\beta .$$

Since $Q(x,y)$ is of degree at most $k - 2$ in $x$, its $(k - 1)$th difference with respect to $x$ is identically zero, by Theorem A of § 1. Also, from

(8) it follows that the indicated $(k-1)$th difference of $H(x, y)$ is just $V(x_1, \cdots, x_{k-1}, y)$, by Theorem B of § 1.

Thus we have

$$(9) \qquad \| \varDelta^k_{x_1 \cdots x_{k-1} y} F(x) - V(x_1, \cdots, x_{k-1}, y) \| \leqq 2^{k-1}\beta .$$

On interchanging $x_j$ with $y$, where $j$ is any positive integer not exceeding $k-1$, and observing that the $k$th difference in (9) is symmetric in all of its increments, we see that

$$(10) \qquad \| \varDelta^k_{x_1 \cdots x_{k-1} y} F(x) - V(x_1, \cdots x_{j-1}, y, x_{j+1}, \cdots x_k, x_j) \| \leqq 2^{k-1}\beta .$$

A combination of the inequalities (9) and (10) results in

$$(11) \qquad \| V(x_1, \cdots, x_{k-1}, y) - V(x_1, \cdots x_{j-1}, y, x_{j+1}, \cdots x_k, x_j) \| \leqq 2^k\beta .$$

We were given that $V$ is additive in each of its first $k-1$ arguments and symmetric in these arguments. In order to prove that *V is additive in its last argument and symmetric in all of its arguments,* we distinguish two cases.

*Case* 1. $k = 2$. Then (11) becomes

$$\| V(x, y) - V(y, x) \| \leqq 4\beta$$

for all $x$ and $y$ in $S$, where $V$ is additive in its first argument. Replacing $x$ by $nx$ and dividing by $n$, where $n$ is any positive integer, we obtain

$$\| V(x, y) - n^{-1} V(y, nx) \| \leqq 4\beta n^{-1} ,$$

and so by letting $n$ tend to infinity,

$$(11') \qquad V(x, y) = \lim_{n \to \infty} n^{-1} V(y, nx) .$$

Thus

$$\begin{aligned}
V(x, y_1 + y_2) &= \lim n^{-1} V(y_1 + y_2, nx) \\
&= \lim n^{-1} V(y_1, nx) + \lim n^{-1} V(y_2, nx) \\
&= V(x, y_1) + V(x, y_2) ,
\end{aligned}$$

so that $V$ is additive in its second argument. The symmetry now follows from (11').

*Case* 2. $k > 2$. In this case there is an index $i \neq j$ with $1 \leqq i \leqq k-1$, and we may replace $x_i$ by $nx_i$ in (11), divide this inequality by $n$, and take the limit as $n \to \infty$ to see that $V$ is symmetric in all of its arguments. Obviously $V$ must be additive in its last argu-

ment.

If we define $g(x, y) = F(x + y) - F(x)$, then from inequality (9) we have

$$\| \Delta^{k-1}_{x_1 \cdots x_{k-1}} g(0, y) - V(x_1, \cdots, x_{k-1}, y) \| \leqq 2^{k-1} \beta .$$

Now take each $x_j = nx$, divide the last inequality by $(k - 1)! n^{k-1}$, and then let $n$ tend to infinity. By (8) the result is

$$H(x, y) = \frac{1}{(k - 1)!} V(x, \cdots, x, y) = \frac{1}{(k - 1)!} \lim_{n \to \infty} n^{-k+1} \Delta^{k-1}_{nx} g(0, y) .$$

In a similar way, if we define $H(x) = k^{-1} H(x, x)$ and use the fact that $V$ is additive in each of its arguments, we obtain from (9) that

$$(12) \qquad H(x) = \frac{1}{k!} V(x, \cdots, x) = \frac{1}{k!} \lim_{n \to \infty} \frac{1}{n^k} \Delta^k_{nx} F(0) ,$$

which completes the proof of Lemma 1.

LEMMA 2. *For any $k > 1$, let $F, Q, H$ satisfy the conditions of Lemma 1, and put $F'(x) = F(x) - H(x)$. Then the transformation $F'(x)$ satisfies the condition of Lemma 1 with $k$ replaced by $k - 1$. That is, there exist a transformation $H'(x, y)$ which is either identically zero or a rational-homogeneous form of degree $k - 2$ in $x$ for each $y$, and a transformation $Q'(x, y)$ of degree at most $k - 3$ in $x$ which vanishes for $x = 0$, such that the inequality*

$$\| F'(x + y) - F'(x) - F'(y) + F'(0) - Q'(x, y) - H'(x, y) \| \leqq \beta$$

*is satisfied for all $x$ and $y$ in $S$.*
(Note: In case $k = 2$, $Q'(x, y) \equiv H'(x, y) \equiv 0$).

*Proof.* By hypothesis we have $F'(x) = F(x) - H(x)$, where $F$ satisfies (7), $H(x, y)$ is related to the multi-additive transformation $V$ by (8), and $H(x)$ is given by (12). In terms of $F'$, (7) becomes

$$(13) \qquad \| F'(x + y) - F'(x) - F'(y) + F'(0) - Q(x, y) - H(x, y)$$
$$+ H(x + y) - H(x) - H(y) \| \leqq \beta .$$

Now by (12),

$$H(x + y) - H(x) - H(y) = \frac{1}{k!} V(x + y, x + y, \cdots, x + y)$$

$$- \frac{1}{k!} V(x, \cdots, x) - \frac{1}{k!} V(y, \cdots, y) ,$$

and by the analog of the binomial theorem,

$$V(x + y, \cdots, x + y) = V(x, \cdots, x) + kV(x, \cdots x, y)$$

$$+ \binom{k}{2} V(x, \cdots x, y, y) + \cdots + V(y, \cdots, y) .$$

Hence, using (8) we have

$$H(x + y) - H(x) - H(y) = \frac{1}{(k - 1)!} \, V(x, \cdots, x, y) + q(x, y)$$

$$= H(x, y) + q(x, y) ,$$

where $q(x, y)$ is of degree at most $k - 2$ in $x$, and vanishes for $x = 0$. Substituting the last result into (13), we see that

$$\| F'(x + y) - F'(x) - F'(y) + F'(0) - Q(x, y) + q(x, y) \| \leq \beta .$$

Since $Q$ and $q$ are each of degree at most $k - 2$ in $x$ and since each vanishes for $x = 0$, the same properties hold for their difference. It follows that we may write $Q(x, y) - q(x, y) = Q'(x, y) + H'(x, y)$, where $H'(x, y)$ is either identically zero or else is a rational-homogeneous form of degree $k - 2$ in $x$, and $Q'(x, y)$ is of degree at most $k - 3$ in $x$, and vanishes for $x = 0$. This proves Lemma 2.

THEOREM II. *Let $S$ be the cone defined in §1 and let $B$ be a Banach space. If $\beta$ is a fixed positive number and if $f$ is a transformation on $S$ to $B$ satisfying the inequality*

$$\| \Delta^m_{h_1 \cdots h_m} f(x) \| \leq \beta ,$$

*for all $x$ and $h_j$ in $S$ $(j = 1, \cdots, m)$, then there exists a transformation $P_{m-1}(x)$ on $S$ to $B$ which is of degree at most $m - 1$, such that for all $x$ in $S$,*

(a) $$\| f(x) - P_{m-1}(x) \| \leq \beta .$$

*Moreover, $P_{m-1}$ is given by the formula*

(b) $$P_{m-1}(x) = f(0) + H_1(x) + \cdots + H_{m-1}(x) ,$$

*where each $H_k$ is either a rational-homogeneous form of degree $k$ or else identically zero.*

*Finally, the $H_k$ are given by the formulas:*

$$H_{m-1}(x) = \lim_{n \to \infty} \frac{1}{(m - 1)! \, n^{m-1}} \, \Delta^{m-1}_{nx} f(0) ,$$

(c) $$H_k(x) = \lim_{n \to \infty} \frac{1}{k! \, n^k} \left\{ \Delta^k_{nx} f(0) - \sum_{j=k+1}^{m-1} \Delta^k_{nx} H_j(0) \right\}$$

*for $1 \leq k \leq m - 2$ .*

*Proof.* By Theorem I, Theorem II holds for $m = 2$, with $H_1(x) = V(x)$. We shall proceed by induction. Assuming that the theorem holds for a given positive integer $m$, we shall prove it with $m$ replaced by $m + 1$. By the hypothesis of the theorem, then,

$$\| \Delta^{m+1}_{h_1 \cdots h_{m+1}} f(x) \| \leqq \beta$$

for all $x$ and $h_j$ in $S$ $(j = 1, \cdots, m + 1)$.

Put

(14) $$G(x, y) = \Delta_y f(x) = f(x + y) - f(x) .$$

Then, treating $y$ as a fixed parameter we take differences with respect to $x$ and get

$$\Delta_{x_1 \cdots x_m} G(x, y) = \Delta^{m+1}_{x_1 \cdots x_m y} f(x) ,$$

so that

(15) $$\| \Delta^m_{x_1 \cdots x_m} G(x, y) \| \leqq \beta ,$$

for each fixed $y$ and all $x$ and $x_j$ in $S$ $(j = 1, \cdots, m)$.

By the induction hypothesis together with (15) it follows that there exists, for each fixed $y \in S$, a transformation $P(x, y)$ defined for all $x$ in $S$ which is of degree at most $m - 1$ in $x$, such that

(16) $$\| G(x, y) - P(x, y) \| \leqq \beta$$

for all $x$ and $y$ in $S$. Moreover, $P(x, y)$ has the form

(17) $$P(x, y) = G(0, y) + Q(x, y) + H(x, y) ,$$

where $H(x, y)$ is a rational-homogeneous form of degree $m - 1$ or else is identically zero, while $Q(x, y)$ is a transformation of degree at most $m - 2$ in $x$, and $Q(0, y) = 0$. By substituting (14) and (17) into (16) we have

(18) $$\| f(x + y) - f(x) - f(y) + f(0) - Q(x, y) - H(x, y) \| \leqq \beta$$

for all $x$ and $y$ in $S$.

Since $f$ satisfies (18), we can apply Lemma 1 to $f$ with $k = m$, and put $H_m(x) = m^{-1} H(x, x)$, so that

(19) $$H_m(x) = \frac{1}{m!} \lim_{n \to \infty} \frac{1}{n^m} \Delta^m_{nx} f(0)$$

is either identically zero or else a homogeneous form of degree $m$.

According to Lemma 2, if we put

(20) $$f_1(x) = f(x) - H_m(x) ,$$

then the transformation $f_1(x)$ satisfies the conditions of Lemma 1 for $k = m - 1$; consequently, there exists the transformation $H_{m-1}(x)$ given by

$$(21) \qquad H_{m-1}(x) = \frac{1}{(m-1)!} \lim_{n \to \infty} \frac{1}{n^{m-1}} \Delta_{nx}^{m-1} f_1(0)$$

$$= \lim_{n \to \infty} \frac{1}{(m-1)! \, n^{m-1}} \{\Delta_{nx}^{m-1} f(0) - \Delta_{nx}^{m-1} H_m(0)\}$$

which is either identically zero or else a homogeneous form of degree $m - 1$.

Again by Lemma 2, if we put

$$(22) \qquad f_2(x) = f_1(x) - H_{m-1}(x) = f(x) - H_{m-1}(x) - H_m(x) \ ,$$

then $f_2$ satisfies the conditions of Lemma 1 for $k = m - 2$, which leads to the existence of the limit

$$H_{m-2}(x) = \frac{1}{(m-2)!} \lim \frac{1}{n^{m-2}} \Delta_{nx}^{m-2} f_2(x) \ ,$$

etc.

Continuing in this manner, we finally arrive at a transformation

$$(23) \qquad f_{m-2}(x) = f(x) - H_3(x) - \cdots - H_m(x) \ ,$$

where the $H_k(x)$ are given by formula (c) in the statement of our theorem, and where $f_{m-2}$ satisfies the inequality

$$(24) \qquad \|f_{m-2}(x + y) - f_{m-2}(x) - f_{m-2}(y) + f_{m-2}(0) - h(x, y)\| \le \beta \ ,$$

in which $h(x, y)$ is either identically zero or a homogeneous form of degree one in $x$. Applying Lemma 1 once more, and putting $H_2(x) = 1/2 \, h(x, x)$, we have

$$H_2(x) = \frac{1}{2!} \lim_{n \to \infty} \frac{1}{n^2} \Delta_{nx}^2 f_{m-2}(0) \ ,$$

which, in view of (23), also agrees with formula (c) of the theorem. Finally, on putting

$$(25) \qquad \begin{aligned} f_{m-1}(x) &= f_{m-2}(x) - H_2(x) \\ &= f(x) - H_2(x) - H_3(x) - \cdots - H_m(x) \ , \end{aligned}$$

and applying Lemma 2 for the case $k = 2$, we get the inequality

$$(26) \qquad \|f_{m-1}(x + y) - f_{m-1}(x) - f_{m-1}(y) + f_{m-1}(0)\| \le \beta \ ,$$

for all $x$ and $y$ in $S$.

Since $f_{m-1}$ satisfies (26), it follows from the corollary to Theorem 1 that there exists an additive transformation

$$(27) \qquad H_1(x) = \lim_{n\to\infty} n^{-1} \Delta_{nx} f_{m-1}(0)$$

satisfying the inequality

$$(28) \qquad \| f_{m-1}(x) - f_{m-1}(0) - H_1(x) \| \leqq \beta$$

for all $x$ in $S$. Obviously $H_1(x)$ agrees with formula (c) by (27) and (25). By substituting (25) into (28) and observing that $f_{m-1}(0) = f(0)$, we obtain

$$\| f(x) - f(0) - H_1(x) - H_2(x) - \cdots - H_m(x) \| \leqq \beta \, ,$$

which is equivalent to conditions (a) and (b) of our theorem with $m$ replaced by $m + 1$. Thus the induction proof has been completed and Theorem II established.

## 4. Uniqueness.

THEOREM III. *Let $f$ be a transformation satisfying the hypothesis of Theorem II. Then any two transformations of degree at most $m - 1$ which satisfy*

$$(a) \qquad \| f(x) - P_{m-1}(x) \| \leqq \beta$$

*for all $x$ in $S$ differ at most by a constant.*

*Proof.* If $P_{m-1}$ and $P'_{m-1}$ are any two transformations of degree at most $m - 1$ which satisfy (a), then their difference $Q(x) = P_{m-1}(x) - P'_{m-1}(x)$ is a transformation of degree at most $m - 1$ whose norm $\| Q(x) \|$ is bounded by $2\beta$ for all $x$ in $S$. This is easily seen to be impossible unless $Q$ reduces to a constant.

## 5. A condition for regularity.
Let us consider the special case in which the domain of our given transformation $f$ is an entire normed vector spase $E$, and where $f$ is bounded in some open set of $E$.

THEOREM IV. *Let $E$ be a normed linear space, $B$ a Banach space, and $\beta$ a fixed positive number. Let $f$ be a transformation on $E$ to $B$ satisfying the inequality $\| \Delta^m_{h_1 \ldots h_m} f(x) \| \leqq \beta$ for all $x$ and all $h_j$ in $E$ $(j = 1, \cdots, m)$. If $f$ is bounded in some open set contained in $E$, then the transformation $P_{m-1}(x)$ of Theorem II is a polynomial of degree at most $m - 1$. That is, $P_{m-1}$ is continuous at each point of $E$.*

*Proof.* By Theorem II, the transformation $P_{m-1}$ satisfies the inequality

(a) for all $x$ in $E$.   Since $f$ is bounded in some open set it follows immediately from this inequality  that $P_{m-1}$ is also bounded in this open set.

By Proposition 11, p. 179 of [7], it follows that $P_{m-1}(x)$ is continuous at each point of $E$.

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