COLLECTIONS AND SEQUENCES OF CONTINUA IN THE PLANE. II

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1. Introduction. This paper includes a study of some convergence properties of sequences of mutually exclusive continua in E^2, and these results are used to obtain some restrictions on the types of continua in an uncountable collection of mutually exclusive continua in E^2. The concept of *width* of a tree-like continuum is introduced, and it is shown that E^2 does not contain uncountably many mutually exclusive tree-like continua with positive widths. This gives a generalization of R. L. Moore's result that E^2 does not contain uncountably many mutually exclusive triodic continua [13]. The author has presented some related results in [7].

Definitions of trees, chains, tree-like continua, and triods can be found in [8].

2. The width of a tree-like continuum. If G is a tree, then a number $\mathscr{W}(G)$ is associated with G as follows. For each chain C in G and each element X of G, there is a distance[1] $\rho(X, C^*)$ from X to C^*. Let

$$\mathscr{W}(G) = \min_{C \text{ in } G} \left[\max_{X \in G} \rho(X, C^*) \right],$$

where each maximum is obtained with C fixed. A number w is called the width of a tree-like continuum M if, for any cofinal sequence $G_1, G_2, G_3, \cdots$ of trees defining M, the sequence $\mathscr{W}(G_1), \mathscr{W}(G_2), \mathscr{W}(G_3), \cdots$ converges to w.

THEOREM 1. *Every tree-like continuum has a width.*

Proof. Suppose that some tree-like continuum M does not have a width. Then there exist two nonnegative numbers w_1 and w_2 ($w_1 < w_2$) and two cofinal sequences $G_1, G_2, G_3, \cdots$ and $H_1, H_2, H_3, \cdots$ of trees defining M such that the sequence $\mathscr{W}(G_1), \mathscr{W}(G_2), \mathscr{W}(G_3), \cdots$ converges to w_1 and the sequence $\mathscr{W}(H_1), \mathscr{W}(H_2), \mathscr{W}(H_3), \cdots$ converges to w_2. Let ε be a positive number such that

$$(1) \qquad 3\varepsilon < w_2 - w_1 .$$

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[1] The point set which is the sum of the elements of C is denoted by C^*.

Let i be an integer such that the mesh of G_i is less than $\varepsilon/2$ and

$$(2) \qquad |\mathscr{W}(G_i) - w_1| < \varepsilon .$$

There exists an integer j such that H_j is a refinement of G_i and

$$(3) \qquad |\mathscr{W}(H_j) - w_2| < \varepsilon .$$

Let C be a chain in G_i such that

$$(4) \qquad \mathscr{W}(G_i) = \max_{X \in G_i} \rho(X, C^*) .$$

There is a chain D in H_j such that each link of C contains a link of D. Let Y be an element of H_j such that

$$(5) \qquad \rho(Y, D^*) = \max_{X \in H_j} \rho(X, D^*) ,$$

and let Z be an element of G_i that contains Y. Then

$$(6) \qquad \rho(Y, D^*) \leqq \rho(Z, C^*) + \varepsilon .$$

It follows from (2) and (4) that

$$(7) \qquad \rho(Z, C^*) < w_1 + \varepsilon .$$

Now (6) and (7) imply that

$$(8) \qquad \rho(Y, D^*) < w_1 + 2\varepsilon .$$

Hence D is a chain in H_j such that

$$(9) \qquad \max_{X \in H_j} \rho(X, D^*) < w_1 + 2\varepsilon ,$$

and since

$$(10) \qquad \mathscr{W}(H_j) \leqq \max_{X \in H_j} \rho(X, D^*) ,$$

it follows from (9) that

$$(11) \qquad \mathscr{W}(H_j) < w_1 + 2\varepsilon .$$

Now a combination of (3) and (11) gives

$$(12) \qquad w_2 < w_1 + 3\varepsilon ,$$

and this is contrary to (1).

COROLLARY. *Every linearly chainable continuum has width zero.*

REMARK. There exists a tree-like continuum which has width zero and which is not linearly chainable. A continuum which is the sum of a simple triod T and a ray spiralling around T is such an example. Any tree-like continuum which is almost chainable [9] has width zero.

THEOREM 2. *If the tree-like continuum M has width zero, then every homeomorphic image of M has width zero.*

Proof. In order that a tree-like continuum K should have width zero it is necessary and sufficient that, for every positive number ε, there should exist an ε-tree G covering K and a chain C in G such that every point of K is within a distance ε of some link of C. Hence, Theorem 2 follows from the fact that every homeomorphism of M is uniformly continuous.

THEOREM 3. *Every tree-like triod has a positive width.*

Proof. Suppose that some tree-like triod M has width zero. Then for each positive number ε, there exist an ε-tree G covering M and a chain C in G such that every point of M is within a distance ε of some link of C. A contradiction can be reached by using an argument similar to the proof of Theorem 6 of [9].

3. Convergent sequences of continua in E^2. A sequence of continua $M_1, M_2, M_3, \cdots$ is said to converge homeomorphically to a continuum M if, for each positive number ε, there exists an integer k such that, for $n > k$, there is a homeomorphism of M_n onto M that moves no point more than a distance ε.

THEOREM 4. *If $M_1, M_2, M_3, \cdots$ is a sequence of mutually exclusive tree-like continua in E^2 converging to a continuum M and, for each i, w_i is the width of M_i, then the sequence $w_1, w_2, w_3, \cdots$ converges to zero.*

The following lemma will be used in the proof of this theorem.

LEMMA. *If n is a positive integer, H is a collection consisting of n mutually exclusive closed disks in E^2, and K is a collection consisting of n^3 mutually exclusive dendrons in E^2 such that each element of K intersects every element of H, then some element of K contains an arc which intersects every element of H.*

Proof. The case where $n = 1$ is trivial, so suppose that $n > 1$. Each element of K contains a dendron which is irreducible among the elements of H. Hence there exists a collection K' consisting of n^3 mutually exclusive dendrons such that each element of K' is irreducible among the elements of H and is a subset of an element of K. Now since n^3 is greater than the product of $2n$ and the number of pairs of elements of H, it follows from [7, Theorem 3] that there exist two elements D_1 and D_2 of H and a collection K'' consisting of $2n$ elements of K' such that each element of K'' intersects each element of H and is a subset of $\mathrm{cl}\,[E^2 - (D_1 + D_2)]$. Now it follows from [6, Theorem 4] that some element of K'' contains an arc which intersects every element of H. Hence some element of K contains such an arc.

Proof of Theorem 4. Suppose that the sequence $w_1, w_2, w_3, \cdots$ does not converge to zero, and for convenience suppose that there is a positive number δ such that each w_i is greater than δ. Let ε be a positive number less than $\delta/4$. There exists a finite set B consisting of n points of M such that every point of M is within a distance ε of some point of B. There exist a collection G of open disks of diameter less than ε and a subcollection G' of G such that

(1) *G is an essential covering of M,*

(2) G' is an essential covering of B, and

(3) the closures of the elements of G' are mutually exclusive.

Let G'' denote the collection of all closed disks which are the closures of the open disks of G'. There exists an integer k such that, for $i \geqq k$, G is an essential covering of M_i. Now for each i $(k \leqq i \leqq k + n^3)$, there exists a tree G_i such that

(1) G_i is an essential covering of M_i,

(2) each element of G_i is an open disk,

(3) G_i is a refinement of G,

(4) no element of G_i intersects an element of G_j for $j \neq i$,

(5) if C_i is a linear chain in G_i, some element of G_i is a distance greater than δ from C_i^*, and

(6) the nerve of G_i can be realized by a dendron K_i which is covered essentially by G_i and which has a width greater than δ.

It follows from the above lemma that for same integer s $(k \leqq s \leqq k + n^3)$, there is an arc T_s in K_s which intersects every element of G''. Requirement (6) implies that some point p of K_s is a distance greater than δ from T_s. Let q be a point of M_s such that $\rho(p, q) = \rho(p, M_s)$, let r be a point of M such that $\rho(q, r) = \rho(q, M)$, and let u be a point of B such that $\rho(r, u) = \rho(r, B)$. Now since $\rho(p, M_s) < \varepsilon$, $\rho(q, M) < \varepsilon$, $\rho(r, B) < \varepsilon$, and $\rho(u, T_s) < \varepsilon$, this leads to the contradiction that $\rho(p, T_s) < \delta$. Hence, the sequence $w_1, w_2, w_3, \cdots$ converges to zero.

THEOREM 5. *If $M_1, M_2, M_3, \cdots$ is a sequence of mutually exclusive tree-like continua in E^2 converging homeomorphically to a continuum M_0, then the width of each M_i is zero.*

Proof. Let ε be a positive number. It follows from Theorem 4 and the homeomorphic convergence of the sequence $M_1, M_2, M_3, \cdots$ that there exist a positive integer n, a tree G_n covering M_n, and a homeomorphism f of M_n onto M_0 such that $\varepsilon/3$ is greater than each of the width of M_n, the number $\mathscr{W}(G_n)$, the mesh of G_n, and the distance any point of M_n is moved under f. Let C_n be a chain in G_n such that

$$(1) \qquad \mathscr{W}(G_n) = \max_{X \in G_n} \rho(X, C_n^*) .$$

Now let G denote the tree which is the collection of all images, under f, of elements of G_n, and let C denote the chain in G which consists of all images, under f, of elements of C_n. It follows that the mesh of G is less than ε and that for each element Y in G_n,

$$(2) \qquad \rho(f(Y), C^*) < \rho(Y, C_n^*) + 2\varepsilon/3 .$$

A combination of (1) and (2) gives

$$(3) \qquad \rho(f(Y), C^*) < \mathscr{W}(G_n) + 2\varepsilon/3 .$$

Now since $\mathscr{W}(G_n) < \varepsilon/3$, it follows from (3) that

$$(4) \qquad \rho(f(Y), C^*) < \varepsilon .$$

Hence it has been shown that for each positive number ε, there is an ε-tree G covering M_0 such that $\mathscr{W}(G) < \varepsilon$, and from this it follows that M_0 has width zero. That the width of each M_i is zero follows from Theorem 2.

THEOREM 6. *If $M_1, M_2, M_3, \cdots$ is a sequence of mutually exclusive continua in E^2 converging homeomorphically to a continuum M_0, then no M_i has more than two complementary domains.*

Proof. Suppose that M_0 has three complementary domains. Let a, b, and c be three points in the complement of M_0 such that no two of them are in the same complementary domain of M_0, and let ε be a positive number that is less than the distance from M_0 to $a + b + c$. There exists an integer k such that, for $n > k$, there is a homeomorphism f_n of M_n onto M_0 that moves no point more than a distance $\varepsilon/2$, and hence, for $n > k$, M_n does not contain one of the points a, b, and c. Now let h and j be two integers greater than k. It follows from a theorem proved by Eilenberg [10, Theorem 5] that each of the continua M_h and M_j separates each two of the points a, b, and c in E^2. On the other hand, M_h and M_j are mutually exclusive so that M_h would lie in some complementary domain of M_j, and hence some two of the points a, b, and c would not be separated by M_h. From this contradiction, it follows that M_0 does not have more than two complementary domains. Consequently, no M_i has more than two complementary domains.

THEOREM 7. *If $M_1, M_2, M_3, \cdots$ is a sequence of mutually exclusive continua in E^2 converging homeomorphically to a continuum M_0 that separates E^2, then each M_i irreducibly separates E^2 into two components.*

Proof. It follows from Theorem 6 that M_0 separates E^2 into two components, so suppose that some proper subcontinuum of M_0 separates E^2. Then some proper subcontinuum K of M_0 would irreducibly separate E^2 into two components. Let p be a point of $M_0 - K$, let q be a point that is separated in E^2 from p by K, and let ε be a positive number less than the distance from K to $p + q$. Let D be an open circular disk with center at p and with radius $\varepsilon/3$. There exist integers h and j such that the continua M_h and M_j are carried onto M_0 by homeomorphisms f_h and f_j, respectively, that move no point more than a distance $\varepsilon/3$. Let K_h and K_j denote the continua $f_h^{-1}(K)$ and $f_j^{-1}(K)$, respectively. Each of the continua M_h and M_j intersects D and, by [10, Theorem 5], each of the continua K_h and K_j separates q from D in E^2. Since K_h and K_j are mutually exclusive, it follows that one of them, say K_h, separates the other, K_j, from D in E^2. This involves the contradiction that M_j intersects both K_j and D but does not intersect K_h. Hence, it follows that each M_i irreducibly separates E^2 into two components.

THEOREM 8. *There does not exist in E^2 a sequence of mutually exclusive triods converging homeomorphically.*

Proof. Let $M_1, M_2, M_3, \cdots$ be a sequence of mutually exclusive continua in E^2 converging homeomorphically to a continuum M. It is sufficient to show that M is not a triod.

Case 1. The continuum M separates E^2. By Theorem 7, M irreducibly separates E^2 into two components so that no proper subcontinuum of M separates M [12]. Hence M is not a triod.

Case 2. The continuum M does not separate E^2. Then M is tree-like as it contains no open subset of E^2 [2]. By Theorem 5, M has width zero, and hence it follows from Theorem 3 that M is not a triod.

4. Uncountable collections of mutually exclusive continua in E^2. Roberts [14] has shown that every linearly chainable continuum has uncountable many mutually exclusive homeomorphic images in E^2. However, this is not the case for tree-like continua with width zero as the continuum described in the remark in § 1 has width zero and contains a simple triod. Anderson [1] has indicated the existence of an uncountable collection of mutually exclusive tree-like continua in E^2 such that no one of them is chainable. By Theorem 9, for any uncountable collection G of mutually exclusive homeomorphic continua in E^2, there exists a sequence of elements of G converging homeomorphically to an element of G. This suggests the following question, which is left unanswered. If M is a tree-like continuum in E^2 such that there exists a sequence of mutually exclusive continua in E^2 converging homeomorphically to M, does M have uncountably many mutually exclusive homeomorphic images in E^2?

THEOREM 9. *If G is an uncountable collection of mutually exclusive homeomorphic continua in E^n, then there exists a sequence of elements of G which converges homeomorphically to an element of G.*

By using Borsuk's theorem that, under the metric $d(f, g) = \max_{x \in M} \rho(f(x), g(x))$, the space of all continuous transformations of a compact metric space M into a separable metric space is separable [5, Theorem 2], Theorem 9 can be proved by the method Bing [4] has indicated for the case where G is a collection of arcs in E^2.

THEOREM 10. *If G is an uncountable collection of mutually exclusive tree-like continua in E^2, then all except a countable number of continua of G have width zero.*

Proof. It is sufficient to show that some continuum of G has width zero. Suppose that no continuum of G has width zero. It follows from Theorem 1 that there is a positive number δ and an uncountable subcollection G' of G such that each continuum of G' has a width greater than δ. But this is contrary to Theorem 4 since there is a convergent sequence of elements of G'.

5. A remark on homogeneous decomposable plane continua. F. B. Jones [11] has shown that every nondegenerate homogeneous decomposable plane continuum has a continuous decomposition G such that G is a simple closed curve with respect to its elements and each element of G is a homogeneous tree-like continuum. Jones' question as to whether each element of G would be a pseudo-arc has not been answered, but Bing [3] has shown that this would be the case if each element of G were linearly chainable. It follows from Theorem 10 that each element of G has width zero. This suggests the following question. Is a homogeneous tree-like continuum chainable if it has width zero?

Added in proof. The author has recently shown that every homogeneous tree-like continuum in E^2 has width zero hereditarily and that a tree-like continuum has width zero hereditarily if and only if it is atriodic. These results will be presented in another paper.

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