THE WAVE EQUATION FOR DIFFERENTIAL FORMS

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1. The Problem. Let M be a compact C^∞ Riemannian manifold of dimension N, having a positive definite metric. The operator $\Delta = d\delta + \delta d$ (see [13] for notation) maps p-forms $(0 \leqq p \leqq N)$ into p-forms and it reduces, when $p = 0$, to minus the Laplace-Beltrami operator. Let $c(P)$ be a C^∞ function which is nonpositive for $P \in M$, and consider the Cauchy problem of solving the system

$$\left(L + \frac{\partial^2}{\partial t^2}\right)v \equiv \left(\Delta + c + \frac{\partial^2}{\partial t^2}\right)v = f(P, t) \tag{1.1}$$

$$v(P, 0) = g(P)\,, \qquad \frac{\partial}{\partial t} v(P, 0) = h(P)\,, \tag{1.2}$$

where f, g, h are C^∞ forms of degree p. The main purpose of the present paper is to solve the system (1.1), (1.2) by the method of Fourier.

The Cauchy problem for second order self-adjoint hyperbolic equations was solved by Fourier's method by Ladyzhenskaya [8] and more recently (with some improvements) by V. A. Il'in [6]. In [8], other methods are also described, namely: finite differences, Laplace transforms, and analytic approximations using a priori inequalities. Higher order hyperbolic equations were treated by Petrowski [12], Leray [9] and Garding [5].

The Fourier method can be based on the fact that the series

$$\sum_{\lambda_n > 0} \frac{|\varphi_n(x)|^2}{\lambda_n^\alpha}, \quad \sum_{\lambda_n > 0} \frac{|\partial\varphi_n(x)/\partial x|^2}{\lambda_n^{\alpha+1}}, \quad \sum_{\lambda_n > 0} \frac{|\partial^2\varphi_n(x)/\partial x^2|^2}{\lambda_n^{\alpha+2}} \tag{1.3}$$

are uniformly convergent. Here $\{\varphi_n\}$ and $\{\lambda_n\}$ are the sequences of eigenfunctions and eigenvalues of the elliptic operator appearing in the hyperbolic equation. In [6] the convergence of (1.3) is proved for $\alpha = [N/2] + 1$. Our proof of the analogous result for eigenforms is different from that of [6] and yields a better (and sharp) value for α, namely, $\alpha = N/2 + \varepsilon$ for any $\varepsilon > 0$. It is based on asymptotic formulas which we derive for $\sum_{\lambda_n \leqq \lambda} |\partial^j\varphi_n(x)/\partial x^j|^2$ as $\lambda \to \infty$.

In § 2 we recall various definitions and introduce the fundamental solution for $L + \partial/\partial t$ which was constructed by Gaffney [4] in the case $c(P) \equiv 0$. In § 3 we derive some properties of the fundamental solution. These properties are used in § 4 to derive the asymptotic formulas for $\sum_{\lambda_n \leqq \lambda} |\partial^j\varphi_n(x)/\partial x^j|^2$, by which the convergence of the series in (1.3) for any $\alpha > N/2$ follows. In § 5 we solve the problem (1.1), (1.2); first for f, g, h

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infinitely differentiable and then under much weaker differentiability assumptions with regard to M, c, f, g, h. In § 6 we briefly treat the Cauchy problem for the parabolic system

$$Lu + \frac{\partial u}{\partial t} = f(P, t) \tag{1.4}$$

$$u(P, 0) = g(P) . \tag{1.5}$$

2. Preliminaries. The first one to use fundamental solutions of the heat equation in the study of the asymptotic distributions of eigenvalues and eigenfunctions was Minakshisundaram [11]. Gaffney [4] extended his method to derive asymptotic formulas for eigenvalues and eigenforms. We shall describe here some well known facts and some of the results of [4] which we will need later on. Slight modifications will be made due to the fact that in [4] $c \equiv 0$.

As is well known, there exists a sequence of eigenvalues $\{\lambda_n\}$ ($0 \leqq \lambda_1 \leqq \cdots \leqq \lambda_k \to \infty$ as $k \to \infty$) and a sequence of the corresponding eigenforms $\{\omega_n\}$ of degree p ($0 \leqq p \leqq N$, p is fixed throughout the paper) of L, that is, $L\omega_n = \lambda_n \omega_n$, such that the eigenforms form a complete orthonormal set in $L_p^2(M)$ (square integrable p-forms on M). The $\omega_i(p)$ are C^∞ forms. The fundamental solution $\Theta(P, Q, t)$ of

$$\left(L + \frac{\partial}{\partial t}\right)\omega = 0 \tag{2.1}$$

is a double p-form which is twice differentiable in Q, once differentiable in t, satisfies (2.1) in (Q, t), $Q \in M$, $t > 0$, (for any fixed P) and, for any $P \in M$,

$$\lim_{t \to 0} \int_M \Theta(P, Q, t) * \alpha(Q) = \alpha(P) \tag{2.2}$$

for any L^2 p-form α which is continuous at P. As in [4] one easily derives the expansion (provided Θ is known to exist)

$$\Theta(P, Q, t) = \sum_{i=1}^{\infty} \omega_i(P)\omega_i(Q)e^{-\lambda_i t} \tag{2.3}$$

where the series on the right is pointwise convergent for all $P, Q \in M$, $t > 0$ (that is, the series of each component is pointwise convergent).

A p-form α can be written locally as

$$\alpha = \sum_{i_1 < \cdots < i_p} A_{i_1 \cdots i_p} dx^{i_1} \cdots dx^{i_p} = \Sigma' A_I dx^I$$

where $'$ indicates summation on $I = (i_1, \cdots, i_p)$ with $i_1 < \cdots < i_p$. The absolute value of α at P is given by

$$|\alpha(P)| = [\Sigma' A_I(x) A^I(x)]^{1/2}$$

where x is the local coordinate of P. Similarly, for a double p-form having local representation $\alpha(P, Q) = \Sigma' A_{IJ}(x, y) dx^I dy^J$ where y is the local coordinate of Q, we define the absolute value by

$$|\alpha(P, Q)| = [\sum_{I,J}{}' A_{IJ}(x, y) A^{IJ}(x, y)]^{1/2} .$$

The right "half-norm" is defined by

$$|\alpha||(P) = \left[\int_M |\alpha(P, Q)|^2 dV_Q\right]^{1/2} .$$

Given two double p-forms α and β, a new double p-form is defined by

$$[\alpha, \beta] = [\alpha, \beta](P, Q) = \int_M \alpha(P, W) * \beta(Q, W) .$$

One then verifies:

$$|[\alpha, \beta](P, Q)| \leqq |\alpha||(P)|\beta||(Q) . \tag{2.4}$$

The following inequalities are immediate:

$$|\alpha + \beta| \leqq |\alpha| + |\beta|, |\alpha + \beta|| \leqq |\alpha|| + |\beta|| , \tag{2.5}$$

where α, β are any double p-forms.

In order to construct Θ, one first constructs a parametrix. Gaffney [4] constructs a parametrix by generalizing the method of Minakshisandaram [11], making use of some calculation of Kodaira [7]. Given a point P, let $y = (y^i)$ be normal coordinates about P (with coordinates x^i). A p-form can be written as a vector X with $\binom{N}{p}$ components and then

$$\Delta X = -\Sigma g^{ij}\partial_i\partial_j X + \Sigma A^i \partial_i X + BX \tag{2.6}$$

where (g_{ij}) is the metric tensor, (g^{ij}) is the inverse matrix, $\partial_i = \partial/\partial x^i$, and A^i, B are matrices depending on the g_{ij} and their first two derivatives. If $X = f(r^2) W(x, y)$ where r is the geodesic distance from x to y (each component of X is now a vector so that W is a square matrix), then

$$\Delta_y[f(r^2) W] = f(r^2)\Delta_y W - f'(r^2)\left\{2N - 4K + 4r\frac{\partial}{\partial r}\right\} W - 4r^2 f''(r^2) W , \tag{2.7}$$

where $K = K(x, y)$ is a C^∞ matrix which vanishes for $y = x$.

There exists a C^∞ matrix M satisfying

$$r\frac{\partial}{\partial r} M = KM \ (x \text{ fixed}), \qquad M(x, x) = I \tag{2.8}$$

where I is the identity matrix. Using (2.8), (2.7) is simplified to

$$M^{-1}\Delta_y(fMW) = f(M^{-1}\Delta M)_y W - f'\left\{2N + 4r\frac{\partial}{\partial r}\right\}W - 4r^2 f'' W . \tag{2.9}$$

(2.9) will now be applied with

$$f(r^2, t) = \frac{1}{(4\pi t)^{N/2}} e^{-r^2/4t} \qquad (t > 0 \text{ fixed}) .$$

Setting

$$H_m = \sum_{j=0}^{m} fMU_j t^j, \qquad U_0 = I$$

one then gets

$$\Delta H_\infty = fM\sum_{j=0}^{\infty}\left\{(M^{-1}\Delta M)U_j t^j + \frac{1}{4t}\left(2N + 4r\frac{\partial}{\partial r}\right)U_j t^j - \frac{r^2}{4t^2}U_j t^j\right\} .$$

Calculating also $\partial H_\infty/\partial t$, one then obtains

$$\left(L_y + \frac{\partial}{\partial t}\right)H_\infty = fM\sum_{j=0}^{\infty}\left\{(M^{-1}\Delta M + c)U_j + \left(r\frac{\partial}{\partial r} + j + 1\right)U_{j+1}\right\}t^j$$

which leads to the successive definitions:

$$U_j = -\frac{1}{r^j}\int_0^r (M^{-1}\Delta M + c)U_{j-1}dr \ (1 \leqq j < \infty), \quad \text{where} \quad U_0 = I . \tag{2.10}$$

We conclude that, for any $m \geqq 0$,

$$\left(L_y + \frac{\partial}{\partial t}\right)H_m = \frac{1}{(4\pi)^{N/2}} e^{-r^2/4t} t^{m-N/2} L_y(MU_m) . \tag{2.11}$$

H_m is a local parametrix. Note that when P, Q vary in a sufficiently small neighborhood V (contained in one coordinate patch), H_m is defined and is C^∞ in (P, Q, t) if $t > 0$. Let $\eta_\varepsilon(r)$ be a C^∞ function of r which is equal to 1 for $r < \varepsilon$ and is equal to 0 for $r > 2\varepsilon$. If ε is sufficiently small then the support of $\eta_\varepsilon(r)H_m(P, Q, t)$ (where r is the distance from P to Q) as a form in Q lies in V, provided $P \in W$, where W is a given open subset of V, $\bar{W} \subset V$. We can cover the manifold M by a finite number of sets W, call then W_i. Let the H_m corresponding to (the corresponding) V_i be denoted by H_m^i. If $\{\alpha_i\}$ is a C^∞ partition of unity subordinate to $\{W_i\}$, then the support of $\alpha_i(P)\eta_\varepsilon(r)H_m^i(P, Q, t)$ as a form of (P, Q) lies in $W_i \times V_i$ and hence this form is C^∞ in (P, Q, t) if $t > 0$.

The global parametrix is given by

$$\Theta_m(P, Q, t) = \Sigma\alpha_i(P)\eta_\varepsilon(r)H_m^i(P, Q, t) . \tag{2.12}$$

The fundamental solution should then formally be

$$(2.13)\qquad \Theta(P, Q, t) = \Theta_m(P, Q, t) + \int_0^t [\gamma_m(P, U, t), \Theta_m(Q, U, t-\tau)]d\tau$$

where γ_m is defined by

$$(2.14)\qquad \gamma_m(P, Q, t) = \sum_{i=1}^{\infty} (-1)^i \delta_m^i(P, Q, t)$$

$$(2.15)\qquad \delta_m^i(P, Q, t) = \int_0^t [\delta_m^{i-1}(P, U, \tau), \delta_m^i(Q, U, t-\tau)]d\tau,$$

$$\delta_m^1 = \left(L_y + \frac{\partial}{\partial t}\right)\Theta_m \,.$$

Using (2.4) and the inequality

$$(2.16)\qquad \left|\int_0^t \alpha(P, Q, \tau)d\tau\right| \leqq \binom{N}{p}\int_0^t |\alpha|\, d\tau\ ,$$

Gaffney establishes the uniform convergence of the right side of (2.14) and then proves that Θ, as defined in (2.13), is a fundamental solution, for any $m \geqq 0$, written in matrix form. We shall use the matrix notation of Θ and the usual double form notation for Θ interchangably; the same for Θ_m.

3. **Properties of the fundamental solution.** We denote by $\partial_P^h\Theta(P, Q, t)$ an hth derivative of Θ with respect to the coordinates of P, in a given coordinate system. If $h = (h_1, \cdots, h_N)$, set $|h| = h_1 + \cdots + h_N$. From the formulas defining Θ it is clear that $\partial_P^h\Theta(P, Q, t)$ exists and is continuous (in fact C^∞) in $P, Q \in M$ and $t > 0$. Let

$$(3.1)\qquad \partial_P^h\Theta(P, Q, t) \sim \sum_{i=1}^{\infty} B_i(P, t)\omega_i(Q)$$

be the Fourier expansion of $\partial_p^h\Theta$, for (P, t) fixed. Then (recalling (2.3))

$$(3.2)\qquad B_i(P, t) = \int_M \partial_P^h\Theta(P, U, t) * \omega_i(U) = \partial_P^h\int_M \Theta(P, U, t) * \omega_i(U)$$
$$= \partial^h\omega_i(P)e^{-\lambda_i t}\ ,$$

where ∂_P^h is abbreviated by ∂^h when there is no confusion.

By the (easily verified) Parseval's equality we get

$$(3.3)\qquad \psi(P, Q, t) \equiv \left[\partial_P^h\Theta\left(P, U, \frac{t}{2}\right), \partial_Q^h\Theta\left(Q, U, \frac{t}{2}\right)\right]$$
$$= \sum_{i=1}^{\infty} \partial_P^h\omega_i(P)\partial_Q^h\omega_i(Q)e^{-\lambda_i t}$$

and the series is pointwise convergent for $P, Q \in M, t > 0$.

We need the following notations. Let α be a double p-form. If it is locally represented by $\Sigma' A_{IJ} dx^I dy^J$, then we set

$$[\alpha(P, P)] = \Sigma' A_I^I .$$

If β is also a double p-form, then we define $[[\alpha(P, U), \beta(P, U)]_U]$ to be $[\gamma(P, P)]$ where $\gamma(P, Q) = [\alpha(P, U), \beta(Q, U)]$.

Using (2.13) and the definition of ψ in (3.3) we have

$$\begin{aligned}(3.4)\quad & \sum_{i=1}^{\infty} |\partial^h \omega_i(P)|^2 e^{-\lambda_i t} = [\psi(P, P, t)] \\ & = \Big[\Big[\partial_P^h \Theta_m\Big(P, U, \frac{t}{2}\Big), \partial_P^h \Theta_m\Big(P, U, \frac{t}{2}\Big)\Big]_U\Big] \\ & + 2\Big[\Big[\int_0^{t/2}\Big[\partial_P^h \gamma_m(P, W, \tau), \Theta_m\Big(U, W, \frac{t}{2} - \tau\Big)\Big] d\tau, \partial_P^h \Theta_m\Big(P, U, \frac{t}{2}\Big)\Big]_U\Big] \\ & + \Big[\Big[\int_0^{t/2}\Big[\partial_P^h \gamma_m(P, W, \tau), \Theta_m\Big(U, W, \frac{t}{2} - \tau\Big)\Big] d\tau, \\ & \qquad \int_0^{t/2}\Big[\partial_P^h \gamma_m(P, W, \tau), \Theta_m\Big(U, W, \frac{t}{2} - \tau\Big)\Big] d\tau\Big]_U\Big] \\ & \equiv J_1(P, t) + 2J_2(P, t) + J_3(P, t) .\end{aligned}$$

We proceed to estimate the J_i. We shall make use of the inequality [4]

$$(3.5)\qquad [\alpha(P, P)] \leqq \binom{N}{p} |\alpha(P, P)|^2 ,$$

and of the inequality [1]

$$\begin{aligned}(3.6)\qquad & \int_0^t \int_{-\infty}^{\infty} \frac{\exp\{-\lambda |x - z|^2/(t - \tau)\}}{(t - \tau)^\mu} \frac{\exp\{-\lambda |z - y|^2/\tau\}}{\tau^\nu} dz d\tau \\ & \leqq \text{const.}\ \frac{\exp\{-\lambda |x - y|^2/t\}}{t^{\mu+\nu-1-N/2}}\end{aligned}$$

where $dz = dz^1 \cdots dz^N$ and $\lambda > 0$, $\mu < N/2 + 1$, $\nu < N/2 + 1$. The following, easily verified, inequality will also be used:

$$\begin{aligned}(3.7)\qquad & \int_{-\infty}^{\infty} \exp\{-\lambda |x - z|^2/t\} \exp\{-\lambda |z - y|^2/t\} dz \\ & \leqq \text{const.} \exp\{-\mu |x - y|^2/t\} t^{N/2}\end{aligned}$$

where $dz = dz^1 \cdots dz^N$ and $\lambda > \mu > 0$. We shall denote by A_j constants which (unless otherwise stated) may depend only on h and on the manifold M.

Using (3.6) one can prove by induction on i that

$$(3.8)\qquad |\partial_P^h \delta_m^i(P, U, t)| \leqq \frac{A_1^{i+1}}{i!} t^{i(m+1-|h|/2)-1-N/2} e^{-r^2/5t} .$$

The case $i = 1$ follows by (2.11), (2.12). (In deriving (3.8) we also use the elementality inequality $\lambda e^{-\alpha\lambda} \leqq \text{const. } e^{-\delta\lambda}$ for all $\lambda > 0$, where α, δ are constants and $\alpha > \delta \geqq 0$.) In (3.8) it is understood that t° (if it occurs) must be replaced by -log t. From now on we take m such that

$$m + 1 - \frac{|h|}{2} > 0 .$$

Using the definition (2.14) we then conclude from (3.8) that

$$(3.9) \qquad |\,\partial_P^h \gamma_m(P, Q, t)\,| \leqq A_2 e^{-r^2/5t} t^{m-(|h|+N)/2} .$$

Next, from the definition of Θ_m one derives

$$(3.10) \qquad |\,\partial_P^h \Theta_m(P, Q, t)\,| \leqq A_3 e^{-r^2/5t} t^{-(|h|+N)/2} .$$

Combining (3.9) and (3.10) ($h = 0$) and applying (3.6), we get

$$(3.11) \qquad \left| \int_0^{t/2} \left[\partial_P^h \gamma_m(P, W, \tau), \Theta_m\left(U, W, \frac{t}{2} - \tau\right) \right] d\tau \right| \leqq A_4 e^{-2r^2/5t} t^{m+1-(|h|+N)/2} .$$

Using (3.10), (3.11) one easily derives, applying (3.7),

$$(3.12) \qquad J_2(P, t) \leqq A_5 t^{m+1-|h|-N/2} .$$

Similary one gets

$$(3.13) \qquad J_3(P, t) \leqq A_6 t^{2(m+1)-|h|-N/2} .$$

Evaluation of $J_1(P, t)$. From the construction of Θ_m it follows that for every sufficiently small neighborhood V we may take it to be of the form

$$(3.14) \qquad \Theta_m(P, U, t) = H_m(P, U, t) + R_m(P, U, t) \quad \text{for all} \quad P \in V$$

where H_m is constructed in § 2 and where, for some $\alpha' > 0$,

$$(3.15) \qquad |\,\partial_P^h R_m(P, U, t)\,| \leqq A_7 e^{-\alpha'/t} t^{|h|+N/2} \leqq A_8 t^{\zeta}$$

for any $\zeta > 0$. A_8 depends also on ζ. Next,

$$(3.16) \qquad \partial_P^h H_m(P, U, t) = \sum_{j=0}^{m} t^j \sum_{|\nu|=0}^{|h|} \binom{h}{\nu} \partial_P^\nu f \partial_P^{h-\nu}(MU_j)$$

where $\binom{h}{\nu} = \binom{h_1}{\nu_1} \cdots \binom{h_N}{\nu_N}$. It is easily seen that

$$(3.17) \qquad \partial_P^\nu f(r^2, t) = \sum_{|\mu|=0}^{\nu_0} H_{\nu\mu}\left(\frac{y-x}{\sqrt{t}}\right) f(r^2, t) t^{|\nu|/2+|\mu|/2}$$

where y^i, x^i are the coordinates of U, P respectively, and $H_{\nu\mu}(z)$ is a polynomial in $z = (z^1, \cdots, z^N)$ with C^∞ coefficients which, for $H_{\nu 0}$, are functions of x only. Substituting (3.17) into (3.16) and recalling that $M(P, U)\nu_2$ becomes (δ_I^J) at $P = U$, we obtain

$$\partial_P^h H_m(P, U, t) = H_{h0}\Big(\frac{y-x}{\sqrt{t}}\Big) f(r^2, t) t^{-|h|/2} Y + S_h(P, U, t) \tag{3.18}$$

where Y is the matrix (δ_I^J) and

$$| S_h(P, U, t) | \leqq A_9 e^{-r^2/2t} t^{(1-|h|-N)/2} . \tag{3.19}$$

Combining (3.14), (3.15), (3.18), (3.19) we conclude that

$$\partial_P^h \Theta_m(P, U, t) = H_{h0}\Big(\frac{y-x}{\sqrt{t}}\Big) f(r^2, t) t^{-|h|/2} Y + T_h(P, U, t) \tag{3.20}$$

and

$$| T_h(P, U, t) | \leqq A_{10} t^{(1-|h|-N)/2} .$$

Using the definition of J_1, and substituting (3.20) in the part of the integral $[\partial_P^h \Theta_m(P, U, t/2), \partial_P^h \Theta_m(P, U, t/2)]_U$ taken over a coordinate patch V_0 containing $\bar{V}$: $y^i - x^i = \xi^i \sqrt{t}$, we find that

$$J_1(P, t) = (C_h(P) + B_0(P, t)) t^{-|h|-N/2} \tag{3.21}$$

where $C_h(P)$ is a continuous function of P, and $| B_0(P, t) | \leqq A_{11}\sqrt{t}$ for $P \in V, 0 < t \leqq b$, for any $b > 0$. A_{11} depends on b.

Combining the evaluation of J_1 with (3.12), (3.13), we obtain from (3.4),

$$\sum_{i=1}^{\infty} | \partial^h \omega_i(P) |^2 e^{-\lambda_i t} = C_h(P) t^{-|h|-N/2} + D_h(P, t) t^{-|h|-(N-1)/2} \tag{3.22}$$

where $D_h(P, t)$ is a uniformly continuous function of $(P, t), P \in V$ and $0 < t \leqq b$ for any $b > 0$. Thus

$$| D_h(P, t) | \leqq A_{12} \tag{3.23}$$

where A_{12} depends on b.

Note that the A_i, in particular A_{12}, are independent of P which varies in V.

4. Asymptotic formulas. To derive asymptotic formulas from the equation (3.22) we use a Tauberian theorem due to Karamata, specialized to Dirichlet series [14; p. 192]. It states:

Let $a_k \geqq 0$ and $0 \leqq \lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots$, and assume that the Dirichlet series $f(t) = \sum_{k=1}^{\infty} a_k e^{-\lambda_k t}$ converges for $t > 0$ and satisfies

$$f(t) \sim \frac{A}{t^\gamma} \textit{ as } t \searrow 0 \qquad (\gamma \geqq 0) .$$

Then the function $\alpha(x) = \sum_{\lambda_k \leqq x} a_k$ *satisfies*

$$\alpha(x) \sim \frac{Ax^\gamma}{\Gamma(\gamma + 1)} \quad \textit{as} \quad x \to \infty .$$

Applying it to (3.22) (using (3.23)), we get

$$\sum_{\lambda_i \leqq \lambda} |\partial^h \omega_i(P)|^2 = \frac{C_h(P)}{\Gamma(|h| + 1 + N/2)} \lambda^{|h|+N/2}[1 + o(1)] \; (\lambda \to \infty) \tag{4.1}$$

and $o(1) \to 0$ as $\lambda \to \infty$, uniformly in $P \in V$.

Let $\lambda_1 = \cdots = \lambda_{q-1} = 0, \lambda_q > 0$. Using the asymptotic formula (4.1) we shall prove:

THEOREM 1. *For any* h *and for any* $\varepsilon > 0$, *the series*

$$\sum_{i=q}^{\infty} \frac{|\partial^h \omega_i(P)|^2}{\lambda_i^{N/2+|h|+\varepsilon}} \tag{4.2}$$

is uniformly convergent in $P \in M$.

Proof. We introduce the function

$$B(P, \lambda) \equiv \sum_{\lambda_q \leqq \lambda_i \leqq \lambda} |\partial^h \omega_i(P)|^2 .$$

Then, we can write the series (4.2) in the form

$$\int_{\lambda'}^{\infty} \frac{dB(P, \lambda)}{\lambda^{N/2+|h|+\varepsilon}} \quad \text{for any} \quad 0 < \lambda' < \lambda_q .$$

Integrating by parts we get

$$\lim_{\mu \to \infty} \left[\frac{B(P, \lambda)}{\lambda^{N/2+|h|+\varepsilon}} \right]_{\lambda=\lambda'}^{\lambda=\mu} - \left(\frac{N}{2} + |h| + \varepsilon \right) \int_{\lambda'}^{\infty} \frac{B(P, \lambda)}{\lambda^{N/2+|h|+\varepsilon+1}} d\lambda . \tag{4.3}$$

Since, by (4.1), $B(P, \lambda) \leqq A_{13} \lambda^{|h|+N/2}$ and since $B(P, \lambda') = 0$, the first term in (4.3) vanishes. The integral in (4.3) converges uniformly in P in view of the bound on $B(P,\lambda)$ just given. The proof of Theorem 1 is thereby completed.

5. Solution of the system (1.1), (1.2). We first derive the formal solution. Substituting

$$g(P) = \sum_{n=1}^{\infty} g_n \omega_n(P), \; h(P) = \sum_{n=1}^{\infty} h_n \omega_n(P), \; f(P, t) = \sum_{n=1}^{\infty} f_n(t) \omega_n(P) \tag{5.1}$$

$$(5.2) \qquad v(P, t) = \sum_{n=1}^{\infty} v_n(t)\omega_n(P)$$

into (1.1), (1.2) we arrive at the equations

$$(5.3) \qquad v_n''(t) + \lambda_n v_n(t) = f_n(t)$$

$$(5.4) \qquad v_n(0) = g_n,\ v_n'(0) = h_n .$$

If $\lambda_n = 0$ the solution is

$$v_n(t) = g_n + h_n t + \int_0^t f(\tau)(t-\tau)d\tau .$$

If $\lambda_n > 0$ the solution is

$$v_n(t) = g_n \cos\sqrt{\lambda_n}\, t + \frac{h_n}{\sqrt{\lambda_n}} \sin\sqrt{\lambda_n}\, t + \frac{1}{\sqrt{\lambda_n}}\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau .$$

Hence, the formal solution of (1.1), (1.2) is

$$(5.5) \qquad \begin{aligned} v(P, t) = {} & \sum_{n=1}^{\infty} g_n\omega_n(P)\cos\sqrt{\lambda_n}\, t + \sum_{n=1}^{q-1} h_n\omega_n(P)t \\ & + \sum_{n=q}^{\infty}\frac{h_n}{\sqrt{\lambda_n}}\omega_n(P)\sin\sqrt{\lambda_n}\, t + \sum_{n=1}^{q-1}\omega_n(P)\int_0^t f_n(\tau)(t-\tau)d\tau \\ & + \sum_{n=q}^{\infty}\frac{1}{\sqrt{\lambda_n}}\omega_n(P)\int_0^t f_n(\tau)\sin\sqrt{\lambda_n}\,(t-\tau)d\tau . \end{aligned}$$

To prove that the formal solution is a genuine one we observe that if $\lambda_n > 0$

$$(5.6) \qquad g_n = \int_M g(Q)*\omega_n(Q) = \frac{1}{\lambda_n^m}\int_M L^m g(Q)*\omega_n(Q)$$

for any positive integer m. Applying Bessel's inequality, we get

$$(5.7) \qquad \sum_{n=1}^{\infty}\lambda_n^{2m} g_n^2 \leqq \int_M L^m g(Q)*L^m g(Q) = \| L^m g \|^2 .$$

Similarly,

$$(5.8) \qquad \sum_{n=1}^{\infty}\lambda_n^{2m} h_n^2 \leqq \| L^m h \|^2,\ \sum_{n=1}^{\infty}\lambda_n^{2m}(f_n(t))^2 \leqq \| L^m f(\cdot, t) \|^2 .$$

It will be enough to show that the part of the first series on the right side of (5.5), where summation is on $\lambda_n > 0$, when differentiated term-by-term twice with respect to P is uniformly convergent in $P \in M$, $0 \leqq t \leqq b$, for any $b > 0$. Now the series obtained is majorized by

$$\Sigma\,|\, g_n \,|\,|\, \partial^2 w_n(P) \,| \leqq \Sigma\lambda_n^k\,|\, g_n \,|\,\frac{|\,\partial^2\omega_n(P)\,|}{\lambda_n^k} \leqq \Sigma\lambda_n^{2k} g_n^2 \Sigma\,\frac{|\,\partial^2\omega_n(P)\,|^2}{\lambda_n^{2k}} .$$

Hence that series is uniformly convergent if $k > N/2 + 1$.

It is clear that each series in (5.5) can actually be differentiated term-by-term any number of times and the resulting series is uniformly convergent.

By a solution of (1.1), (1.2) we mean a p-form which is (a) twice continuously differentiable in (P, t) for $P \in M, t > 0$ (b) once continuously differentiable in t for $P \in M, t \geqq 0$ and (c) satisfies (1.1), (1.2).

The uniqueness of the solution can be proved as for the classical wave equation. Assuming $g \equiv 0$, $h \equiv 0$, $f \equiv 0$ and using the rule $\int du^*\omega = \int u^*\delta\omega$ one finds that if u is a solution then

$$\frac{\partial}{\partial t}\int_M [u_t * u_t + \delta u * \delta u + du * du - cu * u] = 0 .$$

Since the integral vanishes for $t = 0$, it vanishes for all $t > 0$. Since the integrand is nonnegative, $u_t * u_t \equiv 0$, which implies $u_t \equiv 0$ and hence, $u \equiv 0$.

We have thus completed the proof of the following theorem.

THEOREM 2. *Let g, h be C^∞ p-forms and let f be a C^∞ p-form such that $\partial_P^\lambda f$ is continuous in (P, t), for any λ. Then the Cauchy problem* (1.1), (1.2) *has one and only one solution. The solution is a C^∞ p-form and is given by* (5.5).

The assumption that the manifold M is C^∞ can be weakened. Indeed, the theory of differential forms used above remains valid under the assumption that the metric tensor is C^5 (Gaffney [3]; see also Friedrichs [2]). The assumptions on f, g, h can also be weakened without any modification of the preceding proof of Theorem 2.

We need the assumptions:

(A) The metric tensor g_{ij} belongs to $C^{[N/2]+2}$ and to C^5, and c belongs to $C^{[N/2]+1}$ (recall that $c \leqq 0$).

(B) The form g belongs to $C^{[N/2]+3}$ and $L^{[(N+4)/4]}g$ belongs to C^1.

(C) The form h belongs to $C^{[N/2]+2}$ and $L^{[(N+2)/2]}h$ belongs to C^1.

(D) The form f and its first $[N/2] + 2$ p-derivatives are continuous for $P \in M$, $0 \leqq t \leqq b$ (for any $b > 0$); $L^{[(N+2)/2]}f$ and its first p-derivatives are continuous for $P \in M$, $0 \leqq t \leqq b$.

THEOREM 2′. *Under the assumptions* (A) – (D), *there exists one and only one solution of the Cauchy problem* (1.1), (1.2). *It is given by* (5.5).

The assertion of Theorem 2′ remains valid if we further weaken the assumptions (A) – (D) by replacing the classes of continuous derivatives C^q by classes of "strong" derivatives W_2^q (see [6]), assuming that $g_{ij} \in C^5$.

6. The heat equation. The method of § 5 can easily be extended to solve the system (1.4), (1.5). The formal solution is

$$u(P, t) = \sum_{n=1}^{\infty} g_n \omega_n(P) e^{-\lambda_n t} + \sum_{n=1}^{\infty} \omega_n(P) \int_0^t f_n(\tau) e^{-\lambda_n(t-\tau)} d\tau . \tag{6.1}$$

We shall need the assumptions:

(A′) g_{ij} belong to $C^{[N/2]+1}$ and to C^5, and e belongs to $C^{[N/2]}$.

(B′) The form g belongs to $C^{[N/2]+1}$ and $L^{[N/4]}g$ belongs to C^1.

THEOREM 3. *Under the assumption* (A′), (B′), (D) *there exists a unique solution of the system* (1.4), (1.5). *It is given by* (6.1).

REMARK 1. The assumption $c \leqq 0$ is not needed for the validity of Theorem 3 since it can be achieved by a transformation $u = e^{\alpha t}u$ for any constant $\alpha \geqq c$.

REMARK 2. Assuming $c \leqq 0, f \equiv 0$, we can rewrite (6.1) as an operator equation

$$T_t = H + \sum_{k=1}^{\infty} e^{-\mu_k t} H_k \tag{6.2}$$

where $\{\mu_k\}$ is the sequence $\{\lambda_j\}$ taken without multiplicities, H_k is the projection into the space of eigenforms corresponding to μ_k, H corresponds to $\mu_0 = 0$, and T_t is the operator which maps g into the solution u, that is, $u(P, t) = T_t g(P)$. Formula (6.2) was derived, in a different way (for $c \equiv 0$) by Milgram and Rosenbloom [10].

REFERENCES

1. F. G. Dressel, *The fundamental solution of the parabolic equation II*, Duke Math. J., **13** (1946), 61-70.

2. K. O. Friedichs, *Differential forms on Riemannian manifolds*, Comm. Pure Appl. Math., **8** (1955), 551-590.

3. M. P. Gaffney, *Hilbert space methods in the theory of harmonic integrals*, Trans. Amer. Math. Soc., **78** (1955), 426-444.

4. M. P. Gaffney, *Asymptotic distributions associated with the Laplacian forms*, Comm. Pure Appl. Math., **11** (1958), 535-545.

5. L. Garding, *Cauchy's Problem for hyperbolic equations*, University of Chicago Notes, 1957.

6. V. A. Il'in, *On solving mixed problems for hyperbolic and parabolic equations*, Uspehi Math. Nauk, **15**, no. 2 (1960), 97-154.

7. K. Kodaira, *Harmonic fields in Riemannian manifolds, generalized potential theory*, Ann. of Math., **50** (1949), 587-665.

8. O. A. Ladyzhenskaya, *The mixed problem for hyperbolic equations*, Gosudarvstv. Izdat.

Tehn.-Teor. Lit., Moscow, 1953.
9. J. Leray, *Lectures on hyperbolic equations with variable coefficients*, Princeton, Inst. for Advanced Study, 1952.
10. A. N. Milgram and P. C. Rosenbloom, *Harmonic forms and heat conduction I: Closed Riemannian manifolds*, Proc. Nat. Acad. Sci., **37** (1951), 180-184.
11. S. Minakshisundaram, *Eigenfunctions on Riemannian manifolds*, J. Indian Math. Soc., **17** (1953), 159-165.
12. I. G. Petrowski, *Über des Cauchysche Problem für Systeme von partiellen Differentialgleichungen*, Rec. Math. (Moscow) N.S., **2** (1937), 814-868.
13. G. De Rham, *Variétés Différentiables*, Hermann, Paris, 1955.
14. D. Widder, *The Laplace Transform*, Princeton Univ. Press, Princeton, 1946.

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