ARITHMETICAL NOTES, III. CERTAIN EQUALLY DISTRIBUTED SETS OF INTEGERS

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1. Introduction. In this note we shall generalize the following two results in the classical theory of numbers. Let n denote a positive integer with distinct prime divisors $p_1, \cdots, p_m$,

$$n = p_1^{e_1} \cdots p_m^{e_m} \quad (m > 0), \quad n = 1\,(m = 0)\ , \tag{1.1}$$

and place $\Omega(n) = e_1 + \cdots + e_m$, $\Omega(1) = 0$, so that $\Omega(n)$ is the total number of prime divisors of n. For real $x \geqq 1$, let $S'(x)$ denote the number of square-free numbers $n \leqq x$ such that $\Omega(n)$ is even, and let $S''(x)$ denote the number of square-free $n \leqq x$ such that $\Omega(n)$ is odd. It is well-known [6, §161] that

$$S'(x) \sim \frac{3x}{\pi^2}, \quad S''(x) \sim \frac{3x}{\pi^2} \qquad \text{as } x \to \infty\ . \tag{1.2}$$

Correspondingly, let $T'(x)$ denote the total number of integers $n \leqq x$ suct that $\Omega(n)$ is even and $T''(x)$ the total number of $n \leqq x$ with $\Omega(n)$ odd. Then [6, §167]

$$T'(x) \sim \frac{x}{2}, \quad T''(x) \sim \frac{x}{2} \quad \text{as } x \to \infty\ . \tag{1.3}$$

The proof of (1.2) is based upon the deep estimate [6, §155] for the Möbius function $\mu(n)$,

$$M(x) \equiv \sum_{n \leqq x} \mu(n) = o(x)\ , \tag{1.4}$$

while the proof of (1.3) is based upon the analogous estimate [6, §167] for Liouville's function $\lambda(n)$,

$$L(x) \equiv \sum_{n \leqq x} \lambda(n) = o(x)\ . \tag{1.5}$$

In Theorem 3.3 we prove a generalization of (1.2) and in Theorem 3.4 the corresponding generalization of (1.3). The respective proofs are based upon an estimate (Theorem 3.1) corresponding to (1.4) for an appropriate extension of $\mu(n)$ and an estimate (Theorem 3.2) corresponding to (1.5) for the analogous extension of $\lambda(n)$. The proofs of these estimates are in the manner of Delange's proofs [3, $I(b), (c)$] of (1.4) and (1.5), both being based upon a classical Tauberian theorem (Lemma 3.2) for the Lambert summabillty process. We also require some elementary

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estimates contained in §2, and a lemma on inversion functions (Lemma 2.1).

2. Preliminary results. For an arbitrary set A of positive integers n, the *characteristic function* $a(n)$ and *inversion function* $b(n)$ of A are defined by

$$\sum_{d|n} b(d) = a(n) \equiv \begin{cases} 1 & (n \in A) \\ 0 & (n \notin A) . \end{cases}$$

The *enumerative function* $A(x)$ of A is the number of $n \leqq x$ contained in A, and the *generating function* is the function $f(s) = \sum_{n=1}^{\infty} a(n)/n^s$, $s > 1$.

We shall be concerned with several special sets of integers. Let Z denote the set of positive integers, $k \in Z$. Then P_k will represent the set of kth powers of Z, and Q_k the set of k-free integers of Z. The set of k-full intergers, that is, the integers (1.1) with each $e_i \geqq k$, will be denoted R_k. We shall use S_k to denote the integers (1.1) in which each e_i has the value 1 or k. Finally, the set of integers (1.1) such that $e_i \equiv 0$ or 1 (mod k), $i = 1, \cdots, m$, will be denoted T_k. The characteristic functions P_k, Q_k, R_k, S_k, and T_k will be denoted respectively $p_k(n)$, $q_k(n)$, $r_k(n)$, $s_k(n)$, and $t_k(n)$; the corresponding enumerative functions will be denoted $P_k(x)$, $Q_k(x)$, $R_k(x)$, $S_k(x)$, $T_k(x)$. Also let $Q = Q_2$, $Q(x) = Q_2(x)$, and $q(n) = q_2(n)$. All of the sets defined are understood to include the integer 1.

REMARK 2.1. It will be observed that $T_1 = Z$, $S_1 = Q_2$, $S_2 = Q_3$.

In addition to the above notation, we shall use $\lambda_k(n)$ to denote the inversion function of P_k and $\mu_k(n)$ the inversion function of R_k or Q_k according as $k > 1$ or $k = 1$. By familiar properties of $\mu(n)$ and $\lambda(n)$, [4, Theorem 263 and 300], it follows that

$$\mu_1(n) = \mu(n) , \qquad \lambda_2(n) = \lambda(n) . \tag{2.1}$$

LEMMA 2.1. *The functions $\mu_k(n)$, $\lambda_k(n)$ are multiplicative. If p is a prime and e a positive integer, then for $k \geqq 1$,*

$$\mu_k(p^e) = \begin{cases} 1 & \textit{if } e = k \neq 1 , \\ -1 & \textit{if } e = 1 , \\ 0 & \textit{otherwise} , \end{cases} \tag{2.2}$$

while for $k > 1$,

$$\lambda_k(p^e) = \begin{cases} 1 & \textit{if } e \equiv 0 \pmod k , \\ -1 & \textit{if } e \equiv 1 \pmod k , \\ 0 & \textit{otherwise} . \end{cases} \tag{2.3}$$

REMARK 2.2. The multiplicativity property in connection with (2.2) and (2.3) completely determine $\mu_k(n)$, $k \geqq 1$, and $\lambda_k(n)$ for $k \geqq 2$.

Proof. By definition, if $k > 1$,

$$\sum_{d \mid n} \mu_k(d) = r_k(n) = \begin{cases} 1 & \text{if } n \in R_k \\ 0 & \text{if } n \notin R_k . \end{cases} \tag{2.4}$$

Hence, application of the Möbius inversion formula yields

$$\mu_k(n) \sum_{d \mid n} \mu(d) r_k\left(\frac{n}{d}\right), \qquad k > 1 . \tag{2.5}$$

Since $\mu(n)$ and $r_k(n)$ are multiplicative, it follows by (2.5) that $\mu_k(n)$ is also multiplicative (cf. [4, Theorem 265]). Also by (2.5), $\mu_k(p^e) = r_k(p^e) - r_k(p^{e-1})$, from which (2.2) results in case $k > 1$. The case $k = 1$ of (2.2) is a consequence of (2.1). The proof of (2.3) is similar and can be omitted.

We recall next some known elementary estimates for $P_k(x)$, $Q_k(x)$, and $R_k(x)$. Let $\zeta(s)$, $s > 1$, denote the Riemann ζ-function.

LEMMA 2.2. *If $k > 1$, then*

$$P_k(x) = \sqrt[k]{x} + O(1) , \tag{2.6}$$

$$Q_k(x) = \frac{x}{\zeta(k)} + O(\sqrt[k]{x}) , \tag{2.7}$$

$$R_k(x) = c_k \sqrt[k]{x} + O\left(\frac{1}{x^{k+1}}\right) , \tag{2.8}$$

where c_k is a certain nonzero constant depending upon k.

The result (2.6) is trivial, (2.7) is the classical estimate of Gegenbauer (cf. [2, §2]), and (2.8) is a well-known result of Erdös and Szekeres (cf. [1]). In particular, we have

LEMMA 2.3. *If $k > 1$, then*

$$P_k(x) \sim \sqrt[k]{x} , \qquad R_k(x) \sim c_k \sqrt[k]{x} \qquad \text{as } x \to \infty , \tag{2.9}$$

$$Q_k(x) \sim \frac{x}{\zeta(k)} , \qquad \left(Q(x) \sim \frac{6x}{\pi^2}\right) \quad \text{as } x \to \infty . \tag{2.10}$$

We now deduce, for application in §3, estimates for $S_k(x)$ and $T_k(x)$ corresponding to those in Lemma 2.3 for $P_k(x)$, $Q_k(x)$, and $R_k(x)$.

LEMMA 2.4. *If $k > 1$, then*

$$T_k(x) \sim \frac{6\zeta(k)x}{\pi^2} \quad as \ x \to \infty \ ; \tag{2.11}$$

if $k \geqq 1$, *then*

$$S_k(x) \sim \frac{6\alpha_k x}{\pi^2} \quad as \ x \to \infty \ , \tag{2.12}$$

where

$$\alpha_k = \begin{cases} \zeta(k) \prod_p \left(1 - \dfrac{1}{p^{k+1}} + \dfrac{1}{p^{k+2}} - \cdots - \dfrac{1}{p^{2k-1}}\right) \\ \dfrac{\zeta(2k)}{\zeta(k)} \prod_p \left(1 + \dfrac{2}{p^k} - \dfrac{1}{p^{k+1}} + \dfrac{1}{p^{k+2}} - \dfrac{1}{p^{k+3}} + \cdots + \dfrac{1}{p^{2k-1}}\right), \\ 1, \end{cases} \tag{2.13}$$

according as k *is even,* k *is odd and* $\neq 1$, *or* $k = 1$, *the products ranging over the primes* p.

REMARK 2.3. It will be noted that $\alpha_2 = \zeta(2)/\zeta(3) = \pi^2/6\zeta(3)$.

Proof. The elementary estimate (2.11) was proved in [1, Corollary 2.1]. The result in (2.12), in the cases $k = 1$ and $k = 2$, is a consequence of (2.10) and Remarks 2.1 and 2.3. To complete the proof of (2.12) one may therefore suppose that $k > 2$.

Under this restriction, we consider the generating function $f_k(s)$ of $s_k(n)$. In particular, if $s > 1$, we have (cf. [4, §17.4])

$$\begin{aligned} f_k(s) \equiv \sum_{n=1}^{\infty} \frac{s_k(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{ks}}\right) \\ &= \prod_p \left(1 + \frac{1}{p^s}\right)\left[1 + \frac{1}{p^{ks}}\left(1 + \frac{1}{p^s}\right)^{-1}\right]. \end{aligned} \tag{2.14}$$

Since

$$\sum_p \frac{1}{p^{ks}}\left(1 + \frac{1}{p^s}\right)^{-1} \leqq \sum_p \frac{1}{p^{ks}} \leqq \sum_{n=1}^{\infty} \frac{1}{n^{ks}} = \zeta(ks) \ , \quad ks > 1 \ ,$$

it follows from (2.14) that

$$f_k(s) = \left(\frac{\zeta(s)}{\zeta(2s)}\right) g_k(s) \ , \qquad s > 1 \ , \tag{2.15}$$

where

$$g_k(s) \equiv \sum_{n=1}^{\infty} \frac{a_k(n)}{n^s} = \prod_p \left(1 + \frac{1}{p^{sk}} - \frac{1}{p^{s(k+1)}} + \cdots\right), \quad s > \frac{1}{k} \ , \tag{2.16}$$

the product, and hence the series, in (2.16) being absolutely convergent for $s > 1/k$. By Dirichlet multiplication [4. §17.1] one deduces from (2.15) and (2.16) that

$$s_k(n) = \sum_{d\delta=n} q(d)a_k(\delta) ,$$

because $\zeta(s)/\zeta(2s)$ is the generating function of $q(n)$, [cf. [4, Theorem 302]). Applying (2.7) in the case $Q(x) \equiv Q_2(x)$, it follows that

$$S_k(x) \equiv \sum_{m\leq x} s_k(n) = \sum_{d\delta\leq x} q(d)a_k(\delta) = \sum_{n\leq x} a_k(n)Q\left(\frac{x}{n}\right) ,$$

and hence that

$$S_k(x) = \frac{6x}{\pi^2} \sum_{n\leq x} \frac{a_k(n)}{n} + O\left(x^{1/2} \sum_{n>x} \frac{|a_k(n)|}{n^{1/2}}\right) .$$

Recalling that the series in (2.16) converges absoltutely for $s > 1/k$, one obtains, since $k > 2$,

$$s_k(x) = \frac{6x}{\pi^2} \sum_{n=1}^{\infty} \frac{a_k(n)}{n} + o\left(x \sum_{n>x} \frac{a_k(n)}{n}\right) + o(x^{1/2}) ,$$

so that

$$(2.17) \qquad S_k(x) = \frac{6\beta_k x}{\pi^2} + o(x) , \qquad \beta_k = g_k(1) .$$

It is readily verified, using (2.16) with $s = 1$, that $\beta_k = \alpha_k$, which completes the proof of (2.12).

3. **The principal results.** We introduce some further definitions and notation. A divisor d of n will be called *unitary* if $d\delta = n$, $(d, \delta) = 1$. The function $\Omega'(n)$ is defined by $\Omega'(n) = \Omega(g)$ where g is the maximal, unitary, square-free divisor of n. Let S'_k and S''_k, denote, respectively, the subsets of S_k for which $\Omega'(n)$ is even or odd, $n \in S_k$. Analogously, let T'_k and T''_k denote the respective subsets of T_k for which $\Omega(n)$ is even or odd, $n \in T_k$, k even. In addition, we shall use $S'_k(x), S''_k(x)$, $T'_k(x)$, $T''_k(x)$ to denote the enumerative functions of S'_k, S''_k, T'_k, T''_k, respectively.

REMARK 3.1. It will be observed that $S'_1(x) = S'(x)$, $S''_1(x(= S''(x)$, $T'_2(x) = T'(x)$, $T''_2(x) = T''(x)$. In addition, we have, by Lemma 2.1, $\mu_k(n) = (-1)^{\Omega'(n)}s_k(n)$, and in case n is even, $\lambda_k(n) = (-1)^{\Omega(n)}t_k(n)$.

In addition to the lemmas of §2 we shall need the following three known theorems.

LEMMA 3.1 (cf. [5, 259, p. 449]). *For bounded coefficients* a_n, *the series,*

$$\sum_{n=1}^{\infty} a_n \left(\frac{x^n}{1-x^n}\right)$$

is convergent, provided $|x| < 1$.

LEMMA 3.2 ([3, p, 38]). *If the series*

$$\sum_{n=1}^{\infty} na_n \left(\frac{x^n}{1-x^n}\right) = S ,$$

converges for $0 \leqq x < 1$, *and*

$$\lim_{x\to 1^-} (1-x) \sum_{n=1}^{\infty} na_n \left(\frac{x^n}{1-x^n}\right) = S ,$$

then the series $\sum_{n=1}^{\infty} a_n$ *converges with sum* S *provided* $a_n = O(1/n)$.

LEMMA 3.3 ([7, p. 225]). *Suppose that the series* $\sum_{n=1}^{\infty} a_n x^n$ *converges for* $0 \leqq x < 1$ *and diverges for* $x = 1$. *If further,* $s_n \equiv a_1 + \cdots + a_n > 0$ *for all* n, *and* $s_n \sim Cn$ *(*C *constant) as* $n \to \infty$, *then*

$$\lim_{x\to 1^-} (1-x) \sum_{n=1}^{\infty} a_n x^n = C .$$

THEOREM 3.1. *If* $k \geqq 1$, *then*

$$M_k(x) \equiv \sum_{n\leqq x} \mu_k(n) = o(x) . \tag{3.1}$$

Proof. By Lemmas 2.1 and 3.1, and the definition of $\mu_k(n)$,

$$\sum_{n=1}^{\infty} \mu_k(n) \left(\frac{x^n}{1-x^n}\right) = \sum_{n=1}^{\infty} \mu_k(n) \sum_{m=1}^{\infty} x^{nm} = \sum_{h=1}^{\infty} \left(\sum_{d\mid h} \mu_k(d)\right) x^h$$

$$= \begin{cases} \sum_{h=1}^{\infty} r_k(h) x^h = \sum_{n\in R_k} x^n & \text{if} \quad k > 1 , \\ x & \text{if} \quad k = 1 . \end{cases}$$

By (2.9), the set R_k has density 0; hence Lemma 3.3 with $C = 0$ can be applied to the power series so that

$$\lim_{x\to 1^-} (1-x) \sum_{n=1}^{\infty} \mu_k(n) \left(\frac{x^n}{1-x^n}\right) = 0 , \qquad k \geqq 1 .$$

Since $|\mu_k(n)| \leqq 1$, Lemma 3.2 is applicable with $a_n = \mu_k(n)/n$, and one concludes that

$$\sum_{n=1}^{\infty} \frac{\mu_k(n)}{n} = o . \tag{3.2}$$

Put $A_k(x) \equiv \sum_{n\leqq x} (\mu_k(n)/n)$; then by partial summation,

$$M_k(x) = -\sum_{n \leqq x} A_k(n) + A_k(x)([x]) + 1) . \tag{3.3}$$

Since $A_k(x) = o(1)$ by (3.2), the theorem results from (3.3).

THEOREM 3.2. *If $k \geqq 2$, then*

$$L_k(x) \equiv \sum_{n \leqq x} \mu_k(n) = o(x) . \tag{3.4}$$

The proof is similar to that of Theorem 3.1 and is therefore omitted. Note that (3.1) reduces to (1.4) in case $k = 1$ and that (3.4) to (1.5) in case $k = 2$.

THEOREM 3.3. *If $k \geqq 1$, then*

$$S_k'(x) \sim \frac{3\alpha_k x}{\pi^2} , \qquad S_k''(x) \sim \frac{3\alpha_k x}{\pi^2} \qquad \text{as} \quad x \to \infty , \tag{3.5}$$

α_k being defined by (2.13).

Proof. By (2.12), Remark 3.1, and (3.1), one obtains

$$S_k'(x) + S_k''(x) = S_k(x) = \frac{6\alpha_k x}{\pi^2} + o(x) ,$$
$$S_k'(x) - S_k''(x) = M_k(x) = o(x) ,$$

and (3.5) results immediately.

Similarly, one may deduce from (2.11), Remark 3.1 and (3.4),

THEOREM 3.4. *If $k > 1$, k even, then*

$$T_k'(x) \sim \frac{3\zeta(k)x}{\pi^2} , \qquad T_k''(x) \sim \frac{3\zeta(k)x}{\pi^2} \qquad \text{as} \quad x \to \infty . \tag{3.6}$$

Finally, it will be observed that (3.5) becomes (1.2) in case $k = 1$; while (3.6) becomes (1.3) when $k = 2$.

It is possible to extend (3.6) so as to hold for all $k > 1$. Let g^* denote the largest unitary divisor of $n \in T_k$, such that all prime factors of g^* have multiplicity $e \equiv 1 \pmod{k}$. Place $\Omega^*(n) = \omega(g^*)$, where $\omega(n)$ is the number of distinct prime divisors of n, and let $T_k^*(x)$ and $T_k^{**}(x)$ denote the number of $n \leqq x$ contained in T_k according as $\Omega^*(n)$ is even or odd, respectively. Then

THEOREM 3.4′. *If $k > 1$,*

$$T_k^*(x) \sim \frac{3\zeta(k)x}{\pi^2} , \qquad T_k^{**}(x) \sim \frac{3\zeta(k)x}{\pi^2} \qquad \text{as} \quad x \to \infty . \tag{3.7}$$

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