GROUP MEMBERSHIP IN SEMIGROUPS

HUBERT C. KENNEDY

An element a of a semigroup S is said to be a *group member* in S if there is a subgroup G of S which contains a. This notion was introduced in 1909 by Arthur Ranum in discussing group membership in certain sets of matrices. In 1927 he extended the notion to group membership in finite dimensional algebras. Using somewhat different methods, the subject was revived separately by two sets of authors, W. E. Barnes and H. Schneider in 1957, and H. K. Farahat and L. Mirsky in 1956 and 1958. They discussed the notion of group membership in polynomials in an element algebraic over a field and in rings of various types. With regard to rings, we say that an element a is a group member in a ring R if a is a group member of the semigroup formed by elements of R and its multiplicative operation. Barnes and Schneider posed the following question: If S is a subring of a ring R and an element a of S is a group member in R, under what conditions is it a group member in S [1, p. 168]? Farahat and Mirsky echo this question, in a note added in proof to their 1958 article, and state that one of their theorems gives a "partial answer" to this question [3, p. 244]. We propose to return to another of Ranum's notions, that of "associated group member," and develop the theory for semigroups. We shall see that this will lead to a more complete solution of the question proposed, and one which applies equally well to semigroups in general.

It is known [3, p. 232] that if a is a group member in a semigroup S, then there is a maximal subgroup $M(a)$ of S, containing a, whose identity is the identity of every subgroup containing a. Further, the inverse of a, relative to this identity, is the same in every subgroup containing a. Distinct maximal subgroups are disjoint. We say that n is the *group index* of a in S if a^n is a group member in S and n is the smallest positive integer for which this is true. (Thus the group index of a group member is 1.) The following theorem is perhaps not so well known.

THEOREM 1. *If S is a semigroup and n is the group index of an element a in S, then a^t is a group member in S if and only if $t \geqq n$. Furthermore, if $t \geqq n$, then $M(a^t) = M(a^n)$.*

Proof. Let a^t be a group member in S. By the definition of group index, t cannot be less than n. Hence $t \geqq n$.

Conversely, let $t \geqq n$. If $t = n$, then $a^t = a^n$ and so a^t is a group

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member in S since a^n is a group member in S. Now suppose $t > n$. Denote the identity of $M(a^n)$ by e, and for simplicity of notation, let b be the inverse of a relative to e. Then e is a two-sided identity for a^t, since

$$a^t e = (a^{t-n}a^n)e = a^{t-n}(a^n e) = a^{t-n}a^n = a^t ,$$

and

$$ea^t = e(a^n a^{t-n}) = (ea^n)a^{t-n} = a^n a^{t-n} = a^t .$$

We next show that a^t is invertible relative to e. Let p be an integer such that $pn > 2t$. Then $a^{pn-t}b^p$ is a right inverse for a^t, since

$$a^t(a^{pn-t}b^p) = (a^t a^{pn-t})b^p = a^{pn}b^p = (a^n b)^p = e^p = e .$$

We have also that $b^p a^{pn-t}$ is a left inverse for a^t, since

$$(b^p a^{pn-t})a^t = b^p(a^{pn-t}a^t) = b^p a^{pn} = (ba^n)^p = e^p = e .$$

But e is a two-sided identity for $a^{pn-t}b^p$, because

$$(a^{pn-t}b^p)e = (a^{pn-t}b^{p-1})(be) = (a^{pn-t}b^{p-1})b = a^{pn-t}b^p ,$$

and

$$e(a^{pn-t}b^p) = (ea^t)(a^{pn-2t}b^p) = a^t(a^{pn-2t}b^p) = a^{pn-t}b^p .$$

Similarly, we may show that e is a two-sided identity for $b^p a^{pn-t}$. The uniqueness of the inverse of a^t follows from

$$\begin{aligned} a^{pn-t}b^p &= e(a^{pn-t}b^p) = e^p(a^{pn-t}b^p) = (ba^n)^p(a^{pn-t}b^p) = (b^p a^{pn})(a^{pn-t}b^p) \\ &= (b^p a^{pn-t})(a^{pn}b^p) = (b^p a^{pn-t})(a^n b)^p = (b^p a^{pn-t})e^p \\ &= (b^p a^{pn-t})e = b^p a^{pn-t} . \end{aligned}$$

Thus a^t is invertible relative to e. Hence a^t is an element of $M(e)$, and so a^t is a group member in S. It follows that $M(a^t) = M(e) = M(a^n)$.

THEOREM 2. *If S is a semigroup, a has group index n in S, and e is the identity of $M(a^n)$, then eae is a group member in S and $(eae)^n = a^n$.*

Proof. Clearly e is a two-sided identity for eae. Let b be the inverse of a^n relative to e. We shall show that $a^{n-1}b$ is a two-sided inverse for eae relative to e, and that e is a two-sided identity for $a^{n-1}b$. We have that $a^{n-1}b$ is a right inverse for eae relative to e, for

$$\begin{aligned} (eae)(a^{n-1}b) &= (eae)[a^{n-1}(eb)] = (eae)[a^{n-1}(a^n b)b] \\ &= (eae)[a^n(a^{n-1}b)b] = (ea)(ea^n)(a^{n-1}b)b \end{aligned}$$

$$= (ea)a^n(a^{n-1}b)b = (ea^n)a(a^{n-1}b)b$$
$$= a^n(a^nb)b = a^n(eb) = a^nb = e .$$

Similarly, we may show that ba^{n-1} is a left inverse for eae relative to e. That e is a two-sided identity for $a^{n-1}b$ follows from

$$(a^{n-1}b)e = a^{n-1}(be) = a^{n-1}b ,$$

and

$$e(a^{n-1}b) = e[a^{n-1}(eb)] = e[a^{n-1}(a^nb)b] = e[a^n(a^{n-1}b)b]$$
$$= (ea^n)(a^{n-1}b)b = a^n(a^{n-1}b)b = a^{n-1}(a^nb)b$$
$$= a^{n-1}(eb) = a^{n-1}b .$$

Similarly, we may show that e is a two-sided identity for ba^{n-1}. The uniqueness of the inverse of eae now follows from

$$ba^{n-1} = (ba^{n-1})e = (ba^{n-1})(a^nb) = (ba^n)(a^{n-1}b) = e(a^{n-1}b) = a^{n-1}b .$$

We have thus shown that eae is a group member in S. We now show that $(eae)^n = a^n$. We note first that $(eae)^n = ea^ne$, for

$$(eae)^2 = (eae)(eae) = eaeae = eaea(a^nb) = ea(ea^n)(ab)$$
$$= eaa^nab = ea^2(a^nb) = ea^2e ,$$

and by induction, $(eae)^n = ea^ne$. But e is a two-sided identity for a^n, so that $(eae)^n = ea^ne = a^n$.

The element eae of the above theorem is said to be the *associated group member* of a in S. (Note that a is a group member in S if and only if a is identical with its associated group member.)

THEOREM 3. *Let S be a semigroup, let T be a sub-semigroup of S, and let a belong to T. If a has group index n in S and group index t in T, then $n = t$.*

Proof. Clearly $n \leqq t$. Now let e be the identity of the group in T to which a^t belongs. But a^t belongs to the maximal group $M(a^n)$ in S, so that e is the identity of any group in S to which a^n belongs. Then $ea^ne = a^n$. But ea^ne is the associated group member in T of a^n. Hence, by Theorem 2, a^n is a group member in T. Thus $t = n$.

We are now in a position to answer the question proposed in the first paragraph.

THEOREM 4. *Let S be a semigroup and let T be a sub-semigroup of S. If an element a in T is a group member in S, then it is a group member in T if and only if some power of a is a group member in T.*

Proof. Suppose n is the group index of a in T. Since a is a group member in S, its group index in S is 1. By Theorem 3, $n = 1$. Hence, a is a group member in T.

The following theorem points out the wide application of this solution. (Proofs may be found in the places cited in the footnotes.)

THEOREM 5. *Every element of each of the following has finite group index*:

(1) *finite semigroup* [4, pp. 12, 13]

(2) *division ring or a direct sum of division rings* [4, pp. 19, 20]

(3) *semi-divided ring* [3, p. 234]

(4) *finite dimensional algebra* [6, p 292].

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