SOME EXTREMAL PROPERTIES OF LINEAR COMBINATIONS OF KERNELS ON RIEMANN SURFACES

GEORGES G. WEILL

1. Introduction. Let Γ_a be the Hilbert space of analytic differentials of finite Dirichlet norm on an open Riemann surface. We shall consider analytic singularities which are finite linear combinations of elements of the type

$$s_j dz = \sum_{k=0}^{\infty} \frac{c_k^j dz}{(z-\xi_j)^{k+2}} + \frac{d^j dz}{z-\xi_j} .$$

Let

$$sdz = \sum_{j=1}^{N} s_j dz , \qquad \sum_{j=1}^{N} d^j = 0 .$$

To a given singularity sdz there correspond Bergman kernels

$$k_s(z, \zeta)dz \quad \text{and} \quad h_s(z, \zeta)dz$$

for the space Γ_a.

We now consider various subspaces $\Gamma_\alpha \subset \Gamma_a$, and show that linear combinations of the kernels for Γ_α of the form

$$h_s dz + \lambda k_s dz ,$$

where λ is complex, extremalize an explicity given functional.

We proved in our thesis [2] that, for the space Γ_{ae} of analytic exact differentials on a *planar* Riemann surface,

$$k_s dz = \frac{1}{2}\frac{\partial}{\partial z}(p_1 - p_0)dz$$

$$h_s dz = \frac{1}{2}\frac{\partial}{\partial z}(p_1 + p_0)dz$$

where p_1 and p_0 are Sario's principal functions with the corresponding singularities [1, Chapter III].

Here we show that the right hand sides still enjoy the same properties on an arbitrary Riemann surface, for the subspace $\Gamma_p \cap \Gamma_{ase}$, where $\Gamma_{ase} = \left\{adz\colon adz \in \Gamma_a, \int_\gamma adz = 0,\ \gamma \text{ any dividing cycle}\right\}$, and Γ_p is generated over the complex numbers by $\{\Gamma_p\} = \{adz\colon adz = \partial p/\partial z,\ p$ a single-valued harmonic function on W, with finite Dirichlet integral.$\}$

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2. Inner products and singular differentials. We shall be concerned here with the Hilbert space Γ_a of analytic differentials on a given Riemann surface W. The inner product of two analytic differentials $adz = \alpha dx + \beta dy$ and $a_1dz = \alpha_1 dx + \beta_1 dy$ is defined as:

$$(adz, a_1dz)_W = -i\int_W a\bar{a}_1 dz\overline{dz} = \int_W (\alpha\bar{\alpha}_1 + \beta\,\bar{\beta}_1)dxdy\ .$$

If we now consider differentials analytic on W, except for a singularity of the type $dz/(z-\zeta)^{m+2}$, $m \geqq 0$, we delete a disk δ of radius r about $z = \zeta$ and define for differentials bdz and b_1dz analytic except for a singularity of the above type, the inner product

$$(bdz, b_1dz)_W = \lim_{r\to 0}(bdz, b_1dz)_{W-\delta}\ ,$$

which amounts to considering the Cauchy principal value for the inner product. In the case of a singularity $dz/(z-\zeta_1) - dz/(z-\zeta_2)$, we replace δ by disks about $z = \zeta_1$ and $z = \zeta_2$, plus a narrow strip along a cut joining $z = \zeta_1$ to $z = \zeta_2$ and define in the same fashion the inner product by a Cauchy limit.

The previous remarks may be extended to finite linear combinations of singularities of the type

$$s_jdz = \sum_{k=0}^{\infty}\frac{c_k^jdz}{(z-\zeta_j)^{k+2}} + \frac{d^jdz}{(z-\zeta_j)}\ ,$$

provided $\sum_{j=1}^{N} d^j = 0$.

3. Extremal properties of the kernels. Let $sdz = \sum_{j=1}^{N} s_jdz$ be a singularity differential and k_sdz, h_sdz be the Bergman kernels correspond to that singularity. We shall consider linear combinations

$$(h_s + \lambda k_s)dz$$

which are normalized in the sense that they all exhibit the same singularity.

We recall that for $l(z)dz \in \Gamma_a$, the Bergman kernels corresponding to a singularity sdz, enjoy the following properties:

$$\text{for } sdz = \frac{dz}{(z-\zeta)^{m+2}}\ ,\ m \geqq 0 \qquad (ldz, k_sdz) = \frac{2\pi l^{(m)}(\zeta)}{(m+1)!}$$

$$(ldz, h_sdz) = 0$$

$$\text{for } sdz = \frac{dz}{z-\zeta_1}\ \frac{dz}{z-\zeta_2} \qquad (ldz, k_sdz) = -(ldz, h_sdz) = 2\pi\int_c ldz$$

where c is a path from ζ_1 to ζ_2.

For $sdz = as_1dz + bs_2dz$, (a, b constant),

$$k_s dz = ak_{s1}dz + bk_{s2}dz$$
$$h_s dz = ah_{s1}dz + bh_{s2}dz\ .$$

Such a linear property is a consequence of the uniqueness of the kernels. Notice that in particular: $(ldz, k_s dz) = \bar{a}(ldz, k_{s1}dz) + \bar{b}(ldz, k_{s2}dz)$. Let now $a_s dz$ be a differential, analytic except for the singularity sdz. We form

$$\begin{aligned}(1)\qquad \| a_s dz - (h_s + \lambda k_s)dz \|^2 &= \| a_s dz\|^2 - \| h_s dz \|^2 + |\lambda|^2 \| k_s dz\|^2 \\ &+ 2Re((h_s - a_s)dz, h_s dz) + 2Re\bar{\lambda}((h_s - a_s)dz, k_s dz)\ .\end{aligned}$$

Assume now that in a disk about $z = \zeta_j$

$$h_s dz = s_j dz + \sum_{k=0}^{\infty} b_k^j (z - \zeta_j)^k dz$$
$$a_s dz = s_j dz + \sum_{k=0}^{\infty} a_k^j (z - \zeta_j)^k dz\ .$$

We then compute:

$$2Re((h_s - a_s)dz, h_s dz) = -4\pi \sum_{j=1}^{N} Re\bar{d}^j \int_{c_j} (h_s - a_s)dz\ ,$$
$$2Re\bar{\lambda}((h_s - a_s)dz, k_s dz) = 4\pi \sum_{j=1}^{N} Re\bar{\lambda}\Big[\sum_{k=1}^{\infty} \frac{(b_k^j - a_k^j)\bar{c}_k^j}{k+1} + \bar{d}^j \int_{c_j} (h_s - a_s)dz$$

using the linear property of the kernels, with respect to the coefficients of the singularity. We now write (1) in the following form:

$$\begin{aligned}\| a_s dz \|^2 - 4\pi \sum_{j=1}^{N} Re\Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} a_k^j \bar{c}_k^j}{k+1} + (\bar{\lambda} - 1)\bar{d}_j \int_{c_j} (a_s - s)dz\Big] &= \| h_s dz \|^2 \\ - |\lambda|^2 \| k_s dz \|^2 - 4\pi \sum_{j=1}^{N} Re\Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} b_k^j \bar{c}_k^j}{k+1} + (\bar{\lambda} - 1)\bar{d}^j \int (h_s - s)dz\Big] \\ + \| a_s dz - (h_s + \lambda k_s)dz \|^2\ .\end{aligned}$$

We can now study the value of the bracket in the functional, and prove that

$$\sum_{j=1}^{N}\Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} b_k^j \bar{c}_k^j}{k+1} + \bar{\lambda}\bar{d}^j \int_{c_j} (h_s - s)dz\Big] = 0\ .$$

We shall summarize our results in a theorem:

THEOREM III A. *Let* $sdz = \sum_{j=1}^{N} s_j dz$ *where*

$$s_j dz = \sum_{k=0}^{\infty} \frac{c_k^j dz}{(z - \zeta_j)^{k+2}} + \frac{d^j dz}{(z - \zeta_j)}$$

be an analytic singularity with $\sum_{j=1}^{N} d^j = 0$.

Let $k_s dz$, $h_s dz$ be the Bergman kernels corresponding to sdz, and let λ be a complex parameter.

Then the linear combination $(h_s + \lambda k_s)dz$ minimizes the functional:

$$\| a_s dz \|^2 - 4\pi \sum_{j=1}^{N} Re\Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} a_k^j \bar{c}_k^j}{k+1} + (\bar{\lambda} - 1)\bar{d}^j \int_{c_j} (a_s - s)dz\Big]$$

over the class of differentials $a_s dz$, analytic except for the singularity sdz. The minimum is

$$\| h_s dz \|^2 + 4\pi \sum_{j=1}^{N} Re \bar{d}^j \int_{c_j} (h_s - s)dz + |\lambda|^2 \| k_s dz \|^2 ,$$

and the deviation from the minimum is

$$\| a_s dz - (h_s + \lambda k_s)dz \|^2 .$$

Proof. $h_s dz + \lambda e^{i\theta} k_s dz$ for θ real is a competing function; therefore:

$$\| h_s dz \|^2 - |\lambda|^2 \| k_s dz \|^2 - 4\pi \sum_{j=1}^{N} Re\Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} \bar{c}_k^j b_k^j}{k+1} + (\bar{\lambda} - 1)\bar{d}^j \int_{c_j} (h_s - s)dz\Big]$$

$$\leqq \| h_s dz \|^2 - |\lambda|^2 \| k_s dz \|^2 - 4\pi \sum_{j=1}^{N} \Big[\sum_{k=1}^{\infty} \frac{\bar{\lambda} e^{-i\theta} \bar{c}_k^j \bar{b}_k^j}{k+1} + (\bar{\lambda} e^{-i\theta} - 1)\bar{d}^j \int_{cj} (h_s - s)dz\Big] .$$

It follows that

$$\sum_{j=1}^{N} Re \Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} \bar{c}_k^j b_k^j}{k+1} + \bar{\lambda} \bar{d}^j \int_{c_j} (h_s - s)dz\Big]$$

$$\geqq \sum_{j=1}^{N} Re \Big\{e^{-i\theta} \Big[\sum_{k=0}^{\infty} \frac{\bar{\lambda} \bar{c}_k^j b_k^j}{k+1} + \bar{\lambda} d^j \int_{c_j} (h_s - s)dz\Big]\Big\}$$

which is only possible if the bracket is real. It cannot be real fer all λ except if it is equal to zero.

4. Particular cases-applications. Assume now that $adz = (\partial p/\partial z)dz$, where p is a single-valued harmonic function on W, except for a singularity $Re\, S(z) = \sum_{j=1}^{N} Re\, S_j(z)$, with

$$Re S_j(z) = d^j \log |z - \zeta_j| + Re \Big[\sum_{k=0}^{\infty} \frac{c_k^j}{(-k-1)(z - \zeta_j)^{k+1}}\Big] ,$$

where d^j is real. The singularity of $(\partial p/\partial z)dz$ is then $sdz = \sum_{j=1}^{N} s_j dz$, with

$$s_j dz = \frac{d^j dz}{z - \zeta_j} + \sum_{k=0}^{\infty} \frac{c_k^j dz}{(z - \zeta_j)^{k+2}} .$$

Moreover if $p = Re\{S_j(z) + \sum_{k=0}^{\infty} A_k^j(z-\zeta_j)^k\}$ near $z = \zeta_j$ and

$$\frac{\partial p}{\partial z} dz = s_j dz + \sum_{k=0}^{\infty} a_k^j (z-\zeta_j)^k ,$$

it follows that $A_{k+1}^j = a_k^j / k+1$ for $k \geqq 0$. We notice furthermore that $\| adz \|^2 = 2(B(p) - A(p))$, where $B(p) = \int_\beta p dp^*$ (β the ideal boundary of ω) and $A(p) = 2\pi \sum_{j=1}^{N} d^j \int_{c_j} (a_s - s) dz$. The functional to be minimized becomes:

$$2\left[B(p) - 2\pi \sum_{j=1}^{N} Re\left[\sum_{k=0}^{\infty} \frac{\bar{\lambda} a_k^j \bar{c}_k^j}{k+1} + \bar{\lambda} d^j \int_{c_j} (a_s - s) dz\right]\right].$$

We notice that the differentials $adz = (\partial p/\partial z)dz$ with p single valued harmonic function generate a subspace $\Gamma_p \subset \Gamma_a$. If $k_{sp}dz$ and $h_{sp}dz$ are the Bergman kernels for Γ_p, they correspond to two functions K_s harmonic and H_s harmonic except for the singularity $Re\, S(z)$ and such that:

$$k_{sp} dz = \frac{\partial K_s}{\partial z} dz$$

$$h_{sp} dz = \frac{\partial H_s}{\partial z} d_s .$$

We can write the value of the minimum as:

$$2B(H_{sp}) + |\lambda|^2 \| k_{sp} dz \|^2 .$$

We now shall prove the following theorem.

THEOREM IV A: *Let $(\partial_{p0}/\partial z)dz$ and $(\partial_{p1}/\partial z)dz$ be the analytic differentials with singularity sdz, corresponding to the principal functions p_0 and p_1. Then*

$$\tfrac{1}{2}\partial/dz(p_1 - p_0)dz = k_{sp}dz$$

$$\tfrac{1}{2}\partial/dz(p_1 + p_0)dz = h_{sp}dz ,$$

where $h_{sp}dz$ and $k_{sp}dz$ are the orthogonal and reproducing kernels for $\Gamma_p \cap \Gamma_{ase}$, corresponding to the singularity sdz.

Proof. First, we know from the definition of p_0 and p_1, that $(\partial_{p0}/\partial z)dz$ and $(\partial_{p1}/\partial z)dz$ are elements of $\Gamma_p \cap \Gamma_{ase}$. Second, from (1. Chapter III. Theorem 9E where only the notation is different), $(\partial_{p0}/\partial z)dz$ minimizes the same functional as $h_{sp}dz - k_{sp}dz$ (which corresponds to $\lambda = -1$), and $(\partial_{p1}/\partial z)dz$ minimizes the same functional as $(h_{sp}dz + k_{sp})dz$, (which corresponds to $\lambda = 1$). The theorem follows.

We shall consider here a family of functions P harmonic, except for a singularity of the type $ReS(z)$; the periods of P vanish along all dividing cycles. It follows that the differentials $(\partial P/dz)dz$ are elements of $\Gamma_P \cap \Gamma_{ase}$, except for a singularity $s(z)dz$.

We shall call H_s the function corresponding to $h_{sp}dz$, and K_s the one corresponding to $k_{sp}dz$. The following results are consequences of the main Theorem.

THEOREM IV B: *Among all functions P with singularity* $1/(z-\zeta)$, $H_s + \lambda K_s$ *minimizes the functional* $B(P) - 2\pi Re\bar{\lambda}A_1$.

THEOREM IV C: *Among all functions P with singularity* $\log|(z-\zeta_1)/(z-\zeta_2)|$, $H_s + \lambda K_s$ *minimizes* $B(P) - 2\pi Re\bar{\lambda}(A_0^1 - A_0^2)$.

THEOREM IV D: *Among all functions P with singularity $ReS(z)$, H_s minimizes the functional $B(P)$.*

We shall now consider exact differentials, analytic except for some singularity $s(z)dz = \sum_{j=1}^{N} s_j(z)dz$, which may be written $f'(z)dz = df(z)$, where f is a function analytic except for a singularity $S(z) = \sum_{j=1}^{N} s_j(z)$ such that $S'(z)dz = s(z)dz$; then $f = S_j(z) + \sum_{k=0}^{\infty} \alpha_k(z-\zeta_j)^k$ near $z = \zeta_j$. We proved [II] the existence of a non-zero reproducing kernel if $W \notin 0_{AD}$. We shall now find a sufficient condition for the existence of an orthogonal kernel. We recall that in the case of a planar Riemann surface

$$\Gamma_h = \Gamma_{he} + \Gamma_{he}^* \cap \Gamma_{ho}^* .$$

We shall consider here Riemann surfaces on which

$$\Gamma_h = \Gamma_{he} + \Gamma_{he}^* .$$

We call such surfaces type W_E. On a surface of type W_E

$$\Gamma_{h0} \cap \Gamma_{ho}^* = [\Gamma_{he} + \Gamma_{he}^*]^\perp = 0 .$$

We then get the following lemma:

LEMMA IV E: *On a surface of type W_E, given a singularity $s(z)dz = dz/(z-\zeta)^{m+2}$, $m \geqq 0$, there exists a differential analytic exact, except for the corresponding singularity.*

Proof. Let Θ be constructed as in [1, Chapter V. 18.19]. The differential $\Theta - i\Theta^*$ is square integrable and hence has the decomposition?.

$$\Theta - i\Theta^* = \omega_h + \omega_{eo} + \omega_{eo}^* = \omega_{he} + \omega_{he}^* + \omega_{eo} + \omega_{eo}^* .$$

It follows that

$$\eta = \Theta - \omega_{eo} - \omega_{he} = i\Theta^* + \omega_{he}^* + \omega_{eo}^*$$

is harmonic exact except for the singularity and so is η^*. We may write $\eta = \phi + \bar{\psi}$ where ψ is analytic and ϕ is analytic except for the singularity. It follows that ϕ is the differential mentioned in the lemma; $\phi = dF_m$ where F_m is an analytic function except for the singularity

$$\frac{-1}{(m+1)(z-\zeta)^{m+1}}$$

and from [2] there exists an orthogonal kernel dH_m for Γ_{ae} on W_E.

Note. An analogous proof works for differentials with $s(z)dz = dz/(z-\zeta_1) - dz/(z-\zeta_2)$; we have only to discard the periods about $z = \zeta_1$ and $z = \zeta_2$.

From the existence of orthogonal kernels for Γ_{ae} we can state the following theorems; here $B(f) = \frac{1}{2}\int_\beta f d\bar{f}$; H_s and K_s are analytic functions whose differentials are respectively the orthogonal and reproducing kernels for Γ_{ae}, corresponding to the singularity.

THEOREM IV F: *Among all functions f analytic except for a simple pale at $z = \zeta$ with expansion $f = c_1/(z-\zeta) + \alpha_1(z-\zeta) + \cdots$ in a neighborhood of $z = \zeta$, $H_s + \lambda K_s$ minimizes the functional $B(f) + 2\pi Re\bar{\lambda}\bar{c}_1\alpha_1$.*

THEOREM IV G: *Among all functions $f(z)$ analytic except for the singularity*

$$S(z) = \sum_{j=1}^{N}\sum_{k=0}^{\infty} \frac{c_k^j}{(z-\zeta_j)^{k+1}(-k-1)},$$

the function H_s minimizes $B(f)$.

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