ANALYTIC FUNCTIONS WITH VALUES IN A FRECHET SPACE

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We wish to extend certain results in the theory of analytic functions of several complex variables to the case of analytic functions with values in a Frechet space F. To do this, we prove (Theorem 1 below) that such a function φ has an expansion of the form

$$(*) \qquad \varphi = \sum_{n=1}^{\infty} P_n \circ \varphi ,$$

where $\{P_n\}$ is a sequence of continuous mutually annihilating projections on F whose ranges are all one-dimensional subspaces of F. This representation reduces the study of φ, for many purposes, to the study of the functions $P_n \circ \varphi$, which are essentially scalar-valued analytic functions. We actually prove the stronger (and more useful) result that if $\{\varphi_k\}$ is a sequence of analytic functions with values in F then a single sequence $\{P_n\}$ can be found to give an expansion $(*)$ for every φ_k. Expansions of vector-valued functions of a different type have been considered by Grothendick [6].

Theorem 1 is applied to generalize Theorem B of H. Cartan [3]. We consider a coherent analytic sheaf S on a Stein manifold M and introduce the notion of the *vectorization* S_F of S (relative to a given Frechet space F).

If 0 denotes the sheaf of locally-defined analytic functions and 0_F denotes the sheaf of locally-defined analytic functions with values in F, then S_F is defined to be the tensor product $S \otimes 0_F$ of the 0-modules S and 0_F. For the important case of a coherent analytic subsheaf S of the sheaf 0^k of locally-defined k-tuples of analytic functions, S_F turns out to be canonically isomorphic to the sheaf S'_F determined by assigning to each open set U the module of all k-tuples $(f_1, \cdots, f_k)$ of analytic functions from U to F which have the property that for each u in F^* the k-tuple $(u \circ f_1, \cdots, u \circ f_k)$ is a cross-section of S over U. For instance, if S is the sheaf of all locally-defined analytic functions which vanish on a given analytic set A then it is evident that S'_F is the sheaf of all locally-defined analytic functions with values in F which vanish on A.

One of the main results, an extension of Theorem B of [3], will be that the cohomology groups $H^N(M, S_F)$ vanish in all dimensions $N \geqq 1$, where S_F is the vectorization of a coherent analytic sheaf S on a Stein manifold M. Using this theorem and the isomorphism of S_F to the sheaf S'_F defined above one could show, for instance, that the usual

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sheaf—theoretic solutions to Cousin's problems carry over to the case of analytic functions with values in a Frechet space. Special cases were treated by totally different methods in [2], but the techniques of that paper seem to be inadequate to obtain general results.

The proofs are all Banach-space theoretic. That is, only Banach space theory is necessary to obtain the above extension of Theorem B and to prove the necessary facts about vectorizations. We begin with a theorem which is given without proof on p. 278 of Banach [1], who attributes it to H. Auerbach. A proof can be found in Taylor [7]. Since complex Banach spaces are considered here, we give the proof.

THEOREM (Auerbach). *An n-dimensional Banach space B has a basis of unit vectors whose dual basis also consists of unit vectors.*

Proof. Choose a basis $(b^1, \cdots, b^n)$ of B and for any x in B let $(x_1, \cdots, x_n)$ be the coordinates of x relative to the chosen basis. Let T be the set of all n-tuples $(x^1, \cdots, x^n)$ of unit vectors in B. For each $(x^1, \cdots, x^n)$ in T let $\alpha(x^1, \cdots, x^n)$ be the absolute value of the determinant $\det(x^i_j)$. Thus α is a continuous function on the compact space T. Now $\alpha(x^1, \cdots, x^n) \neq 0$ if and only if $(x^1, \cdots, x^n)$ is a basis. Thus α attains its maximum for T at some point $(y^1, \cdots, y^n)$ in T which is a basis of unit vectors. Let $(u^1, \cdots, u^n)$ be the dual basis in B^*. Now $\|u^i\| \geqq 1$ because $\langle y^i, u^i \rangle = 1$. Assume $\|u^i\| > 1$ for some i. Thus there exists t in B with $\|t\| = 1$ and $\langle t, u^i \rangle = c > 1$. Thus $\langle t - cy^i, u^i \rangle = 0$, so that $t - cy^i$ is a linear combination of the vectors of the basis $(y^1, \cdots, y^n)$ other than y^i. If we let $(z^1, \cdots, z^n)$ be the basis $(y^1, \cdots, y^n)$ with y^i replaced by t it follows that $\alpha(z^1, \cdots, z^n) = c\alpha(y^1, \cdots, y^n)$. Since the basis $(z^1, \cdots, z^n)$ consists of unit vectors this contradicts the choice of $(y^1, \cdots, y^n)$. Thus $\|u^i\| = 1$ for all i, and the theorem is proved.

COROLLARY. *If B_0 is a finite-dimensional subspace of dimension n of a Banach space B there exist n mutually annihilating projections (idempotent continuous linear operators) on B, each of norm 1, whose ranges are one-dimensional subspaces of B_0 and whose sum is a projection of B onto B_0 of norm at most n.*

Proof. Let $(y^1, \cdots, y^n)$ be a basis of unit vectors of B_0 such that the dual basis $(u^1, \cdots, u^n)$ of B_0^* also consists of unit vectors. Let v^i be an extension of u^i to a linear functional on B of norm 1. The operators $P_1, \cdots, P_n$ on B defined by

$$P_i x = \langle x, v^i \rangle y^i$$

are the desired projections.

We recall that a Frechet space is a locally convex topological linear space F which admits a countable family $\{\|\ \|_k\}$ of continuous semi-norms such that a basis for the neighborhoods of 0 in F is given by the sets

$$\{x \in F : \|x\|_k < 1\}\ .$$

If $\|\ \|$ is any continuous semi-norm on F it follows that for some k $\|x\| \leqq \|x\|_k$ for all x in F. If necessary it may be assumed that $\{\|\ \|_k\}$ is a monotonely nondecreasing sequence of semi-norms, in which case we shall call it a *defining sequence* of semi-norms for F.

LEMMA 1. *Let F be a Frechet space with a defining sequence $\{\|\ \|_k\}$ of semi-norms. Let $\{a_n\}$ be a sequence of vectors in F, $\{\delta_k\}$ a sequence of nonnegative real numbers, and $\{k_j\}$ a strictly increasing sequence of positive integers. Then there exists a sequence $\{P_n\}$ of mutually annihilating continuous projections on F, whose ranges are subspaces of F of dimensions at most* 1, *and a sequence $\{\varepsilon_k\}$, with $0 < \varepsilon_k < \delta_k$ for all k, with the following properties. For each positive integer j the operator*

$$Q_j = \sum_{n=1}^{k_j} P_n$$

is a projection on the subspace B_j of F spanned by the vectors $a_1, \cdots, a_{k_j}$. For each positive integer n the sum

$$\|a\|_0 = \sum_{k=1}^{\infty} \varepsilon_k \|a\|_k$$

is finite for $a = a_n$. For each positive integer j and all $n \leqq k_j$ we have $\|P_n\|_0 \leqq (1 + k_1^2) \cdots (1 + k_j^2)$, where

$$\|P_n\|_0 = \sup\{\|P_n b\|_0 : b \in F,\quad \|b\|_0 = 1\}\ .$$

Proof. We may assume the δ_k to be so small that $\sum_{k=1}^{\infty} \delta_k \|a_n\|_k < \infty$ for all n. By induction we construct a sequence $\{P_n\}$ of mutually annihilating continuous projections, a sequence $\{\varepsilon_k\}$ of positive real numbers, and an increasing sequence $\{N_j\}$ of positive integers such that

(a) $0 < \varepsilon_k < \delta_k$,

(b) For each j the operator Q_j is a projection onto B_j,

(c) $\|P_n\|^j < (1 + k_1^2) \cdots (1 + k_i^2)$ for $1 \leqq n \leqq k_i$ and all $i \leqq j$.

We explain what is meant by (c). First of all, $\|\ \|^j$ is the continuous semi-norm on F defined by

$$\|b\|^j = \sum_{k=1}^{N_j} \varepsilon_k \|b\|_k\ .$$

Secondly, $\|P_n\|^j$ is defined by

$$\|P_n\|^j = \sup\{\|P_n b\|^j : \|b\|^j = 1\}\ .$$

Assuming that $P_1, \cdots, P_{k_j}$ and $N_1 \cdots, N_j$, and $\varepsilon_1, \cdots, \varepsilon_{N_j}$ have been found with the relevant properties, we show how to continue to the next stage $j+1$. First choose $N_{j+1} > N_j$ so large that $\|\ \|_{N_{j+1}}$ is a norm (and not merely a semi-norm) on B_{j+1}. Choose then ε_i, $N_j < i \leqq N_{j+1}$, so small that $0 < \varepsilon_i < \delta_i$ and $\| P_n \|^{j+1} < (1 + k_1^2) \cdots (1 + k_i^2)$ for $n \leqq k_j$ and all $i \leqq j$. To see that this can be done, notice that because $\|\ \|_{N_j}$ is a norm on B_j there exists $r > 0$ so that $r \| a \|^j > \| a \|_m$ for all a in B_j and all $m \leqq N_{j+1}$. Thus

$$\| P_n \|^{j+1} \leqq \sup \{\| P_n b \|^{j+1} : \| b \|^j = 1\} \leqq (1 + \sum_{m=N_j+1}^{N_{j+1}} \varepsilon_m) \| P_n \|^j .$$

Now use (c).

Now let Q'_j be the restriction of Q_j to B_{j+1} and let I_{j+1} be the identity operator on B_{j+1}. Thus $I_{j+1} - Q'_j$ is a projection of B_{j+1} onto a subspace S_{j+1}. Clearly B_j and S_{j+1} are complementary subspaces of B_{j+1}, so that $\dim S_{j+1} \leqq k_{j+1} - k_j$. By the above corollary there exists a projection E_{j+1} with $\| E_{j+1} \|^{j+1} \leqq k_{j+1}$ of F onto B_{j+1}. Also by the above corollary there exist mutually annihilating projections R_n, $k_j < n \leqq k_{j+1}$, of S_{j+1} onto subspaces of dimensions at most 1 such that $\| R_n \|^{j+1} \leqq 1$ for all n and such that ΣR_n is the identity projection of S_{j+1} onto itself. For $k_j < n \leqq k_{j+1}$ we define

$$P_n = R_n(I_{j+1} - Q'_j)E_{j+1} .$$

Thus the P_n are mutually annihilating projections for $1 \leqq n \leqq k_{j+1}$. Also Q_{j+1} is a projection onto B_{j+1}. Finally for $k_j < n \leqq k_{j+1}$ we have

$$\begin{aligned} \| P_n \|^{j+1} &\leqq \| R_n \|^{j+1} \| I_{j+1} - Q'_j \|^{j+1} \| E_{j+1} \|^{j+1} \\ &\leqq (1 + \sum_{n=1}^{k_j} \| P_n \|^{j+1}) k_{j+1} \\ &< [1 + k_j(1 + k_1^2) \cdots (1 + k_j^2)] k_{j+1} \\ &\leqq (1 + k_1^2) \cdots (1 + k_{j+1}^2) . \end{aligned}$$

The same is true for $n \leqq k_j$, by the above construction. Thus the construction has been continued another step. By induction it follows that sequences $\{P_n\}$, $\{N_j\}$, and $\{\varepsilon_k\}$ can be chosen satisfying properties (a), (b), and (c). It is immediate that the sequences $\{P_n\}$ and $\{\varepsilon_k\}$ satisfy the requirements of the lemma.

LEMMA 2. *Let $\{a_n\}$ be a sequence of elements of a Frechet space F, $\{\|\ \|_k\}$ a defining sequence of semi-norms on F, and $\{\delta_k\}$ a sequence of positive real numbers. Then there exist a sequence $\{\varepsilon_k\}$ of positive real numbers and a sequence $\{P_n\}$ of mutually annihilating projections on F whose ranges are subspaces of F of dimensions at most* 1 *having the following properties.*

(i) $0 < \varepsilon_k < \delta_k$ *for all* k,

(ii) *For* $a = a_n$ *the norm* $\| a \|_0 = \sum_{k=1}^{\infty} \varepsilon_k \| a \|_k$ *is finite for all* n,

(iii) $R_m a_n = a_n$ *for all positive integers* m *and* n *with* $m \geqq 2n$, *where* $R_m = \sum_{j=1}^{m} P_j$,

(vi) *For all* $t > 1$ *and* $\varepsilon > 0$ *the sum* $\sum_{n=1}^{\infty} \| P_n \|_0 t^{-n^{\varepsilon}}$ *converges, where* $\| P_n \|_0$ *is defined as above.*

Proof. Define the sequence $\{k_j\}$ by $k_j = 2^j$. Choose the sequences $\{P_n\}$ and $\{\varepsilon_k\}$ as in lemma 1. Clearly (i) and (ii) are satisfied. Now for each positive integer n there is a positive integer j with $2^{j-1} \leqq n < 2^j$. It follows that $a_n \in B_j$. Thus $R_{2^j} a_n = Q_j a_n = a_n$, so that $R_m a_n = a_n$ for all $m \geqq 2^j$ and therefore for all $m \geqq 2n$. This proves (iii).

Now for each n choose j with $2^{j-1} \leqq n < 2^j$. Thus

$$\begin{aligned} \| P_n \|_0 &\leqq (1 + k_j^2)^j = (1 + 2^{2j})^j \\ &\leqq (5n^2)^j \leqq (5n^2)^{\alpha} , \end{aligned}$$

where $\alpha = 1 + \log_2 n$. From this it follows from elementary calculus that (iv) holds, thereby proving the lemma.

LEMMA 3. *Let*

$$\sum_{n_1 \geqq 0, \cdots, n_\alpha \geqq 0} a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha}$$

where $\alpha = \alpha_i$ *and* $1 \leqq i < \infty$, *be a sequence of formal power series with coefficients in a Frechet space* F. *Let* $\{\delta_k\}$ *be a sequence of positive real numbers. Then there exists a sequence* $\{\varepsilon_k\}$ *with* $0 < \varepsilon_k < \delta_k$ *for all* k *and a sequence* $\{P_n\}$ *of mutually annihilating continuous projections of* F *onto subspaces of dimensions at most* 1 *such that*

(a) $R_m a_i(n_1, \cdots, n_\alpha) = a_i(n_1, \cdots, n_\alpha)$ *whenever* $m \geqq 2^{i+2} n^{\alpha}$, *where* $\alpha = \alpha_i$, $n = n_1 + \cdots + n_\alpha$, *and* $R_m = \sum_{j=1}^{m} P_j$,

(b) $P_m a_i(n_1, \cdots, n_\alpha) = 0$ *whenever* $m > 2^{i+2} n^{\alpha}$,

(c) $\sum_{n=1}^{\infty} \| P_n \|_0 t^{-n^{\varepsilon}} < \infty$ *for all* $t > 1$ *and* $\varepsilon > 0$, *where* $\| \ \|_0$ *is defined as above.*

Proof. For each i order the coefficients $a_i(n_1, \cdots, n_\alpha)$ into a sequence $\{\alpha_i^k\}_{k=1}^{\infty}$ according to the size of n. We now define a sequence $\{a_k\}$ of elements of F which is an ordering of the totality of the $a_i(n_1, \cdots, n_\alpha)$. For k given let 2^i be the largest power of 2 dividing k and let $j = 1/2(k2^{-i} + 1)$. Let $a_k = \alpha_i^j$. Now choose the sequences $\{\varepsilon_k\}$ and $\{P_n\}$ as in Lemma 2. Clearly (c) holds. Since (b) is a consequence of (a) we need only check (a). To this end consider a fixed $a_i(n_1, \cdots, n_\alpha)$. Now there exists $j \leqq n^{\alpha}$ with $a_i(n_1, \cdots, n_\alpha) = \alpha_i^j$. In turn $\alpha_i^j = a_k$ for some $k \leqq 2^{i+1} n^{\alpha}$. By (iii) of Lemma 2 it follows that $R_m a_k = a_k$ for $m \geqq 2k$ and therefore for $m \geqq 2^{i+2} n^{\alpha}$, as was to be proved.

We are now prepared to prove a series representation for analytic functions with values in a Frechet space which will be the principal tool in subsequent proofs.

THEOREM 1. *Let F be a Frechet space and let $\{M_i\}$ be a sequence of complex analytic manifolds. For each i let φ_i be an analytic function on M_i with values in F. Then there exists a sequence of vectors $\{b_n\}$ in F and a sequence $\{P_n\}$ of continuous mutually annihilating projections of F onto one-dimensional subspaces having the following properties. For each i the series $\sum_{n=1}^{\infty} P_n \circ \varphi_i$ converges to φ_i on M_i. For each n we have $P_n b_n = b_n$, so that $P_n \circ \varphi_i = \varphi_i^n b_n$, for some analytic function φ_i^n on M_i. For each i the series $\sum_{n=1}^{\infty} \varphi_i^n$ converges absolutely and uniformly on all compact subsets of M_i. For each continuous semi-norm $\|\ \|$ on F the sequence $\{\| b_n \|\}$ is bounded.*

Proof. For each i let $\dim M_i = \alpha = \alpha_i$, so that M_i is coverable by a countable family of analytic homeomorphs Γ of the unit polycylinder

$$U^\alpha = \{z = (z_1, \cdots, z_\alpha) : |z_j| < 1, 1 \leqq j \leqq \alpha\} .$$

Thus in the proof of the theorem we may replace the sequence $\{M_i\}$ by the totality of all such Γ. There is therefore no loss of generality in assuming that each M_i is a polycylinder U^α of dimension $\alpha = \alpha_i$. Let $\{\|\ \|_k\}$ be a defining sequence of semi-norms on F. Now for each i the analytic function φ_i has a power series expansion

$$\varphi_i = \sum_{n_1 \geqq 0, \cdots, n_\alpha \geqq 0} a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha}$$

on the polycylinder $M_i = U^\alpha$. This expansion converges absolutely and uniformly on each compact subset of M_i in each semi-norm $\|\ \|_k$. By the diagonal process there therefore exist constants $\delta_k > 0$ such that the power series for each φ_i converges absolutely and uniformly on each compact subset of M_i in the norm $\sum_{k=1}^{\infty} \delta_k \|\ \|_k$, so that in particular this norm is finite for each coefficient $a_i(n_1, \cdots, n_\alpha)$. Now choose the sequences $\{\varepsilon_k\}$ and $\{P_n\}$ as in Lemma 3 relative to the power series expansions of the φ_i and to the δ_k just obtained. Thus the power series for φ_i converges absolutely and uniformly on compact subsets of M_i in the norm $\|\ \|_0$ defined above. If some of the projections P_n are zero, these may be omitted from the sequence. Thus for each n there is a vector b_n in F with $\| b_n \|_0 = 1$ spanning the range of P_n. To show that the sequences $\{P_n\}$ and $\{b_n\}$ have the desired properties, consider a fixed compact subset T of a fixed M_i. For each n write

$$\gamma_n = \sum_{n_1 + \cdots + n_\alpha = n} \max \{\| a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha} \|_0 : z \in T\} .$$

By the usual convergence criteria we see that there exist $r > 1$ and $c > 0$ such that $r^n \gamma_n < c$ for all n.

If j is any positive integer let k be the largest integer such that $2^{i+2}k^\alpha < j$. Thus for each z in T we have

$$\begin{aligned} &\| P_j \varphi_i(z) \|_0 \\ &= \left\| P_j \sum_{n_1 + \cdots + n_\alpha \geqq k} a_i(n_1, \cdots, n_\alpha) z_1^{n_1} \cdots z_\alpha^{n_\alpha} \right\|_0 \\ &\leqq \| P_j \|_0 \sum_{n \geqq k} \gamma_n \leqq c \| P_j \|_0 \sum_{n \geqq k} r^{-n} \\ &= c(1 - r^{-1})^{-1} \| P_j \|_0 r^{-k} . \end{aligned}$$

Thus

$$\begin{aligned} \varDelta &= \max \left\{ \sum_{j=1}^{\infty} \| P_j \varphi_i(z) \|_0 : z \in T \right\} \\ &\leqq c(1 - r^{-1})^{-1} \sum_{j=1}^{\infty} r^{-k} \| P_j \|_0 . \end{aligned}$$

Now by the definition of k we see that k is the integral part of $(j2^{-i-2})^{1/\alpha}$, so that $k \geqq j^{1/2\alpha}$ for all j sufficiently large. Thus $\varDelta$ is finite if the sum $\sum_{j=1}^{\infty} r^{-j^\varepsilon} \| P_j \|_0$ converges, where $\varepsilon = (2\alpha)^{-1}$. By the choice of the sequence $\{P_j\}$ this series converges so that $\varDelta$ is finite. Now since $\| b_n \|_0 = 1$,

$$\max \{ | \varphi_i^n(z) | : z \in T \} = \max \{ \| P_n \varphi_i(z) \|_0 : z \in T \} .$$

Therefore the series $\sum_{n=1}^{\infty} \varphi_i^n(z)$ converges absolutely and uniformly on T. If $\| \ \|$ is a continuous semi-norm on F then $\| \ \| \leqq K \| \ \|_0$ for some $K > 0$, so that $\{ \| b_n \| \}$ is bounded by K. Finally, we must show that $\sum_{n=1}^{\infty} P_n \circ \varphi_i$ actually converges to φ_i (and not to something else). To see this, note by (a) and (b) of Lemma 3 that $R_m \circ \varphi_i$ and φ_i have power series expansions in the coordinates $z_1, \cdots, z_\alpha$ which agree up to terms of total order n, whenever $m \geqq 2^{i+2} n^\alpha$. This completes the proof of Theorem 1.

Before giving the definition of the vectorization of an analytic sheaf, we indicate the terminology to be used, following Godement [5]. A presheaf S on a topological space X assigns to each open $U \subset X$ a set $S(U)$ and to each open set $V \subset U \subset X$ a map $r_{VU} : S(U) \to S(V)$ satisfying $r_{WV} \circ r_{VU} = r_{WU}$ for $W \subset V \subset U$. In particular the same terminology will be used if S is a sheaf, that is, a presheaf satisfying axioms (F1) and (F2) on page 109 of [5]. To any presheaf S is canonically associated a sheaf S', and each element f in $S(U)$ gives rise to a unique element in $S'(U)$ which will also be denoted by f. If X is a complex analytic manifold a sheaf S on X is called analytic if it is a module over the sheaf 0 of locally defined analytic functions, that is, if for each U the set $S(U)$ is an $0(U)$-module, and if the usual commutation relations between module multiplication and the restriction maps $S(U) \to S(V)$ and $0(U) \to 0(V)$ hold.

DEFINITION 1. Let S be an analytic sheaf on a complex analytic manifold M and let F be a Frechet space. Let 0 be the sheaf of locally-defined analytic functions on M and let 0_F be the sheaf of locally-defined analytic functions on M with values in F, where by definition a continuous function f from an open set $U \subset M$ to F is called analytic if $u \circ f$ is analytic for all u in F^*. Clearly 0_F is an 0-module, i.e., an analytic sheaf. The vectorization S_F of S (relative to F) is defined to be the sheaf $S \otimes 0_F$, the tensor product of the 0-modules S and 0_F. This is defined in [5] as the sheaf determined by the presheaf data

$$U \to S(U) \otimes 0_F(U) ,$$

where $S(U)$ and $0_F(U)$ are considered as $0(U)$-modules, together with the obvious restriction maps.

Note that if T is a continuous linear operator from a Frechet space F into a Frechet space G then the natural homomorphism T_0 of 0_F into 0_G induced by T gives rise to a homomorphism $T' = 1 \otimes T_0$ of S_F into S_G. In particular, if u is an element of F^* (and so a continuous linear operator from F into C) then u induces a homomorphism of S_F into S_C. But S_C is canonically isomorphic to S, in virtue of the canonical isomorphism between the $0(U)$-modules $S(U) \otimes 0(U)$ and $S(U)$. (See [5] p. 8.) If we identify S_C with S it follows that each u in F^* induces a homomorphism u' of S_F onto S.

DEFINITION 2. If S is an analytic subsheaf of the Cartesian product 0^n we define

$$S'_F(U) = \{f \in (0_F(U))^n : u \circ f \in S(U) \text{ for all } u \text{ in } F^*\} .$$

Clearly S'_F so defined is an analytic subsheaf of the Cartesian product $(0_F)^n$.

THEOREM 2. *If S is a coherent analytic subsheaf of 0^n then to each p in $U \subset M$ and each f in $S'_F(U)$ there exists a neighborhood V of p, functions $H_1, \cdots, H_k$ in $S(V)$ and functions $G_1, \cdots, G_k$ in $0_F(V)$ such that*

$$r_{VU} f = \sum_{m=1}^{k} G_m H_m .$$

Proof. Since S is coherent, there exists a neighborhood $V_0 \subset U$ of p and functions $H_1, \cdots, H_k$ in $S(V_0)$ which generate S at each point of V_0. We may assume that $\bar{V}_0$ is a compact subset of U. Let $V_0 \supset V_1 \supset V_2 \supset \cdots$

be a basis for the neighborhoods of p. Let Ω be the subset of $S(V_0)$ consisting of all elements in $S(V_0)$ which as elements of $(0(V_0))^n$ are bounded on V_0. Thus to each h in Ω there exists $G = (G_1, \cdots, G_k)$ in $(0(V_i))^k$ for some i such that the restriction of h to V_i has the form

$$h = \sum_{i=1}^{k} G_i H_i .$$

By choosing i large enough we may assume that

$$\| G \|_i = \sup \{| G_j(q) | : q \in V_i, 1 \leqq j \leqq k\}$$

is finite. Thus if for each pair (i, N) of positive integers we let Ω_{iN} be the family of all h in Ω such that G can be chosen in $(0(V_i))^k$ with $\| G \|_i \leqq N$, we see that $\Omega = \bigcup \Omega_{iN}$ and that each Ω_{iN} is a closed subset of Ω, where Ω has the norm defined by

$$\| h \|_0 = \sup \{| h_i(q) | : 1 \leqq i \leqq n, q \in V_0\}$$

for each $h = (h_1, \cdots, h_n) \in \Omega \subset (0(V_0))^n$. By the Baire category theorem there exists (i, N) such that Ω_{iN} has a nonvoid interior. From this it follows as usual that there exists a constant $K > 0$ such that for each h in Ω there exists G in $(0(V_i))^k$ as above with $\| G \|_i \leqq K \| h \|_0$. Now consider f as in the statement of the theorem, so that $f \in S'_F(U) \subset (0_F(U))^n$. By Theorem 1 there exists a sequence of vectors $\{b_j\}$ in F which is bounded in each continuous semi-norm on F and a sequence $\{P_j\}$ of continuous projections on F having one-dimensional ranges such that $\sum_{j=1}^{\infty} P_j \circ f$ converges uniformly to f on all compact subsets of U and such that for each j we have $P_j \circ f = f_j b_j$ with $f_j \in (0(U))^n$, where $\sum_{j=1}^{\infty} | f_j |$ converges uniformly on all compact subsets of U. Thus $\sum_{j=1}^{\infty} \| f_j \|_0$ is finite, since $\bar{V}_0 \subset U$.

Now for each j there exists u in F^* with $\langle b_j, u \rangle = 1$. Thus

$$f_j = u \circ (f_j b_j) = u \circ (P_j \circ f) = (u \circ P_j) \circ f$$

is in $S(U)$ because $f \in S'_F(U)$ and $u \circ P_j \in F^*$. Thus $f_j \in S(U)$ for all j. By the above for each j there exists $G^j = (G_1^j, \cdots, G_k^j)$ in $(0(V_i))^k$ such that on V_i we have

$$f_j = \sum_{m=1}^{k} G_m^j H_m ,$$

with $\| G^j \|_i \leqq K \| f_j \|_0$. It follows that the series $\sum_{j=1}^{\infty} G^j b_j$ converges uniformly and absolutely on V_i in each continuous semi-norm on F. Thus the sum of this series is an element $G = (G_1, \cdots, G_k)$ in $(0_F(V_i))^k$. Thus in the topology of uniform and absolute convergence on compact subsets of V_i in each continuous semi-norm on F we have

$$
\begin{aligned}
f &= \lim_{t\to\infty} \sum_{j=1}^{t} f_j b_j \\
&= \lim_{t\to\infty} \sum_{j=1}^{t} \sum_{m=1}^{k} G_m^j H_m b_j \\
&= \sum_{m=1}^{k} \Big(\lim_{t\to\infty} \sum_{j=1}^{t} G_m^j b_j\Big) H_m \\
&= \sum_{m=1}^{k} G_m H_m \ ,
\end{aligned}
$$

as was to be proved.

The following consequence of Theorem 2 will be useful later.

LEMMA 4. *If the element f of $S_F(U)$ has the property that $u'f$ is the zero element of $S(U)$ for all u in F^* then $f = 0$.*

Proof. By taking a covering of U by small open sets we reduce to the case in which f has a representation

$$
f = \sum_{i=1}^{k} h_i \otimes g_i \ ,
$$

with h_i in $S(U)$ and g_i in $0_F(U)$. Let R be the sheaf on U of relations of $h_1, \cdots, h_k$. Thus for each u in F^* we see that

$$
\begin{aligned}
0 = u'f &= \sum_{i=1}^{k} h_i \otimes \langle g_i, u\rangle \\
&= \sum_{i=1}^{k} \langle g_i, u\rangle h_i \ .
\end{aligned}
$$

Thus by Definition 2 we see that $g = (g_1, \cdots, g_k) \in R'_F(U)$. By Theorem 2 it follows that each p in U has a neighborhood $V \subset U$ such that there exist $H_1, \cdots, H_t$ in $R(V)$ and $G_1, \cdots, G_t$ in $0_F(V)$ with

$$
r_{VU} g = \sum_{j=1}^{t} G_j H_j \ .
$$

Thus for each i with $1 \leqq i \leqq k$ we have

$$
r_{VU} g_i = \sum_{j=1}^{t} G_j H_j^i \ ,
$$

where $H_j = (H_j^1, \cdots, H_j^k)$. Therefore on V we have

$$
\begin{aligned}
f = \sum_{i=1}^{k} h_i \otimes g_i &= \sum_{i=1}^{k} h_i \otimes \Big(\sum_{j=1}^{t} G_j H_j^i\Big) \\
&= \sum_{i=1}^{k} \Big(\sum_{j=1}^{t} h_i \otimes (G_i H_j^i)\Big) \\
&= \sum_{j=1}^{t} \Big(\sum_{i=1}^{k} H_j^i h_i\Big) \otimes G_j = 0
\end{aligned}
$$

since $H_j \in R(V)$ for all j. This proves Lemma 4.

We next give an important characterization of S_F in case S is a coherent analytic subsheaf of 0^n for some positive integer n.

THEOREM 3. *Let M be a Stein manifold and S a coherent analytic subsheaf of 0^n. Let F be a Frechet space. For each open $U \subset M$ there is a mapping $\tau(U)$ from $S(U) \otimes 0_F(U)$ into $(0_F(U))^n$ which for each $h = (h_1, \cdots, h_n)$ in $S(U)$ and g in $0_F(U)$ maps $h \otimes g$ onto $gh = (gh_1, \cdots, gh_n)$ in $(0_F(U))^n$. For each such g and h the image gh of $h \otimes g$ actually lies in the subset $S'_F(U)$ of $(0_F(U))^n$. The family of such mappings $\tau(U)$ induces an isomorphism τ of the sheaf S_F (which was defined above to be the sheaf determined by the presheaf data $U \rightarrow S(U) \otimes 0_F(U)$) onto the sheaf S'_F. Thus S'_F and S_F are isomorphic.*

Proof. Clearly the map of the Cartesian product $S(U) \times 0_F(U)$ into $(0_F(U))^n$ defined by $(h, g) \rightarrow gh$ induces a group homomorphism of $(S(U), 0_F(U))$—the free abelian group generated by the elements of the Cartesian product $S(U) \times 0_F(U)$—into $(0_F(U))^n$. It is trivial to check that $N(S(U), 0_F(U))$ belongs to the kernel of this map, where $N(S(U), 0_F(U))$ is defined as in [5] p. 8 to be the subgroup of $(S(U), 0_F(U))$ generated by elements of the forms

(i) $(x_1 + x_2, y) - (x_1, y) - (x_2, y)$

(ii) $(x, y_1 + y_2) - (x, y_1) - (x, y_2)$

(iii) $(ax, y) - (x, ay)$

where x, x_1, and x_2 are in $S(U)$, y, y_1, and y_2 are in $0_F(U)$, and $a \in 0(U)$. Thus this map induces a homomorphism $\tau(U)$ of the quotient $(S(U), 0_F(U))/N(S(U), 0_F(U)) = S(U) \otimes 0_F(U)$ into $(0_F(U))^n$. It is trivial to check that $\tau(U)$ is an $0(U)$-homomorphism. Now with g and h as above and u in F^* we have

$$u \circ \tau(U)(h \otimes g) = u \circ (gh) = (u \circ g)h \in S(U) .$$

Thus $\tau(U)(h \otimes g) \in S'_F(U)$. It follows that the range of $\tau(U)$ is a subset of $S'_F(U)$. It is now clear that the family of mappings $\tau(U)$ induces an 0-homomorphism τ of S_F into S'_F. To show that τ is one-to-one we must prove

(a) If $\tau(U)(\sum_{i=1}^N h_i \otimes g_i) = 0$ then each p in U has a neighborhood V such that $r_{VU}(\sum_{i=1}^N h_i \otimes g_i) = 0$.

To show that τ is onto we must prove

(b) If $f \in S'_F(U)$ then each p in U has a neighborhood V such that $r_{VU}f = \tau(V)(\sum_{i=1}^N h_i \otimes g_i)$ for some elements h_i in $S(V)$ and g_i in $0_F(V)$.

We first prove (a). If we let R be the sheaf of relations on U of $h_1, \cdots, h_N$ by the coherence of R there exists a neighborhood V_0 of p and elements $r_1 = (r_1^1, \cdots, r_1^N), \cdots, r_n = (r_n^1, \cdots, r_n^N)$ of $R(V_0)$ which generate R at each point of V_0. Now

$$\sum_{i=1}^{N} g_i h_i = \tau(U)\Big(\sum_{i=1}^{N} h_i \otimes g_i\Big) = 0 .$$

Thus for each u in F^* we have

$$\sum_{i=1}^{N} (u \circ g_i) h_i = 0$$

so that $(u \circ g_1, \cdots, u \circ g_N) \in R(U)$ for all u in F^*. By definition this means that $(g_1, \cdots, g_N) \in R'_F(U)$. Therefore by Theorem 2 we see that there exists a neighborhood V of p and $G = (G_1, \cdots, G_n)$ in $(0_F(V))^n$ such that $(g_1, \cdots, g_N) = G_1 r_1 + \cdots + G_n r_n$. Thus on V we have

$$\begin{aligned}\sum_{i=1}^{N} h_i \otimes g_i &= \sum_{i=1}^{N} h_i \otimes \Big(\sum_{j=1}^{n} G_j r_j^i\Big) \\ &= \sum_{j=1}^{n} \Big(\sum_{i=1}^{N} (r_j^i h_i)\Big) \otimes G_j = 0\end{aligned}$$

since $r_j \in R(V)$ for each j. This proves (a).

To prove (b) notice by Theorem 2 that there exists a neighborhood V of p, elements $h_1, \cdots, h_N$ in $S(V)$, and elements $g_1, \cdots, g_N$ in $0_F(V)$ such that on V we have

$$f = \sum_{i=1}^{N} g_i h_i = \tau(V)\Big(\sum_{i=1}^{N} h_i \otimes g_i\Big) .$$

This completes the proof of Theorem 3.

We state for future reference a version of a theorem of Banach, first giving a definition.

DEFINITION 3. If $\{g_n\}$ is a sequence of vectors in a Frechet space F, the series $\sum_{n=1}^{\infty} g_n$ is called *absolutely convergent* if the series $\sum_{n=1}^{\infty} \| g_n \|$ converges for each continuous semi-norm $\| \quad \|$ on F.

Notice that a continuous linear transformation from a Frechet space F to a Frechet space G takes absolutely convergent sequences into absolutely convergent sequences.

LEMMA 5. *Let σ be a continuous linear map of a Frechet space F onto a Frechet space G. Let $\{g_i\}$ be an absolutely convergent sequence from G. Then there exists an absolutely convergent sequence $\{f_i\}$ in F such that $\sigma(f_i) = g_i$ for all i.*

Proof. Let $\{\| \quad \|_k\}$ be a defining sequence of semi-norms on F. Since the map σ is continuous, we see ([1] p. 40) that for each k the set $\sigma\{f : \|f\|_k \leqq 1\}$ contains a neighborhood $\{g : \|g\|'_k \leqq 1\}$ of 0 in G, where $\| \quad \|'_k$ is some continuous semi-norm on G. Thus for each g in

G and each k there exists f in F with $\sigma(f) = g$ and $\|f\|_k \leqq \|g\|'_k$. Now for each k choose $j = j(k)$ such that

$$\sum_{n=j}^{\infty} \|g_n\|'_k < 2^{-k} ,$$

so that

$$\sum_{k=1}^{\infty} \sum_{n=j(k)}^{\infty} \|g_n\|'_k < \infty .$$

We may assume that $j(1) < j(2) < \cdots$. For each n with $j(k) \leqq n < j(k+1)$ choose f_n in F with $\sigma(f_n) = g_n$ and $\|f_n\|_k \leqq \|g_n\|'_k$. If for each n we let $k(n)$ be the smallest value of k for which $n < j(k+1)$, it follows that

$$\sum_{n=1}^{\infty} \|f_n\|_{k(n)} < \infty .$$

Since for each t we have $\|f_n\|_t \leqq \|f_n\|_k$ for all $k \geqq t$ it follows that

$$\sum_{n=1}^{\infty} \|f_n\|_t$$

is finite for all t. This proves the lemma.

THEOREM 4. *If S is a coherent analytic sheaf on a Stein manifold M and if F is a Frechet space then $H^N(M, S_F) = 0$ for all $N \geqq 1$.*

Proof. Let f be an element of $H^N(M, S_F)$. Consider a locally finite covering $\{U_i\}$ of M by holomorphically convex open sets U_i, so fine that f is represented by an element of $H^N(\{U_i\}, S_F)$. For each finite sequence $K = (i_1, \cdots, i_k)$ of positive integers let $U_K = U_{i_1} \cap \cdots \cap U_{i_k}$. The element f of $H^N(M, S_F)$ can be considered to belong to $H^N(\{U_i\}, S_F)$ and therefore can be represented by a cocycle $f = \{f_I\}$ of $Z^N(\{U_i\}, S_F)$. Here I is any sequence of $N+1$ positive integers, and, for each I, f_I is an element of $S_F(U_I)$. Also $\delta f = 0$, where δ is the coboundary operator from $C^N(\{U_i\}, S_F)$ into $C^{N+1}(\{U_i\}, S_F)$ and $Z^N(\{U_i\}, S_F)$ is the kernel of δ. By choosing the covering $\{U_i\}$ fine enough we may assume that for each K there exist elements $h_{1K}, \cdots, h_{\alpha K}$, with α depending on K, in $S(U_K)$ which generate S at each point of U_K. This implies ([3], expose XVIII, p. 9) that every h in $S(U_K)$ has a representation of the form $h = \sum_{i=1}^{\alpha} g_i h_{iK}$, with $g_i \in 0(U_K)$. We may also choose the covering $\{U_i\}$ so fine that, for each I, f_I can be represented in the form

$$f_I = \sum_{i=1}^{\alpha} h_{iI} \otimes g_{iI}$$

with h_{iI} as above and with g_{iI} in $0_F(U_I)$.

By Theorem 1 there exists a sequence $\{P_n\}$ of continuous mutually annihilating projections on F whose ranges are one dimensional and a sequence $\{b_n\}$ of vectors in F bounded in each continuous semi-norm on F having the following properties. For each I and i the series $\sum_{n=1}^{\infty} P_n \circ g_{iI}$ converges to g_{iI} on U_I. For each I and i we have $P_n \circ g_{iI} = g_{iI}^n b_n$, where $g_{iI}^n \in 0(U_I)$. For each I and i the series $\sum_{n=1}^{\infty} g_{iI}^n$ converges absolutely in the Frechet space $0(U_I)$. Now since for each n the projection P_n induces a homomorphism of the sheaf S_F onto itself, the element $\{P_n f_I\}$ of $C^N(\{U_i\}, S_F)$ is in $Z^N(\{U_i\}, S_F)$. Also

$$P_n f_I = \sum_{i=1}^{\alpha} h_{iI} \otimes P_n g_{iI}$$
$$= \sum_{i=1}^{\alpha} h_{iI} \otimes g_{iI}^n b_n = \left(\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}\right) \otimes b_n \,.$$

If for each n and I we let f_I^n be the element $\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}$ of $S(U_I)$ it follows that for each n the element $f^n = \{f_I^n\}^{i=1}$ of $C^N(\{U_i\}, S)$ belongs to $Z^N(\{U_i\}, S)$. It is also clear that $f^n b_n = P_n f$.

Now there exists a natural Frechet space topology on each $S(U)$, described in [4], expose XVII. This topology has the property that for each h in $S(U)$ the map $g \to gh$ of $0(U)$ into $S(U)$ is continuous. We therefore see that for each I the series

$$\sum_{n=1}^{\infty} f_I^n = \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\alpha} g_{iI}^n h_{iI}\right)$$

converges absolutely in $S(U_I)$ because for each I and i the series $\sum_{n=1}^{\infty} g_{iI}^n$ converges absolutely in $0(U_I)$. Now the space $C^N(\{U_i\}, S)$ is the Cartesian product of the Frechet spaces $S(U_I)$, and therefore possesses a Frechet space structure. Moreover $Z^N(\{U_i\}, S)$ is closed in $C^N(\{U_i\}, S)$ and is therefore also a Frechet space. Since for each I the series $\sum_{n=1}^{\infty} f_I^n$ converges absolutely in $S(U_I)$ it follows that $\sum_{n=1}^{\infty} f^n$ converges absolutely in $Z^N(\{U_i\}, S)$. By Theorem B of [3] and Leray's theorem (see [5] p. 213) we see that the coboundary map δ of the Frechet space $C^{N-1}(\{U_i\}, S)$ into $Z^N(\{U_i\}, S)$ is onto. From [4] we also see that δ is continuous.

Let J stand for an arbitrary N-tuple of positive integers. Thus for each J, by the above, there is a continuous homomorphism.

$$\tau_J : (G_1, \cdots, G_\alpha) \to \sum_{i=1}^{\alpha} G_i h_{iJ}$$

of the Frechet space $(0(U_J))^\alpha$ onto the Frechet space $S(U_J)$. These maps induce a continuous homomorphism τ of the Frechet space Φ onto the Frechet space $C^{N-1}(\{U_i\}, S)$, where Φ is defined to be the product $\prod_J (0(U_J))^\alpha$, with α depending as above on J, of the Frechet spaces $(0(U_J))^\alpha$. Thus

$$\sigma = \delta \circ \tau$$

is a continuous homomorphism of Φ onto $Z^N(\{U_i\}, S)$. Since $\sum_{n=1}^{\infty} f^n$ converges absolutely in $Z^N(\{U_i\}, S)$ it follows from Lemma 5 that there exists an absolutely convergent sequence $\{G^n\}$ in Φ with $\sigma(G^n) = f^n$ for all n. For each n write $G^n = \{G_J^n\}$, where

$$G_J^n = (G_{1J}^n, \cdots, G_{\alpha J}^n) \in (0(U_J))^\alpha .$$

Thus for each J we see that the series $\sum_{n=1}^{\infty} G_J^n$ converges absolutely and uniformly on every compact subset of U_J, so that the series $\sum_{n=1}^{\infty} G_J^n b_n$ converges absolutely in $(0_F(U_J))^\alpha$ to an element

$$G_J = (G_{1J}, \cdots, G_{\alpha J})$$

in $(0_F(U_J))^\alpha$. Thus for each i and J we have $G_{iJ} = \sum_{n=1}^{\infty} G_{iJ}^n b_n$.

For each J let e_J be the element

$$e_J = \sum_{i=1}^{\alpha} h_{iJ} \otimes G_{iJ}$$

of $S_F(U_J)$. Thus $e = \{e_J\} \in C^{N-1}(\{U_i\}, S_F)$. We shall finish the proof by showing that $\delta e = f$. To this end it is sufficient by Lemma 4 to show $u'(\delta e) = u'(f)$ for all u in F^*. We compute:

$$\begin{aligned} u'(e_J) &= \sum_{i=1}^{\alpha} \langle G_{iJ}, u \rangle h_{iJ} \\ &= \sum_{i=1}^{\alpha} \left\langle \sum_{n=1}^{\infty} G_{iJ}^n b_n, u \right\rangle h_{iJ} \\ &= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\alpha} G_{iJ}^n h_{iJ} \right) \langle b_n, u \rangle \\ &= \sum_{n=1}^{\infty} (\tau_J(G_J^n)) \langle b_n, u \rangle \end{aligned}$$

absolutely in $S(U_J)$. Thus

$$u'(e) = \sum_{n=1}^{\infty} (\tau(G^n)) \langle b_n, u \rangle$$

absolutely in $C^{N-1}(\{U_i\}, S)$. Thus

$$\begin{aligned} u'(\delta e) = \delta(u'(e)) &= \sum_{n=1}^{\infty} (\delta \circ \tau)(G^n) \langle b_n, u \rangle \\ &= \sum_{n=1}^{\infty} \sigma(G^n) \langle b_n, u \rangle = \sum_{n=1}^{\infty} f^n \langle b_n, u \rangle . \end{aligned}$$

Also for each I we have

$$u'(f_I) = \sum_{i=1}^{\alpha} \langle g_{iI}, u \rangle h_{iI}$$

$$= \sum_{i=1}^{\alpha} \left\langle \sum_{n=1}^{\infty} g_{iI}^{n} b_n, u \right\rangle h_{iI}$$

$$= \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\alpha} g_{iI}^{n} h_{iI} \right) \langle b_n, u \rangle = \sum_{n=1}^{\infty} f_I^n \langle b_n, u \rangle .$$

Therefore $u'(f) = \sum_{n=1}^{\infty} f^n \langle b_n, u \rangle$. It follows that $u'(f) = u'(\delta e)$ for all u in F^*, so that $f = \delta e$. This completes the proof of Theorem 4.

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