ON THE GENERATION OF DISCONTINUOUS GROUPS

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In a paper in this Journal (v. 11, p. 675) M. I. Knopp remarked that $G(j)$, the principal congruence subgroup of level $j \geqq 2$ of the modular group, can be generated exclusively by parabolic transformations if and only if it is of genus zero. The following natural generalization is easily proved:

Let Γ be a horocyclic[1] group of genus g. Then Γ possesses a system of generators consisting entirely of parabolic and elliptic elements if and only if $g = 0$.

Knopp's result is a special case, since $G(j)$ has no elliptic substitutions.

For the proof we appeal to the classical result that Γ has a canonical fundamental region whose sides are conjugated by elliptic and parabolic substitutions and $2g$ hyperbolic substitutions $A_1, B_1, \cdots, A_g, B_g$ (cf. [**1**], p. 182 ff). These substitutions generate Γ. If $g = 0$, the hyperbolic ones are absent and the conclusion follows.

Conversely, let Γ be generated by elliptic and parabolic transformations $T_1, \cdots, T_s$. Let the domain of existence of Γ be, for example, the upper half-plane H. Denote by H^+ the union of H and the parabolic cusps of Γ. If $g > 0$ there exists an abelian integral of the first kind, that is, a function F regular in H^+ such that

$$(*) \qquad F(Lz) = F(z) + C(L)$$

for all $L \in \Gamma$. Each T_i has a fixed point lying in H^+. Letting z tend to this fixed point in $(*)$, we see that $C(T_i) = 0, \ i = 1, \cdots, s$. Since

$$C(L_1L_2) = C(L_1) + C(L_2)\ ,$$

and the T_i generate Γ, we have

$$C(L) = 0$$

for all $L \in \Gamma$. The abelian integral F has zero periods and is therefore an automorphic *function* on Γ. Since it is regular in the closed fundamental region, it is a constant. Differentiating, we conclude that there are no abelian differentials of the first kind except 0,

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[1] A discontinuous group Γ is called horocyclic (*Grenzkreisgruppe*) if there is a fixed disk (or half-plane) preserved by each element of Γ and every boundary point of the disk is a limit point of Γ.

whence Γ is of genus 0. This completes the proof.

That a group of genus 0 cannot always be generated entirely by parabolic elements is shown by the following example, supplied by Morris Newman. Let H be the group generated by $G = G(3)$ and T, where $T\tau = -1/\tau$. Since T is of period 2 and commutes with G, we have

$$H = G + TG .$$

Now G is of genus 0, as is known. Let $f(\tau)$ be a univalent function on G with a simple pole at $\tau_0 \neq i$. Then $f(\tau) + f(-1/\tau)$ is univalent on H, which is therefore of genus 0. A parabolic element P of H cannot lie in TG, for P has trace ± 2 whereas $TG \equiv T \pmod 3$ has trace divisible by 3. Hence P is in G, and therefore every product of parabolic elements of H is also in G. It follows that H cannot be generated by parabolic elements alone.

Instead of $G(3)$ we could also have used $G(4)$ or $G(5)$.

Reference

1. R. Fricke-F. Klein, *Vorlesungen über die Theorie der automorphen Funktionen*, vol. 1, Teubner, Leipzig, 1897.

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