ON DENSITIES OF SETS OF LATTICE POINTS

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1. Introduction. Let A be a set of positive integers, and for any positive integer x denote by $A(x)$ the number of integers of A which are not greater than x. Then the Schnirelmann density of A is defined [4] to be the quantity

$$\alpha = \operatorname{glb}_x \frac{A(x)}{x} .$$

For any k sets $A_1, \cdots, A_k$ of positive integers, $k \geqq 2$, let the sum set $A_1 + \cdots + A_k$ be the set of all nonzero sums $a_1 + \cdots + a_k$ for which each a_i, $i = 1, \cdots, k$, is either contained in A_i or is 0. Let kA be the set $A + \cdots + A$ with k summands.

Schnirelmann [4] and Landau [2] have shown that if A and B are two sets of positive integers with $C = A + B$, and if α, β, γ are the Schnirelmann densities of A, B, C, respectively, then $\gamma \geqq \alpha + \beta - \alpha\beta$, and if $\alpha + \beta \geqq 1$ then $\gamma = 1$. They have also shown that if A is a set of positive integers whose Schnirelmann density is positive then A is a basic sequence for the set of positive integers, or, in other words, there exists a positive integer k such that every positive integer can be written as the sum of at most k elements of A.

We will show that by using extensions of the methods employed by Schnirelmann and Landau the above results can be generalized to certain sets of vectors in a discrete lattice (for definition and discussion see [3, pp. 28–31] or [5, pp. 141–145]). Without loss of generality it may be assumed that the components of the vectors in such a lattice are rational integers. The usual identification of algebraic integers with lattice points then gives an immediate extension of these results to algebraic integers.

2. Notation and definitions. Let Q_n be the set of all n-dimensional lattice points $(x_1, \cdots, x_n)$, $n \geqq 1$, for which each x_i, $i = 1, \cdots, n$, is a nonnegative integer and at least one x_i is positive. Define the sum of subsets of Q_n in the same manner as was done for sets of positive integers, and for any subsets A and B of Q_n let $A - B$ denote the set of all elements of A which are not in B. If A and S are subsets of Q_n and S is finite let $A(S)$ be the number of elements in $A \cap S$.

DEFINITION 1. A finite nonempty subset R of Q_n will be called a

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fundamental subset of Q_n or, briefly, a *fundamental set*, if whenever an element $(r_1, \cdots, r_n)$ is in R then all elements $(x_1, \cdots, x_n)$ of Q_n such that $x_i \leqq r_i$, $i = 1, \cdots, n$, are also in R.

DEFINITION 2. Let A be any subset of Q_n. The *density* of A is defined to be the quantity

$$\alpha = \operatorname{glb} \frac{A(R)}{Q_n(R)}$$

taken over all fundamental sets R.

3. Extension of the Landau-Schnirelmann results. Throughout this section we let A and B be subsets of Q_n with $C = A + B$, and let α, β, γ be the densities of A, B, C, respectively.

THEOREM 1. *If* $\alpha + \beta \geqq 1$ *then* $\gamma = 1$.

Proof. Assume $\gamma < 1$. Then there exists a fundamental set R for which $C(R) < Q_n(R)$, which in turn implies that there exists an element $(x_1^0, \cdots, x_n^0)$ in $Q_n - C$. Let R_0 be the set of all elements $(x_1, \cdots, x_n)$ in Q_n for which $x_i \leqq x_i^0$, $i = 1, \cdots, n$. Then for any $(x_1, \cdots, x_n)$ in R_0 either $(x_1, \cdots, x_n)$ is in A, or $(x_1, \cdots, x_n) = (x_1^0, \cdots, x_n^0) - (b_1, \cdots, b_n)$ for some $(b_1, \cdots, b_n)$ in $B \cap R_0$, or neither, but not both. In particular, $(x_1^0, \cdots, x_n^0)$ is neither. Hence,

$$A(R_0) + B(R_0) \leqq Q_n(R_0) - 1 ,$$

and

$$\alpha + \beta \leqq \frac{A(R_0) + B(R_0)}{Q_n(R_0)} < 1$$

which is a contradiction. Therefore $\gamma = 1$.

THEOREM 2. $\gamma \geqq \alpha + \beta - \alpha\beta$.

Proof. Let ω_i, $1 \leqq i \leqq n$, be that vector in Q_n for which the ith component is 1 and the other components, if any, are 0. If any one of the vectors $\omega_1, \cdots, \omega_n$ is missing from A then $\alpha = 0$ and the theorem is trivial. Hence we assume all the vectors $\omega_1, \cdots, \omega_n$ are in A. We must show

$$\frac{C(R)}{Q_n(R)} \geqq \alpha + \beta - \alpha\beta \tag{1}$$

for all fundamental sets R. If $C(R) = Q_n(R)$ then (1) holds, since

$(1-\alpha)(1-\beta) \geqq 0$ implies $1 \geqq \alpha + \beta - \alpha\beta$. Therefore we assume $C(R) < Q_n(R)$ and, consequently, $A(R) < Q_n(R)$.

Let $H = R - A$. We will show that there exist vectors $a^{(1)}, \cdots, a^{(s)}$ in A and sets $L_1, \cdots, L_s$ with the following properties.

(i) $L_i \subseteqq H$ and L_i is not empty, $i = 1, \cdots, s$.

(ii) The sets $L'_i = \{x - a^{(i)} \mid x \in L_i\}$ are fundamental sets.

(iii) $L_i \cap L_j = \phi$ for $i \neq j$.

(iv) $H = L_1 \cup \cdots \cup L_s$.

Let the elements of R be ordered so that $(x_1, \cdots, x_n) > (x'_1, \cdots, x'_n)$ if $x_1 > x'_1$ or if $x_1 = x'_1, \cdots, x_p = x'_p, x_{p+1} > x'_{p+1}$. For every $h = (h_1, \cdots, h_n)$ in H, let A_h be the set of all $(a_1, \cdots, a_n)$ in A such that each $a_i \leqq h_i$. The sets A_h are not empty since $\omega_i \in A$ for $i = 1, \cdots, n$. The A_h are finite sets, hence they contain (in our ordering) a largest vector. Let $a^{(1)}, \cdots, a^{(s)}$ be all the distinct vectors that are largest vectors in any A_h. Let L_i be the set of all vectors x in H such that $a^{(i)}$ is the largest vector in A_x.

That (i), (iii), and (iv) are satisfied follows immediately from this definition of the L_i. To prove (ii) consider a vector $y = (y_1, \cdots, y_n)$ such that

$$x_j \geqq y_j \geqq a_j^{(i)} , \tag{2}$$

where $x = (x_1, \cdots, x_n)$ is in L_i and $y \neq a^{(i)}$. Suppose $y \in L_k$, $k \neq i$. Then

$$x_j \geqq y_j \geqq a_j^{(k)} \tag{3}$$

and $a^{(k)} \geqq a^{(i)}$. But (2) and (3) and $x \in L_i$ imply $a^{(k)} \leqq a^{(i)}$, hence $a^{(k)} = a^{(i)}$. Similarly, $y \in A$ implies $y = a^{(i)}$. This proves (ii).

If $b \in B \cap L'_i$ then $a^{(i)} + b$ is in $C \cap L_i$, hence in $C - A$. Therefore,

$$\begin{aligned} C(R) &\geqq A(R) + B(L'_1) + \cdots + B(L'_s) \\ &\geqq A(R) + \beta[Q_n(L'_1) + \cdots + Q_n(L'_s)] \\ &= A(R) + \beta[Q_n(L_1) + \cdots + Q_n(L_s)] \\ &= A(R) + \beta[Q_n(H)] \\ &= A(R) + \beta[Q_n(R) - A(R)] \\ &= (1-\beta)A(R) + \beta[Q_n(R)] \\ &\geqq (1-\beta)\alpha[Q_n(R)] + \beta[Q_n(R)] , \end{aligned}$$

and

$$\frac{C(R)}{Q_n(R)} \geqq \alpha + \beta - \alpha\beta ,$$

which completes the proof.

COROLLARY 1. *Let $A_1, \cdots, A_k$ be any k subsets of Q_n, $k \geqq 2$, let α_i be the density of A_i for $i = 1, \cdots, k$, and let $d(A_1 + \cdots + A_k)$ be the density of $A_1 + \cdots + A_k$. Then*

$$1 - d(A_1 + \cdots + A_k) \leqq (1 - \alpha_1) \cdots (1 - \alpha_k) .$$

Proof. If $k = 2$ then Theorem 2 implies that $1 - d(A_1 + A_2) \leqq 1 - \alpha_1 - \alpha_2 + \alpha_1\alpha_2 = (1 - \alpha_1)(1 - \alpha_2)$. Hence assume $1 - d(A_1 + \cdots + A_{k-1}) \leqq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})$. Then

$$\begin{aligned} 1 - d(A_1 + \cdots + A_{k-1} + A_k) &\leqq [1 - d(A_1 + \cdots + A_{k-1})](1 - \alpha_k) \\ &\leqq (1 - \alpha_1) \cdots (1 - \alpha_{k-1})(1 - \alpha_k) . \end{aligned}$$

COROLLARY 2. *If A is any subset of Q_n with density $\alpha > 0$ then there exists an integer $k > 0$ such that $kA = Q_n$.*

Proof. There exists an integer $m > 0$ such that $(1 - \alpha)^m \leqq 1/2$. Let $d(mA)$ be the density of mA. Then Corollary 1 implies that $1 - d(mA) \leqq (1 - \alpha)^m \leqq 1/2$, or $d(mA) \geqq 1/2$. From Theorem 1, $d(mA) + d(mA) \geqq 1$ implies $d(2mA) = 1$, or $2mA = Q_n$.

4. Remark. We may identify Q_2 with the set of nonzero Gaussian integers $x + yi$ for which x and y are both nonnegative rational integers. Luther Cheo [1] defined density for subsets of this Q_2 as follows, using our notation.

DEFINITION 3. Let $x_0 + y_0 i$ be any element of Q_2 and S the set of all $x + yi$ in Q_2 such that $x \leqq x_0$ and $y \leqq y_0$. Then for any subset A of Q_2 the *density* of A is the quantity

$$\alpha_c = \operatorname*{glb}_S \frac{A(S)}{Q_2(S)} .$$

Cheo proved Theorem 1 for his density and also a theorem which implies that if ji is in A for all $j = 1, 2, \cdots$, and if $\alpha_c, \beta_c, \gamma_c$ are the Cheo densities of $A, B, C = A + B$, respectively, then

$$\gamma_c \geqq \alpha_c + \beta_c - \alpha_c\beta_c .$$

We cannot remove the requirement that all ji be in A by means of an argument like that used to establish Theorem 2 since it would be necessary to partition H in such a way that the sets L'_j are of the type S used in defining the Cheo density, and this is not always possible. Consider, for example, the set $R = \{x + yi: x + yi \text{ is in } Q_2, x \leqq 4, y \leqq 3\}$, and let $A \cap R = \{1, i, 3 + 3i\}$. Then $H = R - A$ cannot be so partitioned, as the reader can easily verify.

References

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