PARTIAL ISOMETRIES

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0. Introduction. For normal operators on a Hilbert space the problem of unitary equivalence is solved, in principle; the theory of spectral multiplicity offers a complete set of unitary invariants. The purpose of this paper is to study a special class of not necessarily normal operators (partial isometries) from the point of view of unitary equivalence.

Partial isometries form an attractive and important class of operators. The definition is simple: a partial isometry is an operator whose restriction to the orthogonal complement of its null-space is an isometry. Partial isometries play a vital role in operator theory; they enter, for instance, in the theory of the polar decomposition of arbitrary operators, and they form the cornerstone of the dimension theory of von Neumann algebras. There are many familiar examples of partial isometries: every isometry is one, every unitary operator is one, and every projection is one. Our first result serves perhaps to emphasize their importance even more; the assertion is that the problem of unitary equivalence for completely arbitrary operators is equivalent to the problem for partial isometries. Next we study the spectrum of a partial isometry and show that it can be almost anything; in the finite-dimensional case even the multiplicities can be prescribed arbitrarily. In a special (finite) case, we solve the unitary equivalence problem for partial isometries. After that we ask how far a partial isometry can be from the set of normal operators and obtain a very curious answer. Generalizing and simplifying a result of Nagy, we show also that if two partial isometries are sufficiently near, then some natural cardinal numbers (dimensions) associated with them are the same. This result yields a partitioning of the metric space of all partial isometries into open-closed sets, and we conclude by proving that these sets are exactly the components.

For any operator A with null-space $\mathfrak{N}$ we write $\nu(A) = \dim \mathfrak{N}$ and we call $\nu(A)$ the *nullity* of A. If A is a partial isometry with range $\mathfrak{R}$, we write $\rho(A) = \dim \mathfrak{R}$ and $\rho'(A) = \dim \mathfrak{R}^{\perp}$; the cardinal numbers $\rho(A)$ and $\rho'(A)$ are the *rank* and the *co-rank* of A. The subspace $\mathfrak{N}^{\perp}$ is the *initial space* of A; the range $\mathfrak{R}$ (which is the same as the image $A\mathfrak{N}^{\perp}$) is the *final space* of A. If A is a partial isometry then so is A^*; the initial space of A^* is the final space of A, and vice versa. It follows that $\nu(A^*) = \rho'(A)$ and

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$\rho'(A^*) = \nu(A)$.

It is natural to define a partial order for partial isometries as follows: $A \leqq B$ in case B agrees with A on the initial space of A. (This implies that the initial space of A is included in the initial space of B.) A partial isometry is maximal with respect to this order if and only if either its initial space or its final space is the entire underlying Hilbert space. It follows that every partial isometry can be enlarged to either an isometry or a co-isometry (the adjoint of an isometry). A necessary and sufficient condition that a partial isometry possess a unitary enlargement (i.e., that there exist a unitary operator that dominates it) is that its nullity be equal to its co-rank. If the underlying Hilbert space is finite-dimensional, this condition is always satisfied; in the infinite-dimensional case it may not be.

1. Reduction. If A is a construction (i.e., if $\|A\| \leqq 1$) on a Hilbert space $\mathfrak{H}$, then $1 - AA^*$ is positive, and, consequently, $1 - AA^*$ has a unique positive square root A'. Assertion: if $M = M(A)$ is the operator matrix $\begin{pmatrix} A & A' \\ 0 & 0 \end{pmatrix}$, interpreted as an operator on $\mathfrak{H} \oplus \mathfrak{H}$, then M is a partial isometry. One quick proof is to compute MM^* and observe that it is a projection; this can happen if and only if M is a partial isometry. Consequence: every contraction on a Hilbert space can be extended to a larger Hilbert space so as to become a partial isometry.

THEOREM 1. *If A and B are unitarily equivalent contractions, then $M(A)$ and $M(B)$ are unitarily equivalent; if, conversely, A and B are invertible contractions such that $M(A)$ and $M(B)$ are unitarily equivalent, then A and B are unitarily equivalent.*

Proof. If U is a unitary operator that transforms A onto B, then U transforms A^* onto B^*, and therefore U transforms A' onto B'; it follows that $\begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix}$ transforms $M(A)$ onto $M(B)$.

Suppose next that A and B are invertible and that $M(A)$ and $M(B)$ are unitarily equivalent. The range of $M(A)$ consists of all column vectors of the form $\begin{pmatrix} Af + A'g \\ 0 \end{pmatrix}$. This set is included in the set of all column vectors with vanishing second coordinate; the invertibility of A implies that the range of $M(A)$ consists exactly of all column vectors with vanishing second coordinate. Since the same is true for $M(B)$, it follows that every unitary operator matrix that transforms $M(A)$ onto $M(B)$ maps the subspace of all vectors of the form $\begin{pmatrix} f \\ 0 \end{pmatrix}$ onto itself. This implies that that subspace reduces every such unitary operator matrix, and hence that every such unitary operator matrix is diagonal. Since the diagonal entries of a diagonal unitary matrix are unitary operators, it follows that A and B are unitarily equivalent, as asserted.

The theorem implies that the problem of unitary equivalence for partial isometries is equivalent to the problem for invertible contractions. The latter problem, in turn, is equivalent to the problem for arbitrary operators. The reason is that by a translation $(A \to A + \alpha)$ and a change of scale $(A \to \beta A)$ every operator becomes an invertible contraction, and translations and changes of scale do not affect unitary equivalence.

Here is a comment on the technique used in the proof. There are many ways that a possibly "bad" operator A can be used to manufacture a "good" one $\left(\text{e.g., } A + A^* \text{ and } \begin{pmatrix} 0 & A \\ A^* & 0 \end{pmatrix}\right)$. None of these ways has ever yielded sufficiently many usable unitary invariants for A. It is usually easy to prove that if A and B are unitarily equivalent, then so are the various constructs in which they appear. It is, however, usually false that if the constructs are unitarily equivalent, then so are A and B. In the case treated by Theorem 1 this converse is true, and its proof is the less trivial part of the argument.

2. Spectrum. What can the spectrum of a partial isometry be? Since a partial isometry is a contraction, its spectrum is included in the closed unit disc. If the partial isometry is invertible (i.e., if 0 is not in the spectrum), then it is unitary, and therefore the spectrum is a non-empty compact subset of the unit circle; well known constructions prove that every such set is the spectrum of some unitary operator. If the partial isometry is not invertible, then its spectrum contains 0; what else can be said about it? The answer is, nothing else. This answer was pointed out to us by Arlen Brown; its precise formulation is as follows.

THEOREM 2. *If a compact subset of the closed unit disc contains the origin, then it is the spectrum of some partial isometry.*

Proof. It is sufficient to prove that if A is a contraction, then the spectrum of $M(A)$ is the union of the spectrum of A and the singleton $\{0\}$. (This is sufficient because every non-empty compact subset of the closed unit disc is the spectrum of some contraction.) It is easy enough to see that 0 always belongs to the spectrum of $M(A)$; indeed every vector of the form $\begin{pmatrix} 0 \\ f \end{pmatrix}$ is in the null-space of $M(A)^*$. It remains to prove that if $\lambda \neq 0$, then a necessary and sufficient condition that $\begin{pmatrix} A-\lambda & A' \\ 0 & -\lambda \end{pmatrix}$ be invertible is that $A-\lambda$ be invertible. This assertion belongs to the theory of formal determinants of operator matrices. Here is a sample theorem from that theory: if C and D commute and if D is invertible, then a necessary and sufficient condition that $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ be invertible is that $AD-BC$ be invertible. For our present purpose it is sufficient to consider the special case $C=0$, in which case the commutativity hypothesis is automatically satisfied; we proceed to give the proof for that case. If A is invertible, then $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ can be proved to be invertible by exhibiting its inverse: it is $\begin{pmatrix} A^{-1} & A^{-1}BD^{-1} \\ 0 & D^{-1} \end{pmatrix}$. (Recall that the invertibility hypothesis on D is in force throughout.) If, conversely, $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$ is invertible, with inverse $\begin{pmatrix} P & Q \\ R & S \end{pmatrix}$ say, then

$$\begin{pmatrix} AP+BR & AQ+BS \\ DR & DS \end{pmatrix} = \begin{pmatrix} PA & PB+QD \\ RA & RB+SD \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that $DR=0$; since D is invertible, this implies that $R=0$, and hence that $AP=PA=1$. The proof is complete.

3. Multiplicity. For finite sets what the preceding argument proves is this: if $\lambda_1, \cdots, \lambda_n$ are distinct complex numbers with $|\lambda_i| \leqq 1$ for all i, and if $\lambda_i = 0$ for at least one i, then there exists a partial isometry whose spectrum is the set $\{\lambda_1, \cdots, \lambda_n\}$. The partial isometry that the proof yields acts on a space of dimension $2n$ and has a large irrelevant null-space. There is an alternative proof that yields much more for finite sets.

THEOREM 3. *If $\lambda_1, \cdots, \lambda_n$ are complex numbers (not necessarily distinct) with $|\lambda_i| \leqq 1$ for all i, and if $\lambda_i = 0$ for at least one i, then there exists a partial isometry on a space of dimension n, whose characteristic roots are exactly the λ's, each with the algebraic multiplicity equal to the number of times it occurs in the list.*

Proof. The proof can be given by induction on n. For $n=1$, the operator 0 on a space of dimension 1 satisfies all the conditions. The induction step is implied by the following assertion: if an $n \times n$ matrix U with 0 in its spectrum is a partial isometry, and if $|\lambda| \leqq 1$, then there exists a column vector f with n coordinates such that $\begin{pmatrix} U & f \\ 0 & \lambda \end{pmatrix}$ is a partial isometry. To prove this, observe that, since 0 is in the spectrum of U, the column-rank of U is less than n. This makes it possible to find a non-zero vector f orthogonal to all the columns of U; to finish the construction, normalize f so that $\|f\|^2 = 1 - |\lambda|^2$.

4. Equivalence. In at least one case, a very special case, the unitary equivalence problem for partial isometries has a simple and satisfying solution.

THEOREM 4. *If two partial isometries on a finite-dimensional space are such that* 0 *is a simple root of each of their characteristic equations, then a necessary and sufficient condition that they be unitarily equivalent is that they have the same characteristic equation (i.e., that they have the same characteristic roots with the same algebraic multiplicities).*

REMARK. The principal hypothesis is that 0 is a root of multiplicity 1 of the characteristic equation. If this were replaced by the hypothesis that 0 is not a root of the characteristic equation at all (i.e., is a root of multiplicity 0), then the statement would become the classical solution of the unitary equivalence problem for normal operators on a finite-dimensional space.

Proof. The necessity of the condition is trivial. Sufficiency can be proved by induction on the dimension. If the dimension is 1, the assertion is trivial. For the induction step, if the dimension is $n+1$, represent the given partial isometries by triangular matrices with 0 in the northwest corner, and write the results in the form

$$U = \begin{pmatrix} U_0 & f \\ 0 & \lambda \end{pmatrix}, \qquad V = \begin{pmatrix} V_0 & g \\ 0 & \lambda \end{pmatrix},$$

where U_0 and V_0 are $n \times n$ matrices, and f and g are n-rowed column vectors. Since both U and V are partial isometries with first column 0 and rank n, it follows that, in both cases, the remaining n columns constitute an orthonormal set, and hence, in particular, that f is orthogonal to the columns of U_0 and g is orthogonal to the columns of V_0. The thing to prove is that if U_0 and V_0 are unitarily equivalent, then so also are U and V. Suppose therefore that W_0 is unitary and $W_0 U_0 W_0^* = V_0$. Assertion: there exists a complex number θ of modulus 1 such that $W = \begin{pmatrix} W_0 & 0 \\ 0 & \theta \end{pmatrix}$ transforms U onto V. Indeed, if $|\theta| = 1$, then

$$WUW^* = \begin{pmatrix} V_0 & \bar{\theta} W_0 f \\ 0 & \lambda \end{pmatrix},$$

Since this matrix is a partial isometry with first column 0 and rank n, it follows that $\bar{\theta} W_0 f$ is orthogonal to the span (of dimension $n-1$) of the columns of V_0. Since g also is orthogonal to the columuns of V_0, it follows that θ can indeed be chosen so that $\bar{\theta} W_0 f = g$. The only case that gives a moment's pause is the one in which $W_0 f = 0$. In that case $f = 0$, and therefore $|\lambda| = 1$; this implies that $g = 0$, and all is well.

5. Distance. Since the unitary equivalence problem is solved for normal operators, it is reasonable to approach its solution in the general case by asking how far any particular operator is from normality. The figurative "how far" can be interpreted literally, and its literal interpretation yields a curious unitary invariant. Let $\boldsymbol{N}$ be the set of all normal operators, and for each (not necessarily normal) operator A consider the distance $d(A, \boldsymbol{N})$ from A to $\boldsymbol{N}$. The distance here is meant in the usual sense appropriate to subsets of metric spaces: $d(A, \boldsymbol{N}) = \inf\{\|A - N\| : N \in \boldsymbol{N}\}$. The definition makes sense for all operators, and, in particular, for partial isometries. We proceed to study one of the simplest questions that the definition suggests: as U varies over the set $\boldsymbol{P}$ of partial isometries, what possible values can $d(U, \boldsymbol{N})$ attain? The answer we obtain is rather peculiar.

THEOREM 5. *The set of all possible values of $d(U, \boldsymbol{N})$, for U in $\boldsymbol{P}$, is the closed interval* $[0, 1/2]$ *together with the single number* 1.

Proof. We begin with the assertion that if a partial isometry U has a unitary enlargement, then $d(U, \boldsymbol{N}) \leqq 1/2$. The proof consists in verifying that if W is a unitary enlargement of U, then

$$\left\| U - \frac{1}{2} W \right\| = \frac{1}{2} \, .$$

Indeed, if $\mathfrak{N}$ is the null-space of U, then U is equal to 0 on $\mathfrak{N}$ and to W on $\mathfrak{N}^\perp$; it follows that $U - \dfrac{1}{2} W$ is equal to $-\dfrac{1}{2} W$ on $\mathfrak{N}$ and to $\dfrac{1}{2} W$ on $\mathfrak{N}^\perp$. This implies that $U - \dfrac{1}{2} W$ is 1/2 times a unitary operator and hence that its norm is 1/2.

It is easy to exhibit a partial isometry U such that $d(U, \boldsymbol{N}) = 1/2$; in fact this can be done on a two-dimensional Hilbert space. A simple example is the operator U_0 given by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. That U_0 is a partial isometry can be verified at a glance. (Its matrix is obtained from a unitary matrix by "erasing" a column.) The preceding paragraph implies that $d(U_0, \boldsymbol{N}) \leqq 1/2$; it remains to prove that if N is normal, then $\| U_0 - N \| \geqq 1/2$. For this purpose, let f be an arbitrary unit vector and note that

$$\begin{aligned}|\,\|U_0f\| - \|U_0^*f\|\,| &\leqq |\,\|U_0f\| - \|Nf\|\,| + |\,\|N^*f\| - \|U_0^*f\|\,| \\ &\leqq 2\|U_0 - N\| .\end{aligned}$$

(Recall that, by normality, $\|Nf\| = \|N^*f\|$.) If f is the column vector $\begin{pmatrix}0\\1\end{pmatrix}$, then $\|U_0f\| = 1$ and $\|U_0^*f\| = 0$; the proof is complete.

For each number t in the interval $[0, 1]$ write $t' = \sqrt{1-t^2}$. The mapping $t \to U_t = \begin{pmatrix}t & t'\\0 & 0\end{pmatrix}$ is a continuous path in the metric space $\boldsymbol{P}$, which joins the partial isometry U_0 to the projection $\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$ (a normal partial isometry). Conclusion: as U varies over all partial isometries, $d(U, \boldsymbol{N})$ can take (at least) all values between 0 and 1/2 inclusive. (The technique of the preceding paragraph can be used to show that $d(U_t, \boldsymbol{N}) = \frac{1}{2}t'$.)

For the next step we need the following lemma: if P is a projection, and if A is an operator such that $P + A$ is one-to-one, then $\nu(A) \leqq \rho(P)$. To prove this, observe that the null-spaces of P and A have only 0 in common, so that the restriction of P to the null-space $\mathfrak{N}$ of A is one-to-one. It follows that the dimension of $\mathfrak{N}$ is less than or equal to the dimension of the entire range of P, which is the desired conclusion. (We use here the assertion that one-to-one bounded linear transformations do not lower dimension; cf. [2, Lemma 3].)

Suppose now that U is a partial isometry such that $\nu(U) < \rho'(U)$. Assertion: no operator at a distance less than 1 from U can be invertible. Suppose, indeed, that $\|U - A\| < 1$, so that $\|U^*U - U^*A\| < 1$. Write $P = 1 - U^*U$; since U is a partial isometry, P is the projection onto the null-space of U. Since $U^*U - U^*A = 1 - (P + U^*A)$, it follows that $P + U^*A$ is invertible, and hence, from the lemma of the preceding paragraph, that $\nu(U^*A) \leqq \rho(P) = \nu(U)$. If A were invertible, then U^*A and U^* would have the same nullity, and it would follow that $\nu(U^*) \leqq \nu(U)$. This contradicts the assumption on U, and it follows that A cannot be invertible.

Since the closure of the set of all invertible operators includes $\boldsymbol{N}$, it follows from the preceding paragraph that if U is a partial isometry with $\nu(U) < \rho'(U)$, then $d(U, \boldsymbol{N}) \geqq 1$. This result quickly implies some minor improvements of itself. To begin with, the hypothesis $\nu(U) < \rho'(U)$ can be replaced by $\nu(U) \neq \rho'(U)$. (If $\nu(U) > \rho'(U)$, then $\rho'(U^*) > \nu(U^*)$, and the original formulation is applicable to U^*.) Next, the conclusion $d(U, \boldsymbol{N}) \geqq 1$ can be replaced by $d(U, \boldsymbol{N}) = 1$. (Since 0 is normal, no partial isometry is at a distance greater than 1 from $\boldsymbol{N}$.) Finally, the result implies the principal assertion: if U is a partial isometry such that $d(U, \boldsymbol{N}) > 1/2$, then $d(U, \boldsymbol{N}) = 1$. Indeed, if $d(U, \boldsymbol{N}) > 1/2$, then $\nu(U) \neq \rho'(U)$, for otherwise U would have a unitary enlargement, and therefore, by the first paragraph of this proof, U would be at a distance not more than 1/2 from $\boldsymbol{N}$. The proof of Theorem 5 is complete.

6. Continuity. Associated with each partial isometry U there are three cardinal numbers: the rank $\rho(U)$, the nullity $\nu(U)$, and the co-rank $\rho'(U)$. Our next purpose is to prove that the three functions ρ, ν, and ρ' are continuous. For the space $\boldsymbol{P}$ of partial isometries we use the topology induced by the norm; for cardinal numbers we use the discrete topology. With this explanation the meaning of the continuity assertion becomes unambiguous: if U is sufficiently near to V, then U and V have the same rank, the same nullity, and the same co-rank. The following assertion is a precise quantitative formulation of the same result.

THEOREM 6. *If U and V are partial isometries such that $\| U - V \| < 1$, then $\rho(U) = \rho(V)$, $\nu(U) = \nu(V)$, and $\rho'(U) = \rho'(V)$.*

Proof. The null-space of U and the initial space of V can have only 0 in common. Indeed, if f is a nonzero vector such that $Uf = 0$ and $\| Vf \| = \| f \|$, then $\| Uf - Vf \| = \| f \|$, and this contradicts the hypothesis $\| U - V \| < 1$. It follows that the restriction of U to the initial space of V is one-to-one, and hence (see [2] again) that the dimension of the initial space of V is less than or equal to the dimension of the entire range of U. In other words, the result is that $\rho(V) \leqq \rho(U)$; the assertion about ranks follows by symmetry. This part of the theorem generalizes (from projections to arbitrary partial isometries) a theorem of Nagy (see [4, § 105]), and, at the time, considerably shortens its proof. The original proof is, in a sense, more constructive; it not only proves that two subspaces have the same dimension, but it exhibits a partial isometry for which the first is the initial space and the second the final space.

The assertion about ν can be phrased this way: if $\nu(U) \neq \nu(V)$, then $\| U - V \| \geqq 1$. Indeed, if $\nu(U) \neq \nu(V)$, say, for definiteness, $\nu(U) < \nu(V)$, then there exists at least one unit vector f in the null-space of V that is orthogonal to the null-space of U. To say that f is orthogonal to the null-space of U is the same as to say that f belongs to the initial space of U. It follows that $1 = \| f \| = \| Uf \| = \| Uf - Vf \| \leqq \| U - V \|$, and the proof of the assertion about nullities is complete.

The assertion about co-ranks is an easy corollary: if $\| U - V \| < 1$, then $\| U^* - V^* \| < 1$, and therefore $\rho'(N) = \nu(U^*) = \nu(V^*) = \rho'(V)$.

If the dimension of the underlying Hilbert space is δ, then the rank, nullity, and co-rank of each partial isometry are cardinal numbers

ρ, ν, and ρ' such that $\rho + \nu = \rho + \rho' = \delta$. If, conversely, ρ, ν, and ρ' are any three cardinal numbers satisfying these equations, then there exist partial isometries with rank ρ, nullity ν, and co-rank ρ'. Let $\boldsymbol{P}(\rho, \nu, \rho')$ be the set of all such partial isometries. Clearly the sets of the form $\boldsymbol{P}(\rho, \nu, \rho')$ constitute a partition of the space $\boldsymbol{P}$ of all partial isometries; it is a consequence of Theorem 6 that each set $\boldsymbol{P}(\rho, \nu, \rho')$ is both open and closed.

7. **Connectivity.** We proved in § 5 that there is a continuous path in the space $\boldsymbol{P}$ joining a normal partial isometry (in fact a projection) to one whose distance from $\boldsymbol{N}$ is 1/2. On the other hand, § 6 shows that $\boldsymbol{P}$ is not connected, and this suggests the question of just how disconnected $\boldsymbol{P}$ is. The following assertion is the answer.

THEOREM 7. *For each ρ, ν, and ρ', the set $\boldsymbol{P}(\rho, \nu, \rho')$ of all partial isometries of rank ρ, nullity ν, and co-rank ρ' is arcwise connected.*

Proof. The principal tool is the theorem that the set $\boldsymbol{P}(\rho, 0, 0)$ of all unitary operators is arcwise connected. This is a consequence of the functional calculus. Indeed, if U is unitary, then there exists a Hermitian operator A such that $U = e^{iA}$, If $U_t = e^{itA}$, $0 \leqq t \leqq 1$, then $t \to U_t$ is a continuous path of unitary operators joining $1 \ (= U_0)$ to $U (= U_1)$. Since each unitary operator can be joined to 1, it follows that any two can be joined to each other. This settles the case $\boldsymbol{P}(\rho, 0, 0)$. A useful consequence is that if two partial isometries are unitarily equivalent, then they can be joined by a continuous path. Indeed if U_0 and U_1 are partial isometries, and if V is a unitary operator such that $V^* U_0 V = U_1$, then let $t \to V_t$ be a continuous path joining 1 to V, and observe that $t \to V_t^* U_0 V_t$ is a continuous path joining U_0 to U_1.

For the next step we need to recall the basic facts about shifts (see [1] or [3]). A simple shift (more precisely, a simple unilateral shift) is an isometry V for which there exists a unit vector f such that the vectors f, Vf, $V^2 f, \cdots$ form an orthonormal basis for the space. A shift (not necessarily simple) is, by definition, the direct sum of simple ones. It is easy to see that every shift is an isometry whose co-rank is the number of simple direct summands. Two shifts are unitarily equivalent if and only if they have the same co-rank. The fundamental theorem about shifts is that every element of $\boldsymbol{P}(\rho, 0, \rho')$ (i.e., every isometry of co-rank ρ') is either unitary (in which case $\rho' = 0$), or a shift of co-rank ρ', or the direct sum of a unitary operator and a shift of co-rank ρ'.

Suppose now that U_0 and U_1 are in $\boldsymbol{P}(\rho, 0, \rho')$, with $\rho' \neq 0$. If both U_0 and U_1 are shifts, then (since they have the same co-rank) they are unitarily equivalent, and, therefore, they can be joined by a continuous path.

Suppose next that U_0 is a shift (of co-rank ρ') and that $U_1 = V_1 \oplus W_1$, where V_1 is a shift (of co-rank ρ') and W_1 is unitary. Since the dimension of the domain of U_0 is $\rho' \cdot \aleph_0$, and since U_0 and U_1 have the some domain, it follows that the dimension of the domain of W_1 is not more than $\rho' \cdot \aleph_0$. If $\rho' > \aleph_0$, then break up W_1 into ρ' direct summands, each on a space of dimension $\aleph_0$, and match these summands with the ρ' simple direct summands of U_0 and U_1. The result of this procedure is to reduce the problem to the problem of joining a simple shift U to the direct sum of a simple shift V and a unitary operator W on a space of dimension $\aleph_0$ or smaller.

If the dimension of the domain of W is n $(< \aleph_0)$, the problem is easy to describe and to solve in terms of matrices. The shift U is unitarily equivalent to (and therefore it can be joined to) an operator with matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & & \ddots \end{pmatrix},$$

and, similarly, the direct sum $V \oplus W$ can be joined to an operator with matrix

$$\left(\begin{array}{cccccc|cccc} 1 & 0 & 0 & & 0 & 0 & & & & \\ 0 & 1 & 0 & \cdots & 0 & 0 & & & & \\ 0 & 0 & 1 & & 0 & 0 & & 0 & & \\ & \vdots & & & \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & & & & \\ 0 & 0 & 0 & & 0 & 1 & & & & \\ & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & 1 & 0 & 0 & 0 \\ & & & 0 & & & 0 & 1 & 0 & 0 \\ & & & & & & 0 & 0 & 1 & 0 \\ & & & & & & & & & \ddots \end{array}\right).$$

It remains to prove that the first of these two matrices can be joined to the second. For this purpose, note that the (unitary) permutation matrix (with $n + 1$ rows and columns)

$$\begin{pmatrix} 0 & 0 & 0 & & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 1 & 0 \end{pmatrix}$$

can be joined to the identity matrix (with $n+1$ rows and columns). Let $t \to M_t$ be a continuous path of unitary matrices that joins them, and let P be the projection matrix (with $n+1$ rows and columns)

$$\begin{pmatrix} 1 & 0 & 0 & & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & & 0 & 0 \end{pmatrix}.$$

The "product" path $t \to M_t P$ joins

$$\begin{pmatrix} 0 & 0 & 0 & & 0 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & & 0 & 0 \\ & \vdots & & & \vdots & \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & & 1 & 0 \end{pmatrix}$$

to P. Use this path in the northwest corner (of size $n+1$) of the infinite matrices to obtain a path joining the matrix of U to the matrix of $V \oplus W$.

If the dimension of the domain of W is $\aleph_0$, the solution is easier. It is easy to verify that the operator matrix $\begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix}$ (considered as an operator on the direct sum of the underlying space with itself) is unitarily equivalent to U, and the operator matrix $\begin{pmatrix} W & 0 \\ 0 & U \end{pmatrix}$ is unitarily equivalent to $V \oplus W$. Since W can be joined to the identity by a continuous path, it remains to prove that $\begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix}$ can be joined to $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$. If $t \to \begin{pmatrix} \alpha_t & \beta_t \\ \gamma_t & \delta_t \end{pmatrix}$ is a continuous path of numerical unitary matrices that joins $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, then $t \to \begin{pmatrix} \alpha_t & \beta_t U \\ \gamma_t & \delta_t U \end{pmatrix}$ is a continuous path of partial isometries that joins $\begin{pmatrix} 0 & U \\ 1 & 0 \end{pmatrix}$ to $\begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix}$.

What we have proved so far (after successive reductions) implies that any two isometries can be joined by a continuous path, i.e., that the set $\boldsymbol{P}(\rho,0,\rho')$ is arcwise connected.

To prove that $\boldsymbol{P}(\rho, \nu, \rho')$ is always arcwise connected, it is sufficient to consider the case $\nu \leqq \rho'$. (Argue by adjoints.) If U_0 and U_1 are in $\boldsymbol{P}(\rho, \nu, \rho')$, then they can be enlarged to isometries V_0 and V_1. Such enlargements are far from unique; what is important for our purposes is that V_0 and V_1 can be found so that they have the same co-rank. If P_0 and P_1 are the projections onto the initial spaces of U_0 and U_1 (i.e., $P_0 = U_0^* U_0$ and $P_1 = U_1^* U_1$), then P_0 and P_1 have the same rank and co-rank. It follows that there exist paths $t \rightarrow V_t$ and $t \rightarrow P_t$ joining V_0 to V_1 and P_0 to P_1. Since $U_0 = V_0 P_0$ and $U_1 = V_1 P_1$, this implies that $t \rightarrow V_t P_t$ is a continuous path joining U_0 to U_1. The proof of Theorem 7 is complete.

The following consequence of Theorems 6 and 7 is trivial, but worth making explicit: the components of P are exactly the sets $\boldsymbol{P}(\rho, \nu, \rho')$.

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