A TOPOLOGICAL MEASURE CONSTRUCTION

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1. Introduction. The purpose of this paper is to give two results in topological measure theory that generalize two well known results for metric spaces.

The principal one of these, which is given in § 3, concerns the construction of a measure from a nonnegative set function. Carathéodory [1] has done this in a natural way in defining Carathéodory linear measure in a finite dimensional Euclidean space. It is well known that this Carathéodory construction can be applied in metric spaces to produce measures for which the open sets are measurable.[1] Our treatment produces a measure for which the open F_σ sets, in a regular topological space, are measurable, and is identical with the Carathéodory measure in case the topology is metrizable. Since each open set in a metric space is an F_σ this provides a generalization of the metric result.

Our other result, which is given in § 2, concerns a necessary and sufficient condition for the measurability of open sets. A well known condition for this in metric spaces is that the measure be additive on any two sets which are a positive distance apart. When this condition is changed to require the additivity on two sets whose closures do not intersect, it becomes a necessary and sufficient condition for the measurability of the open F_σ sets, for a normal topological space. We show that the condition of normality can be weakened to one of "ϕ Normal" (see definition 2.4.5 below). Since a metric space is normal and therefore ϕ Normal, and since each open set of a metric space is an F_σ, this provides a clear generalization of the metric result.

At first glance the weakened normality condition of 2.4.5 appears to add little to topological measure theory. However, this is just the condition that results from our construction in § 3 (even though the topology is not necessarily normal) and hence the results of § 2 help us to obtain the results of § 3.

Nowhere in this paper is an assumption of local compactness made.

2. Conditions for measurability.

2.1. DEFINITIONS.

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[1] See, for example, Method II, page 105, of [3].

.1 $\sim B =$ the complement of B.

.2 $\sigma F =$ the union of $F = \bigcup \beta \in F \beta = \mathrm{E}x\,(x \in \beta$ for some $\beta \in F)$
$=$ the set of points x for which $x \in \beta$ for some $\beta \in F$.

.3 $x \in$ domain f if and only if $(x, y) \in f$ for some y.

.4 $\omega =$ the set of nonnegative integers.

2.2. DEFINITIONS.

.1 ϕ measures $\mathscr{S}$ if and only if ϕ is such a function of the subsets of $\mathscr{S}$ that:

$$0 \leqq \phi(A) \qquad \text{whenever} \qquad A \subset \mathscr{S} ;$$

and

$$\phi(A) \leqq \sum \beta \in F \phi(\beta)$$

whenever F is a countable family for which

$$A \subset \sigma F \subset \mathscr{S} .$$

.2 A is ϕ measurable if and only if $A \in$ domain ϕ and

$$\phi(T) = \phi(TA) + \phi(T \sim A)$$

for each $T \in$ domain ϕ.

.3 measurable $\phi = \mathrm{E}A\,(A$ is ϕ measurable$)$

.4 section $\phi T =$ the function ψ on domain ϕ such that

$$\psi(A) = \phi(TA)$$

for each $A \in$ domain ϕ.

.5 ψ is a submeasure of ϕ if and only if $\psi =$ section ϕT for some T for which $\phi(T) < \infty$.

2.3. DEFINITIONS.

.1 $\mathfrak{T}$ is a topology if and only if $\mathfrak{T}$ is such a family of sets that

$$\sigma F \in \mathfrak{T} \qquad \text{whenever} \qquad F \subset \mathfrak{T} ,$$

and

$$\alpha \cap \beta \in \mathfrak{T} \qquad \text{whenever } \alpha \in \mathfrak{T} \text{ and } \beta \in \mathfrak{T} .$$

Thus a topology $\mathfrak{T}$ is closed to finite intersections and unrestricted unions. For example, the family of all open sets of a metric space is a topology.

.2 $\mathfrak{T}$ topologizes $\mathscr{S}$ if and only if $\mathfrak{T}$ is a topology and $\mathscr{S} = \sigma\mathfrak{T}$.

.3 C is $\mathfrak{T}$ closed if and only if $C = \sigma\mathfrak{T} \sim A$ for some $A \in \mathfrak{T}$.

.4 Closure $\mathfrak{T}A =$ the intersection of all $\mathfrak{T}$ closed sets which contain A.

.5 $\mathfrak{T}$ is regular if and only if corresponding to each $A \in \mathfrak{T}$ and each $x \in A$ there is a $B \in \mathfrak{T}$ such that $x \in B$ and Closure $\mathfrak{T}B \subset A$.

.6 Fsigma $\mathfrak{T} = \text{E}B$ ($B \in \mathfrak{T}$ and $B = \sigma F$ for some countable family F of $\mathfrak{T}$ closed sets).

.7 Gdelta $\mathfrak{T} = \text{E}C$(C is $\mathfrak{T}$ closed and C is the intersection of a countable subfamily of $\mathfrak{T}$).

Thus Fsigma $\mathfrak{T}$ and Gdelta $\mathfrak{T}$ are the familiar open F_σ's and closed G_δ's.

This paper deals with a fixed topology, $\mathfrak{T}$, which topologizes the space, $\mathscr{S}$. It is assumed that the hypothesis

$$\mathfrak{T} \text{ topologizes } \mathscr{S}$$

is added to every theorem. Also the "$\mathfrak{T}$" will be dropped from such expressions as "C is $\mathfrak{T}$ closed" whenever no confusion will result. Thus we write "Fsigma" for "Fsigma $\mathfrak{T}$," "Closure A" for "Closure $\mathfrak{T}A$," etc.

In definitions 2.4 below the well known topological concepts of compactness and normality, are followed by generalizations involving both topology and measure.

2.4. DEFINITIONS.

.1 A is compact if and only if A is closed and for each $F \subset \mathfrak{T}$ for which $A \subset \sigma F$ there is a finite subfamily H of F for which $A \subset \sigma H$.

.2 A is ϕ compact if and only if A is closed and for each $F \subset \mathfrak{T}$ for which $A \subset \sigma F$, for each submeasure ψ for ϕ, for each $\varepsilon > 0$, there is a finite subfamily H of F for which $\psi(A) \leqq \psi(A\sigma H) + \varepsilon$.

.3 A is normal if and only if A is closed and for each C and B for which C is closed, $C \subset A$, $C \subset B \in \mathfrak{T}$, there exists $D \in \mathfrak{T}$ for which $C \subset D$ and Closure $D \subset B$.

.4 A is ϕ normal if and only if A is closed and for each C and B for which C is closed, $C \subset A$, $C \subset B \in \mathfrak{T}$, and for each submeasure ψ for ϕ, for each $\varepsilon > 0$, there exists $D \in \mathfrak{T}$ and a closed set C' for which $C' \subset C$, $C' \subset D$, Closure $D \subset B$, and

$$\psi(C) \leqq \psi(C') + \varepsilon \,. \tag{1}$$

We define the slightly less general notion of ϕ Normality by changing the condition (1) to condition (2) below.

.5 A is ϕ Normal if and only if A is closed and for each C and B for which C is closed, $C \subset A$, $C \subset B \in \mathfrak{T}$, and for each submeasure ψ of ϕ, for each $\varepsilon > 0$, there exists $D \in \mathfrak{T}$ and a closed set C' for which $C' \subset C$, $C' \subset D$, Closure $D \subset B$, and

$$\psi(C \sim C') \leqq \varepsilon \,. \tag{2}$$

.6 ϕ is $\mathfrak{T}$ additive if and only if $\phi(A \cup B) = \phi(A) + \phi(B)$ whenever Closure $A \cap$ Closure $B = 0$.

.7 ϕ is a ρ metric measure if and only if ρ metrizes $\mathscr{S}$, ϕ measures $\mathscr{S}$, and $\phi(A \cup B) = \phi(A) + \phi(B)$ whenever A and B are subsets of $\mathscr{S}$ which are a positive ρ distance apart.

2.5. THEOREM. *If ϕ measures $\mathscr{S}$ then*:

.1 *if A is compact then A is ϕ compact*;

.2 *if A is normal then A is ϕ Normal*;

.3 *if A is ϕ Normal then A is ϕ normal.*

2.6. THEOREM. *If ρ metrizes $\mathscr{S}$, $\mathfrak{T}$ is the family of ρ-open sets, and ϕ measures $\mathscr{S}$ then*:

.1 *ϕ is $\mathfrak{T}$ additive if and only if ϕ is a ρ metric measure*;

.2 *$\mathfrak{T}$ is a regular topology*;

.3 *$\mathscr{S}$ is normal.*

A well known theorem is the following:

2.7. THEOREM. *If ρ metrizes $\mathscr{S}$ and ϕ measures $\mathscr{S}$ then ϕ is a ρ metric measure if and only if the ρ-open sets are ϕ measurable.*

The primary aim of this section is to generalize Theorem 2.7 to the case where the metric space is replaced by a regular, ϕ Normal, topology. This is done in Theorem 2.19.

2.8. THEOREM. *If ϕ measures $\mathscr{S}$, C is closed and $C \subset A$ then:*

.1 *if A is compact then C is compact;*

.2 *if A is ϕ compact then C is ϕ compact.*

Proof. .1 is well known.

Suppose $F \subset \mathfrak{T}$, $C \subset \sigma F$, ψ is a submeasure of ϕ, $\psi' =$ section ψC, $0 < \varepsilon < \infty$, $F' = \mathrm{E}B\,(B \in F$ or $B = \mathscr{S} \sim C)$. Since $A \subset \sigma F'$, $F' \subset \mathfrak{T}$, ψ' is a submeasure of ϕ, and A is ϕ compact, we can and do choose such a finite subfamily H' of F' that

$$\psi'(A) \leqq \psi'(A\sigma H') + \varepsilon\ .$$

Letting $H = \mathrm{E}B\ (B \in H'$ and $B \neq \mathscr{S} \sim C)$, it follows that H is a finite subfamily of F, and

$$\begin{aligned}\psi(C) = \psi'(A) &\leqq \psi'(A\sigma H') + \varepsilon \\ &= \psi'(A\sigma H) + \varepsilon = \psi(C\sigma H) + \varepsilon\ .\end{aligned}$$

Consequently C is ϕ compact, and .2 is proved.

2.9. THEOREM. *If $\mathfrak{T}$ is regular, and ϕ measures $\mathscr{S}$ then*

.1 *if A is compact then A is normal,*

.2 *if A is ϕ compact then A is ϕ normal.*

Proof. .1 is well known.

Suppose A is ϕ compact, $C \subset A$, C is closed, $C \subset B \in \mathfrak{T}$, ψ is a submeasure of ϕ, and $\varepsilon > 0$. First use the fact that $\mathfrak{T}$ is regular to secure such a function f on C that $f(x) \in \mathfrak{T}$, $x \in f(x)$, Closure $f(x) \subset B$ for each $x \in C$. Now again use the regularity of $\mathfrak{T}$ to secure such a function g on C that $g(x) \in \mathfrak{T}$, $x \in g(x)$, Closure $g(x) \subset f(x)$ for each $x \in C$. Now use the (2.8) facts that C is ϕ compact and $C \subset \bigcup x \in Cg(x)$ to secure such a finite subset Q of C that

$$\psi(C) \leqq \psi(C \cap \bigcup x \in Qg(x)) + \varepsilon\ .$$

Let

$$D = \bigcup x \in Qf(x)\,, \qquad C' = \bigcup x \in Q(C \cap \text{Closure } g(x))$$

and observe that $D \in \mathfrak{T}$, C' is closed, $C' \subset C$,

$$C' = \bigcup x \in Q(C \cap \text{Closure } g(x)) \subset \bigcup x \in Qf(x) = D\,,$$

Closure $D =$ Closure $\bigcup x \in Qf(x) = \bigcup x \in Q$ Closure $f(x) \subset B$, and

$$\begin{aligned}\psi(C) &\leqq \psi(C \cap \bigcup x \in Qg(x)) + \varepsilon \\ &\leqq \psi(C \cap \bigcup x \in Q \text{ Closure } g(x)) + \varepsilon \\ &= \psi(C') + \varepsilon\,.\end{aligned}$$

Consequently A is ϕ normal and the proof is complete.

The following theorem is given in Chapter 5 of [3].

2.10. THEOREM. *If ϕ measures $\mathscr{S}$ and if for each $n \in \omega$ and each submeasure ψ of ϕ,*

$$A_n \subset A_{n+1} \subset \mathscr{S}\,,$$

and

$$\psi(A_n \cup (\mathscr{S} \sim A_{n+1})) = \psi(A_n) + \psi(\mathscr{S} \sim A_{n+1})$$

then

.1 $\bigcup n \in \omega\ A_n$ *is ϕ measurable, and*

.2 $\theta(\bigcup n \in \omega A_n) = \lim_{n\to\infty} \theta(A_n)$, *whenever θ is a submeasure of ϕ.*

By considering complements one easily established the following corollary of 2.10.

2.11. THEOREM. *If ϕ measures $\mathscr{S}$ and if for each $n \in \omega$ and each submeasure ψ of ϕ*

$$A_{n+1} \subset A_n \subset \mathscr{S}\,,$$

and

$$\psi(A_{n+1} \cup (\mathscr{S} \sim A_n)) = \psi(A_{n+1}) + \psi(\mathscr{S} \sim A_n)$$

then

.1 $\bigcap n \in \omega A_n$ *is ϕ measurable, and*

.2 $\theta(\bigcap n \in \omega A_n) = \lim_{n\to\infty} \theta(A_n)$, *whenever θ is a submeasure of ϕ.*

The following lemma is easily verified.

2.12. LEMMA. *If ϕ measures $\mathscr{S}$ then ϕ is $\mathfrak{T}$ additive if and only if ψ is $\mathfrak{T}$ additive for each submeasure ψ of ϕ.*

2.13. THEOREM. *If ϕ measures $\mathscr{S}$, ϕ is $\mathfrak{T}$ additive, $A_n \subset \mathscr{S}$, and* Closure $A_{n+1} \cap$ Closure $(\mathscr{S} \sim A_n) = 0$ *for each $n \in \omega$, then*

.1 $\bigcap n \in \omega A_n$ *is ϕ measurable, and*

.2 $\theta(\bigcap n \in \omega A_n) = \lim_{n\to\infty} \theta(A_n)$, *whenever θ is a submeasure of ϕ.*

Proof. Observe that $A_{n+1} \subset A_n \subset \mathscr{S}$, for $n \in \omega$. Let ψ be a submeasure of ϕ. Since, by 2.12, ψ is $\mathfrak{T}$ additive, it follows that

$$\psi(A_{n+1} \cup (\mathscr{S} \sim A_n)) = \psi(A_{n+1}) + \psi(\mathscr{S} \sim A_n)$$

for each $n \in \omega$. Application of 2.11 completes the proof.

2.14. THEOREM. *If ϕ measures $\mathscr{S}$, $\phi(\mathscr{S}) < \infty$, ϕ is $\mathfrak{T}$ additive, C is ϕ normal, $C \subset B \in \mathfrak{T}$, and $0 < \varepsilon < \infty$, then there exist sets C' and D such that C' is ϕ measurable, $C' \in$* Gdelta, *$C' \subset C$, $D \in \mathfrak{T}$, $C' \subset D$,* Closure *$D \subset B$, and $\phi(C \sim C') \leqq \varepsilon$.*

Proof. Since $\phi(\mathscr{S}) < \infty$ it follows that ϕ is a submeasure of ϕ. We use the facts that C is ϕ normal and $C \subset B$ to inductively obtain such sequences c' and d that $c'_0 = C$, $d_0 = B$, c'_n is closed, $d_n \in \mathfrak{T}$, $c'_{n+1} \subset c'_n \subset d_n$, Closure $d_{n+1} \subset d_n$, and $\phi(c'_n) \leqq \phi(c'_{n+1}) + \varepsilon/2^{n+1}$, for each $n \in \omega$.

Let $C' = \bigcap n \in \omega d_n$, $D = d_1$. Since

$$\text{Closure } d_{n+1} \cap \text{Closure } (\mathscr{S} \sim d_n) = 0$$

for each $n \in \omega$, it follows from 2.13.1 that C' is ϕ measurable. Also

$$C' = \bigcap n \in \omega d_n = \bigcap n \in \omega \text{ Closure } d_{n+1}$$

is closed, $C' \in$ Gdelta, $D \in \mathfrak{T}$, $C' \subset C$, $C' \subset d_1 = D$ and Closure $D =$ Closure $d_1 \subset d_0 = B$.

We now use induction to deduce that, for any $m \in \omega$,

$$\begin{aligned} \phi(C) = \phi(c'_0) &\leqq \phi(c'_m) + \textstyle\sum n \in \omega(\varepsilon/2^{n+1}) \\ &\leqq \phi(c'_m) + \varepsilon \, . \end{aligned} \tag{1}$$

Since $c'_m \subset d_m$, $c'_m \subset C$, and $c'_m \subset C \cap d_m$, for each $m \in \omega$, it follows from (1) that

$$\phi(C) \leqq \phi(c'_m) + \varepsilon \leqq \phi(C \cap d_m) + \varepsilon$$

for each $m \in \omega$. Let $\theta =$ section ϕC and, with the help of 2.13.2. observe that

$$\begin{aligned}\phi(CC') &= \theta(C') \\ &= \lim_{m\to\infty} \theta(d_m) \\ &= \lim_{m\to\infty} \phi(Cd_m) \\ &\geqq \lim_{m\to\infty} \phi(C) - \varepsilon \\ &= \phi(C) - \varepsilon ,\end{aligned}$$

and, since C' is ϕ measurable,

$$\phi(C \sim C') = \phi(C) - \phi(CC') \leqq \varepsilon .$$

2.15. THEOREM. *If ϕ measures $\mathscr{S}$, $\phi(\mathscr{S}) < \infty$, ϕ is $\mathfrak{T}$ additive, C is ϕ normal, and $C \subset B \in \mathfrak{T}$, then for some ϕ measurable set K, $C \subset K \subset B$.*

Proof. Repeatedly use 2.14 to secure such a sequence, k, of ϕ measurable sets that $k_n \subset B$, and $\phi(C \sim k_n) \leqq (1/2^n)$ for each $n \in \omega$. Let

$$K' = \bigcup n \in \omega k_n , \qquad K = K' \cup C .$$

Thus K' is ϕ measurable, $K \subset B$,

$$0 \leqq \phi(C \sim K') \leqq \phi(C \sim k_n) \leqq (1/2^n)$$

for each $n \in \omega$, $\phi(C \sim K') = 0$, $C \sim K$ is ϕ measurable, $K = K' \cup (C \sim K')$ is ϕ measurable and $C \subset K \subset B$.

The following lemma is well known.

2.16. LEMMA. *If ϕ measures $\mathscr{S}$ and $A \subset \mathscr{S}$ then A is ϕ measurable if and only if A is ψ measurable for each submeasure ψ of ϕ.*

2.17. THEOREM. *If ϕ measures $\mathscr{S}$, ϕ is $\mathfrak{T}$ additive and $\mathscr{S}$ is ϕ normal then* Fsigma $\subset$ measurable ϕ.

Proof. Let $B \in$ Fsigma, and let ψ be a submeasure of ϕ. Choose such a countable subfamily F of closed sets that $B = \sigma F$. Check that $\psi(\mathscr{S}) < \infty$, ψ is $\mathfrak{T}$ additive, C is ϕ normal, and C is ψ normal, for each $C \in F$.

Thus we can and do use 2.15 to secure such a function K on F that $K(C)$ is ψ measurable and $C \subset K(C) \subset B$ for each $C \in F$. It follows that

$$\begin{aligned} B = \sigma F &= \bigcup C \in FC \\ &\subset \bigcup C \in FK(C) \\ &\subset B , \\ B &= \bigcup C \in FK(C) , \end{aligned}$$

B is ψ measurable.

Thus B is ψ measurable for each submeasure ψ of ϕ, and, by 2.16, B is ϕ measurable.

The desired conclusion is at hand.

2.18. THEOREM. *If ϕ measures $\mathscr{S}$, $\phi(\mathscr{S}) < \infty$, C is ϕ Normal $C \subset B \in \mathfrak{T}$, and $0 < \varepsilon < \infty$, then there exists such a member C' of* Gdelta *that $C' \subset B$ and $\phi(C \sim C') \leqq \varepsilon$.*

Proof. Repeatedly use the fact that C is ϕ Normal to obtain such sequences c' and d that $c'_0 = C$, $d_0 = B$, c'_n is closed, $d_n \in \mathfrak{T}$, $c'_{n+1} \subset c'_n \subset d_n$, Closure $d_{n+1} \subset d_n$, and

$$\phi(c'_n \sim c'_{n+1}) \leqq (\varepsilon/2^{n+1}) ,$$

for each $n \in \omega$.

Let $C' = \bigcap n \in \omega d_n$ and observe that $C' \in$ Gdelta, $C' \subset B$,

$$\begin{aligned} \phi(C \sim C') &= \phi(C \sim \bigcap n \in \omega d_n) \\ &= \phi(\bigcup n \in \omega(C \sim d_n)) \\ &\leqq \phi(\bigcup n \in \omega(C \sim c'_n)) \\ &= \phi(\bigcup n \in \omega(c'_0 \sim c'_n)) \\ &= \phi(\bigcup n \in \omega(c'_n \sim c'_{n+1})) \\ &\leqq \sum n \in \omega \phi(c'_n \sim c'_{n+1})) \\ &\leqq \sum n \in \omega(\varepsilon/2^{n+1}) = \varepsilon . \end{aligned}$$

Thus $\phi(C \sim C') \leqq \varepsilon$, and the proof is complete.

2.19. THEOREM. *If ϕ measures $\mathscr{S}$, and $\mathscr{S}$ is ϕ Normal then*

$$\text{Fsigma} \subset \text{measurable } \phi$$

if and only if ϕ is $\mathfrak{T}$ additive.

Proof. If ϕ is $\mathfrak{T}$ additive, it follows from 2.17 that

$$\text{Fsigma} \subset \text{measurable } \phi ,$$

since (2.5.3) a set which is ϕ Normal is also ϕ normal.

Now suppose that Fsigma $\subset$ measurable ϕ. Let ψ be a submeasure of ϕ, $0 < \varepsilon < \infty$, $A \subset \mathscr{S}$, $B \subset \mathscr{S}$, $\bar{A} = \text{Closure } A$, $\bar{B} = \text{Closure } B$, $\alpha = \mathscr{S} \sim \bar{B}$, and suppose that $\bar{A}\bar{B} = 0$.

Since $\bar{A} \subset \alpha$ we may use 2.18 to secure such a member C' of Gdelta that $C' \subset \alpha$ and $\psi(\bar{A} \sim C') \leqq \varepsilon$. Thus $\psi(A \sim C') \leqq \varepsilon$, $(A \cup B)C' = AC'$, $(A \cup B) \sim C' = (A \sim C') \cup B$, and, since $C' \in \text{Fsigma} \subset \text{measurable } \phi$,

$$\begin{aligned} \psi(A \cup B) &= \psi((A \cup B)C' + \psi((A \cup B) \sim C') \\ &= \psi(AC') + \psi((A \sim C') \cup B) \\ &\geqq \psi(AC') + \psi(B) \\ &= \psi(A) - \psi(A \sim C') + \psi(B) \\ &\geqq \psi(A) + \psi(B) - \varepsilon \,. \end{aligned}$$

Thus $\psi(A \cup B) \geqq \psi(A) + \psi(B)$, and since $\psi(A \cup B) \leqq \psi(A) + \psi(B)$, it follows that

$$\psi(A \cup B) = \psi(A) + \psi(B) \,.$$

Therefore ϕ is $\mathfrak{T}$ additive, and the proof is complete.

2.20. Remark. Since any metric space is normal and therefore (2.52) ϕ Normal and since every open set of a metric space is an F_σ, it follows that 2.19 is a generalization of the following well known theorem.

Theorem. *If ρ metrizes $\mathscr{S}$, ϕ measures $\mathscr{S}$, and $\mathfrak{T}$ is the family of ρ-open sets, then $\mathfrak{T} \subset$ measurable ϕ if and only if $\phi(A \cup B) = \phi(A) + \phi(A)$ whenever A and B are a positive ρ-distance apart.*

We shall use 2.17 in the next section where the topological space is not known to be normal but is ϕ normal for the measure ϕ that is constructed there.

3. Measure construction.

It is well known that the set function, $\theta = \text{msm } g\rho H$, defined in 3.2 below, is a Borel measure (i.e., the ρ-open sets are θ measurable) in case ρ metrizes $\mathscr{S}$ and g is a nonnegative function of H. It is the purpose of this section to generalize this result to the topolagical case by defining a measure, $\phi = \text{mst } g\mathfrak{T}H$, for which the open F_σ's are ϕ-measurable whenever $\mathfrak{T}$ is regular, and which is equal to θ in case $\mathfrak{T}$ is metrizable.

3.0. Definitions.

.1 mss $g\mathscr{S}H$ = the function ψ, on the subsets of $\mathscr{S}$, such that if $A \subset \mathscr{S}$ then $\psi(A)$ is the infimum of numbers of the form

$$\sum B \in Fg(B)$$

where F is such a countable subfamily of H that $A \subset \sigma F$.

In connection with 3.1. we would like to remind the reader that an empty infimum is ∞.

.2 msm $g\rho H =$ the function θ, on the subsets of $\mathscr{S}$, such that, if $A \subset \mathscr{S}$ then

$$\theta(A) = \lim_{n \to \infty} \text{mss } g\mathscr{S}H_n(A) ,$$

where H_n consists of those members of H whose ρ-diameter is less than $1/2^n$, for each n.

.3 msf $gFH =$ mss $g\mathscr{S}H_F$, where

$$H_F = H \cap \text{E}D(D \subset B \text{ for some } B \in F) .$$

.4 Cover $\mathfrak{T} = \text{E}F(F \subset \mathfrak{T}$ and $\sigma F = \sigma\mathfrak{T})$.

Thus, $E \in$ Cover $\mathfrak{T}$ if and only if $F \subset \mathfrak{T}$ and F covers the space covered by $\mathfrak{T}$.

.5 mst $g\mathfrak{T}H =$ the function ϕ on the subsets of $\mathscr{S}$ such that, if $A \subset \mathscr{S}$ then

$$\phi(A) = \sup F \in \text{Cover } \mathfrak{T}\phi_F(A)$$

where $\phi_F =$ msf gFH, for each $F \in$ Cover $\mathfrak{T}$.

.6 F is a refinement of F' if and only if each member of F is a subset of some member of F'.

.7 $F \cap\cap F' = \text{E}B(B = \alpha\beta$ for some $\alpha \in F$ and some $\beta \in F')$.

3.1. LEMMA. *If $F \in$ Cover $\mathfrak{T}$, $F' \in$ Cover $\mathfrak{T}$, and $F'' = F \cap\cap F'$, then $F'' \in$ Cover $\mathfrak{T}$ and F'' is a refinement of F.*

The following theorem is well known.

3.2. THEOREM. *If g is a nonnegative function on H, and $\psi =$ mss $g\mathscr{S}H$ then:*

.1 *ψ measures $\mathscr{S}$;*

.2 *if $H \subset H'$ and $\psi' =$ mss $g\mathscr{S}H'$, then $\psi'(A) \leqq \psi(A)$ for*

$A \subset \mathscr{S}$;

.3 *if* $\psi(A) < \infty$ *and* $0 < \varepsilon < \infty$ *then there exists such a countable subfamily* G *of* H *that* $A \subset \sigma G$ *and* $\sum D \in G\ g(D) \leqq \psi(A) + \varepsilon$.

3.3. THEOREM. *If* g *is a nonnegative function of* H, F *is a refinement of* F', $\phi_F = \text{msf } gFH$, *and* $\phi_{F'} = \text{msf } gF'H$, *then*

$$\phi_{F'}(A) \leqq \phi_F(A)$$

for each $A \subset S$.

Proof. Let

$$H_F = H \cap \mathrm{E}B\,(B \subset D \text{ for some } D \in F)\,,$$
$$H_{F'} = H \cap \mathrm{E}B\,(B \subset D \text{ for some } D \in F')\,,$$

and note that $\phi_F = \text{mss } g\mathscr{S}H_F$. Application of 3.2.2 completes the proof.

The following theorem is well known.

3.4. THEOREM. *If* F *is nonempty,* ψ *measures* $\mathscr{S}$ *for each* $\psi \in F$, *and*

$$\phi(D) = \sup \psi \in F \psi(D)$$

whenever $D \subset \mathscr{S}$, *then*:

.1 ϕ *measures* $\mathscr{S}$;

.2 *if for each* $\psi' \in F$ *and each* $\psi'' \in F$, *there exists a* $\psi \in F$ *for which*

$$\psi'(D) \leqq \psi(D) \text{ and } \psi''(D) \leqq \psi(D)$$

whenever $D \subset \mathscr{S}$, *then*

$$\bigcap \psi \in F \text{ measurable } \psi \subset \text{measurable } \phi\,.$$

3.5. THEOREM. *If* $\mathfrak{T}$ *is a regular topology,* g *is a nonnegative function on* H, *and* $\phi = \text{mst } g\mathfrak{T}H$ *then* Fsigma $\subset$ measurable ϕ.

Proof. Let $\phi_F = \text{msf } gFH$ for each $F \in \text{Cover } \mathfrak{T}$. Thus

$$\phi(A) = \sup F \in \text{Cover } \mathfrak{T}\phi_F(A)$$

whenever $A \subset \mathscr{S}$. The proof is completed in six parts.

PART I. ϕ *measures* $\mathscr{S}$.

Proof. Since, by 3.2.1, ϕ_F measures $\mathscr{S}$, for each $F \in \text{Cover } \mathfrak{T}$, and since Cover $\mathfrak{T}$ is not empty, it follows from 3.4.1 that ϕ measures $\mathscr{S}$.

PART II. *If $F \in \text{Cover } \mathfrak{T}$, $\phi(A) < \infty$, and $0 < \varepsilon < \infty$ then there exists such a refinement F' of F that $F' \in \text{Cover } \mathfrak{T}$, and $\phi(A) \leqq \phi_{F'}(A) + \varepsilon$.*

Proof. Choose such a member F'' of Cover $\mathfrak{T}$ that $\phi(A) \leqq \phi_{F''}(A) + \varepsilon$, and let $F' = F \cap \cap F''$ (see Definition 3.0.7). Thus $S \subset \sigma F'$, $F' \subset \mathfrak{T}$, $F' \in \text{Cover } \mathfrak{T}$, and using 3.1 and 3.3, we infer that F' is a refinement of F, F' is a refinement of F'', and

$$\phi(A) \leqq \phi_{F''}(A) + \varepsilon \leqq \phi_{F'}(A) + \varepsilon .$$

PART III. *$\mathscr{S}$ is ϕ compact.*

Proof. Recall definition 2.4.2. Let $F \in \text{Cover } \mathfrak{T}$, $\phi(T) < \infty$, $\psi = \text{section } \phi T$, and $0 < \varepsilon < \infty$. Use Part II to choose such a refinement F' of F that $F' \in \text{Cover } \mathfrak{T}$ and

$$\phi(T) \leqq \phi_{F'}(T) + \varepsilon/2 < \infty .$$

Now use 3.2.3 and definition 3.0.3 to secure such a countable subfamily H' of H that $T \subset \sigma H'$, H' is a refinement of F', and $\sum D \in H' g(D) < \infty$; and let H'' be such a finite subfamily of H' that

$$\sum D \in (H' \sim H'') g(D) \leqq \varepsilon/2 .$$

Since H'' is a refinement of F, choose such a finite subfamily G of F that H'' is a refinement of G. Thus

$$\begin{aligned}
\phi_{F'}(T) &\leqq \phi_{F'}(T\sigma H'') + \phi_{F'}(T\sigma(H' \sim H'')) \\
&\leqq \phi(T\sigma H'') + \phi_{F'}(\sigma(H' \sim H'')) \\
&\leqq \phi(T\sigma G) + \textstyle\sum D \in (H' \sim H'') g(D) \\
&\leqq \psi(T) + \varepsilon/2 , \\
\psi(\mathscr{S}) &= \phi(T) \leqq \phi_{F'}(T) + \varepsilon/2 \\
&\leqq \psi(T) + \varepsilon .
\end{aligned}$$

Thus $\mathscr{S}$ is ϕ compact.

PART IV. *ϕ is $\mathfrak{T}$ additive.*

Proof. Recall definition 2.4.6. Let $\bar{A} = \text{Closure } A$, $\bar{B} = \text{Closure } B$, $\bar{A}\bar{B} = 0$. If $\phi(A \cup B) = \infty$ then $\phi(A \cup B) = \phi(A) + \phi(B)$.

Now assume $\phi(A \cup B) < \infty$, $0 < \varepsilon < \infty$, and select such members G' and G'' of Cover $\mathfrak{T}$ that

$$\phi(A) \leqq \phi_{G'}(A) + \varepsilon/3 , \qquad \phi(B) \leqq \phi_{G''}(B) + \varepsilon/3 , \tag{1}$$

and let

$$F' = \mathrm{E}\alpha\,(\alpha = B \cap (\mathscr{S} \sim \bar{B}) \text{ for some } \beta \in G') ,$$
$$F'' = \mathrm{E}\alpha\,(\alpha = \beta \cap (\mathscr{S} \sim \bar{A}) \text{ for some } \beta \in G'') ,$$
$$F = F' \cup F'' .$$

Thus:

$$F \subset \mathfrak{T} ; \quad \mathscr{S} \subset \sigma F ; \quad F \in \text{Cover } \mathfrak{T} ;$$

$$\begin{aligned} &\text{if} \quad D \in F \quad \text{and} \quad DA \neq 0 \quad \text{then} \quad D \in F' ; \\ &\text{if} \quad D \in F \quad \text{and} \quad DB \neq 0 \quad \text{then} \quad D \in F'' . \end{aligned} \tag{2}$$

Now use 3.2.3 to secure such a countable subfamily H''' of H that H''' is a refinement of F, $A \cup B \subset H'''$, and

$$\textstyle\sum D \in H'''g(D) \leqq \phi_F(A \cup B) + \varepsilon/3 ; \tag{3}$$

infer from (2) that

$$\begin{aligned} &\text{if} \quad D \in H''' \quad \text{and} \quad DA \neq 0 \quad \text{then} \quad DB = 0 \\ &\text{if} \quad D \in H''' \quad \text{and} \quad DB \neq 0 \quad \text{then} \quad DA = 0 ; \end{aligned}$$

let $H' = H''' \cap \mathrm{E}D(DA \neq 0)$, $H'' = H''' \cap \mathrm{E}D(DB \neq 0)$.

Thus $H' \cup H'' \subset H''' \subset H$, $H' \cup H''$ is countable, $A \subset \sigma H'$, $B \subset \sigma H''$, $H'H'' = 0$, H' is a refinement of G', H'' is a refinement of G'', and

$$\phi_{G'}(A) \leqq \textstyle\sum D \in H'g(D) , \qquad \phi_{G''}(B) \leqq \textstyle\sum D \in H''g(D) . \tag{4}$$

Consequently, with the help of (1), (4) and (3), we deduce that

$$\begin{aligned} \phi(A) + \phi(\mathrm{B}) &\leqq \phi_{G'}(A) + \phi_{G''}(B) + 2\varepsilon/3 \\ &\leqq \textstyle\sum D \in H'g(D) + \textstyle\sum D \in H''g(D) + 2\varepsilon/3 \\ &\leqq \textstyle\sum D \in H'''g(D) + 2\varepsilon/3 \\ &\leqq \phi_F(A \cup B) + \varepsilon/3 + 2\varepsilon/3 \\ &\leqq \phi(A \cup B) + \varepsilon . \end{aligned}$$

Thus, if $\phi(A \cup B) < \infty$,

$$\phi(A) + \phi(B) \leqq \phi(A \cup B) .$$

Therefore

$$\phi(A \cup B) = \phi(A) + \phi(B) ,$$

whenever $A \subset \mathscr{S}$, $B \subset \mathscr{S}$, and ϕ is $\mathfrak{T}$ additive.

PART V. *$\mathscr{S}$ is ϕ normal.*

Proof. Recall definition 2.4.4, use part IV and 2.9.2.

PART VI. Fsigma $\subset$ measurable ϕ.

Proof. By parts IV and V, ϕ is $\mathfrak{T}$ additive and $\mathscr{S}$ is ϕ normal. Application of 2.17 completes the proof.

The reader will observe that the regularity of $\mathfrak{T}$ was not used in the proofs of Parts I–III.

3.6. LEMMA. *If ρ metrizes $\mathscr{S}$, $\mathfrak{T}$ is the family of ρ-open sets, $F \subset \mathfrak{T}$, $n \in \omega$, and $G_n = \mathrm{E}D$(the ρ-diameter of $D < 1/2^n$), then:*

.1 $\mathfrak{T} \cap G_n \in \mathrm{Cover}\ \mathfrak{T}$;

.2 *if $H' \subset G_n$, if $DA \neq 0$ whenever $D \in H'$, and if for each $x \in A$ there exists a $B \in F$ such that*

$$(\text{the } \rho\text{-distance from } x \text{ to } \mathscr{S} \sim B) > 1/2^n ,$$

then H' is a refinement of F;

.3 *$D \in G_n$ if, and only if, $D \subset B \in G_n$ for some $B \in \mathfrak{T}$.*

Proof. Suppose $D \in G_n$, and let $d =$ the ρ-diameter of D, $\delta = 1/2^n - d$, and

$$B = \mathrm{E}x\,(\rho(x, y) < \delta/3 \quad \text{for some} \quad y \in D) .$$

Thus, we have immediately that $B \in \mathfrak{T}$, $D \subset B$, and one can show that $B \in G_n$.

Therefore if $D \in G_n$ then $D \subset B \in G_n$ for some $B \in \mathfrak{T}$.

The remainder of the proof is straightforward.

3.7. THEOREM. *If ρ metrizes $\mathscr{S}$, $\mathfrak{T}$ is the family of all ρ-open sets, $F \subset \mathfrak{T}$, g is a nonnegative function on H, $G_n = \mathrm{E}D$((the ρ-diameter of $D) < 1/2^n$) for each $n \in \omega$, and $\theta = \mathrm{msm}\ g\rho H$, then:*

.0 $\mathfrak{T} \subset$ measurable ϕ;

.1 *if $\theta(A) < \infty$, $n \in \omega$, and $0 < \varepsilon < \infty$, then there exists a countable subfamily H' of H for which $H' \subset G_n$, $A \subset H'$, and $\sum D \in H' g(D) \leqq \theta(A) + \varepsilon$;*

.2 *if* $\theta(A) < \infty$, $n \in \omega$, $0 < \varepsilon < \infty$, *and if for each* $x \in A$ *there exists a* $B \in F$ *such that*

$$(\textit{the } \rho\textit{-distance from } x \textit{ to } \mathscr{S} \sim B) \geqq 1/2^n ,$$

then there exists a countable subfamily H' *of* H *for which* $H' \subset G_n$, $A \subset \sigma H'$, H' *is a refinement of* F, *and* $\sum D \in H' g(D) \leqq \theta(A) + \varepsilon$.

Proof. .0 is well known.

Proof .1 Let $\theta_n = \text{mss } g\mathscr{S}(HG_n)$. Thus $\theta_n(A) \leqq \theta(A) < \infty$, and we can use 3.2.3 to secure such a countable subfamily H' of $(H \cap G_n)$ that $A \subset \sigma H'$ and $\sum D \in H'g(D) \leqq \theta_n(A) + \varepsilon$. Since $\theta_n(A) \leqq \theta(A)$ the proof is complete.

Proof .2 Use .1 to obtain such a countable subfamily H'' of (HG_n) that $A \subset \sigma H''$ and $\sum D \in H''g(D) \leqq \theta(A) + \varepsilon$, and let $H' = H'' \cap \text{E}D(DA \neq 0)$.

Thus $A \subset \sigma H'$, and $\sum D \in H'g(D) \leqq \sum D \in H''g(D) \leqq \theta(A) + \varepsilon$. Also $DA \neq 0$ for each $D \in H'$, so by 3.6.2 H' is a refinement of F.

3.8. THEOREM. *If* ρ *metrizes* $\mathscr{S}$, $\mathfrak{T}$ *is the family of all* ρ*-open sets,* g *is a nonnegative function on* H, $\phi = \text{mst } g\mathfrak{T}H$, *and* $\theta = \text{msm } g\rho H$ *then* $\phi = \theta$.

Proof. Let $\phi_F = \text{msf } gFH$ for each $F \in \text{Cover } \mathfrak{T}$, and complete the proof in Parts I and II by showing that $\phi(A) \geqq \theta(A)$, and $\phi(A) \leqq \theta(A)$ whenever $A \subset \mathscr{S}$.

PART I. *If* $A \subset \mathscr{S}$ *then* $\phi(A) \geqq \theta(A)$.

Proof. Let

$$G_n = \text{E}D((\text{the } \rho\text{-diameter of } D) < 1/2^n) ,$$

$F_n = \mathfrak{T} \cap G_n$, $H_n = H \cap G_n$, and $\theta_n = \text{mss } g\mathscr{S}H_n$, for each $n \in \omega$. Thus, for each $n \in \omega$, $F_n \in \text{Cover } \mathfrak{T}$, and by definition 3.0.3, and 3.6.3,

$$\begin{aligned} \phi_{F_n} &= \text{msf } gF_nH \\ &= \text{mss } g\mathscr{S}(H \cap \text{E}D(D \subset B \text{ for some } B \in F_n)) \\ &= \text{mss } g\mathscr{S}H_n \\ &= \theta_n . \end{aligned}$$

Consequently,

$$\begin{aligned}\phi(A) &= \sup F \in \text{Cover } \mathfrak{T}\phi_F(A) \\ &\geqq \sup n \in \omega\phi_{F_n}(A) \\ &= \sup n \in \omega\theta_n(A) \\ &= \theta(A) .\end{aligned}$$

PART II. *If* $A \subset \mathscr{S}$ *then* $\phi(A) \leqq \theta(A)$.

Proof. The result being obvious if $\theta(A) = \infty$, we assume that $\theta(A) < \infty$. Let $0 < \varepsilon < \infty$, $F \in \text{Cover } \mathfrak{T}$, and let f' and f be such sequences that

$$\begin{aligned} f'_n &= \bigcup B \in F \ \mathrm{E}x\,((\text{the } \rho\text{-distance from } x \text{ to } \mathscr{S} \sim B) > 1/2^n) , \\ f_0 &= f'_0 , \\ f_{n+1} &= f'_{n+1} \sim f'_n , \end{aligned}$$

whenever $n \in \omega$. We infer: f_n is a ρ-Borel set and, by 3.7.0, f_n is ϕ measurable for each $n \in \omega$; $A = \bigcup n \in \omega(Af_n)$; $\theta(A) = \sum n \in \omega\theta(Af_n)$; for each $n \in \omega$, for each $x \in (Af_n)$, there is a $B \in F$ for which the distance from x to $\mathscr{S} \sim B$ is greater than $1/2^n$. Consequently, we can use 3.7.2 to secure such a sequence, h, that h_n is a countable subfamily of H, $Af_n \subset \sigma h_n$, h_n is a refinement of F, and

$$\sum D \in h_n g(D) \leqq \theta(Af_n) + \varepsilon/2^{n+1} ,$$

whenever $n \in \omega$.

Let $H' = \bigcup n \in \omega h_n$ and note that H' is a countable subfamily of H, $A = \bigcup n \in \omega(Af_n) \subset \bigcup n \in \omega\sigma h_n = \sigma H'$, H' is a refinement of F, and

$$\begin{aligned}\phi_F(A) &\leqq \sum D \in H'g(D) \leqq \sum n \in \omega \sum D \in h_n g(D) \\ &\leqq \sum n \in \omega(\theta(Af_n) + \varepsilon/2^{n+1}) = \sum n \in \omega\theta(Af_n) + \varepsilon = \theta(A) + \varepsilon .\end{aligned}$$

Thus $\phi_F(A) \leqq \theta(A)$ for each $F \in \text{Cover } \mathfrak{T}$, and

$$\begin{aligned}\phi(A) &= \sup F \in \text{Cover } \mathfrak{T}\phi_F(A) \\ &\leqq \theta(A) .\end{aligned}$$

3.9. REMARK. Thus, because of Theorems 3.5.3 and 3.8, the topological measure $\phi = \text{mst } g\mathfrak{T}H$ is a generalization of the metric measure $\theta = \text{msm } g\rho H$.

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