DEGENERATE ELLIPTIC EQUATIONS

R. M. REDHEFFER AND E. G. STRAUS

Let B denote a region of Euclidean n space, with points $x = (x_1, x_2, \cdots, x_n) \in B$, and let $u = u(x)$ be such that each partial derivative, u_i, is a differentiable function of x. If

$$\sum a_{ij}(x)u_{ij} + g(|\operatorname{grad} u|) \geqq 0 \text{ and } (a_{ij}) \geqq 0 ,$$

then appropriate conditions on (a_{ij}) and on the function g ensure that u satisfies the maximum principle. That is, the inequality $u \leqq m$ on ∂S implies $u \leqq m$ in S for every constant m and every compact set $S \subset B$.

For example: Let $g(s)$ be positive, continuous and increasing for $s > 0$, and let

$$\int_0^1 \frac{ds}{g(s)} = \infty .$$

Suppose there exists a function $c(x) \in C^{(2)}$ such that, for $x \in S$,

$$\inf \sum a_{ij}(x)c_i(x)c_j(x) > 0 , \qquad \inf \sum a_{ij}(x)c_{ij}(x) > -\infty .$$

Then the maximum principle holds [1].

If $g(s) = o(s)$ the weaker condition [2]

$$\inf \sum a_{ij}(x)c_{ij}(x) > 0$$

suffices; for example, let (a_{ij}) be continuous and nonvanishing. Even when $g(s) = o(s)$, the maximum principle fails if (a_{ij}) vanishes at one point. But if $g(s) = 0$, a great many zeros can be allowed, and that is the reason for this note.

We shall establish:

THEOREM 1. *Let u be a $C^{(2)}$ solution of $\sum a_{ij}(x)u_{ij} \geqq 0$, where $(a_{ij}) \geqq 0$. Suppose that the set of points $x \in B$ where $(a_{ij}) = (0)$ has no interior points. Then u satisfies the maximum principle.*

The proof depends on the following lemma.

LEMMA 1. *Let $u \in C^{(2)}$ in a bounded region B, and let $u \in C^{(0)}$ be in the closure, $\bar{B}$, of B. Let $\tilde{B}$ be a dense subset of B. If $\sup_{x \in B} u > \sup_{x \in \partial B} u$ then there exists a quadratic polynomial $\theta(x)$ with arbitrarily small coefficients so that $(\theta_{ij}) > 0$ and $u + \theta$ attains*

Received April 19, 1963.

its maximum in $\tilde{B}$.

Proof. Choose $h>0$ so small that $\sup_{\partial B}(u+h|x|^2)<\sup_B(u+h|x|^2)$. Then the function $v = u + h|x|^2$ attains its maximum at a point $x_0 \in B$. The function $w = v - (h/2)|x - x_0|^2$ has x_0 as a unique maximum point and satisfies $(w_{ij}(x_0)) = (v_{ij}(x_0)) - hI \leqq -hI < 0$ and therefore $(w_{ij}(x)) < 0$ in a neighborhood $N: |x - x_0| < \delta$. The surface $S: z = w(x)$ is strictly concave for $x \in N$, while for $x \notin N$ we have $w(x) \leqq w(x_0) - h\delta^2/2$. Since the tangent plane of S at x_0 is horizontal and its direction varies continuously in N, there is a neighborhood $N_1 \subset N$ of x_0 so that tangent plane of S at any point $x_1 \in N_1$ lies entirely above S, except at the point x_1 itself.

Choose $x_1 \in N_1 \cap \tilde{B}$. Then function $w(x) - w(x_1) - \sum_i w_i(x_1)(x^i - x_1^i)$ is negative everywhere in the closure of B except at x_1. Thus, the function

$$\theta(x) = h|x|^2 - \frac{1}{2}h|x - x_0|^2 - \sum_i w_i(x_1)(x^i - x_1^i)$$

has the desired properties, since $(\theta_{ij}) = hI > 0$ and we can choose h and $w_i(x_1)$ arbitrarily small.

Proof of Theorem 1. Let $\tilde{B}$ be the set for which $(a_{ij}) \neq 0$. If for some compact subset S of B we would have u attain its maximum in the interior of S, then according to Lemma 1 we could choose θ so that $u + \theta$ attained its maximum at a point of $\tilde{B} \cap S$. This leads to a contradiction since $(u_{ij}) \leqq -(\theta_{ij}) < 0$ at this point.

The foregoing proof makes essential use of the condition $u \in C^{(2)}$. We now assume only that u is differentiable.

A singularity is a point where one or more of the following undesirable things happen:

(1) Some derivative u_i fails to be differentiable.
(2) The differential inequality $\sum a_{ij}(x)u_{ij} \geqq 0$ fails.
(3) The matrix $(a_{ij}) = (0)$.
(4) The condition $(a_{ij}) \geqq 0$ fails.

A "smooth surface" is a surface of form $\phi(x) = 0$, where $\phi \in C^{(2)}$ and grad $\phi \neq 0$. We can now state:

THEOREM 2. *Let* u *be differentiable for* $x \in B$, *and let the singularities be contained in the union of countably many smooth surfaces. Then* u *satisfies the maximum principle.*

The proof again depends on a small modification of u which moves the maximum outside the surfaces of singularities.

LEMMA 2. *Let u be differentiable in the bounded region B and continuous in the closure of B. Let $\phi^{(k)}(x)$ be twice differentiable with bounded $\phi_{ij}^{(k)}$ and* $\operatorname{grad}\phi^{(k)}(x) \neq 0$ *in B; $k = 1, 2, \cdots$.*

If $\sup_B u > \sup_{\partial B} u$ *then there exists a function $\theta(x)$ twice differentiable in B so that $\theta, \theta_i, \theta_{ij}$ are arbitrarily small in B; $(\theta_{ij}) > 0$ and $u + \theta$ attains its maximum at a point of B which does not lie on any surface $\phi^{(k)}(x) = 0$.*

Proof. We write $\theta = h|x|^2 + \sum c_k\phi^{(k)}(x)$ where $h > 0$ is chosen so small that $\sup_B (u + h|x|^2) > \sup_{\partial B}(u + h|x|^2) + h$ and the c_k are determined successively as follows. Set $\theta^{(0)} = h|x|^2$ and $\theta^{(n)} = h|x|^2 + \sum_{k=1}^{n} c_k\phi^{(k)}(x)$. If $u + \theta^{(n)}$ does not attain its maximum on $\phi^{(n+1)}(x) = 0$ then we set $c_{n+1} = 0$. If $u + \theta^n$ does attain its maximum on $\phi^{(n+1)}(x) = 0$ then we choose $c_{n+1} > 0$ but so small that

$$(1)\quad c_{n+1}(\phi_{ij}^{(n+1)}(x)) < \frac{h}{2^{n+1}} I ,$$

$$(2)\quad c_{n+1}|\phi^{(n+1)}(x)| < \frac{1}{2^{n+1}}(\max_B (u + \theta^{(k)}) - \max_{\phi^{(k)}=0} (u + \theta^{(k)}) , \qquad k = 1, 2, \cdots, n ,$$

$$(3)\quad c_{n+1}|\phi^{(n+1)}(x)| < \frac{h}{2^{n+1}} , \quad c_{n+1}|\phi_i^{(n+1)}(x)| < \frac{h}{2^{n+1}}$$

for all $x \in B$.

Since grad $\phi^{(n+1)} \neq 0$ it follows that $u + \theta^{(n+1)}$ does not attain its maximum on $\phi^{(n+1)}(x) = 0$ while condition (2) guarantees that it also does not attain its maximum on $\phi^{(k)}(x) = 0$, $k = 1, \cdots, n$. Conditions (1) and (3) guarantee the convergence of θ to a twice differentiable function which together with its first and second derivatives is small for small choices of h. By condition (2) $u + \theta$ does not attain its maximum on any surface $\phi^{(k)}(x) = 0$, but since $|\theta| < h|x|^2 + h$ it attains its maximum in B. Finally, condition (1) makes

$$(\theta_{ij}) > 2hI - \sum c_k(|\phi_{ij}^{(k)}|) > 2hI - \sum \frac{h}{2^k} I = hI .$$

The proof of Theorem 2 now proceeds exactly as the proof of Theorem 1.

Combining the ideas of Theorems 1 and 2 we obtain the following generalization of Theorem 1.

THEOREM 3. *Let u be differentiable in B, and have continuous second derivatives except on the union of countably many smooth surfaces. If the conditions*

$$\sum a_{ij}(x)u_{ij} \geqq 0 , \quad (a_{ij}) \geqq 0 , \quad (a_{ij}) \neq (0)$$

hold on a dense subset of B, then u satisfies the maximum principle.

Proof. According to Lemma 2 we can find a function, θ so that $(\theta_{ij}) > 0$ and $u + \theta$ attains its maximum at a point of continuity of (u_{ij}). The construction in the proof of Lemma 1 therefore yields a function $\tilde{\theta}$ so that $u + \theta + \tilde{\theta}$ attains its maximum at a point of the set of points in B at which $(a_{ij}) \neq 0$, and $(\theta_{ij}) + (\tilde{\theta}_{ij}) > 0$.

It is fairly obvious that these theorems are in many ways best possible. Certainly if the set at which $(a_{ij}) = 0$ has interior points the maximum principle fails.

The integral of a singular (Cantor) function satisfies $u_{11} = 0$ except at points of the Cantor set, but it need not satisfy the maximum principle. Thus the restriction to a denumerable number of surfaces of singularities in Theorems 2 and 3 cannot be substantially relaxed.

REFERENCES

1. R. M. Redheffer, *An extension of certain maximum principles*, Monatsh. f. Math., **62** (1962), 56-75.

2. ———, *Bemerkungen über Monotonie und Fehlerabschätzung bei nichtlinearen partiellen Differentialgleichungen*, Arch. Rat. Mech. and Anal., **10** (1962), 427-457.

UNIVERSITY OF CALIFORNIA, LOS ANGELES