# POLYNOMIALS WITH MINIMAL VALUE SETS

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Let $\mathscr{K}$ be a finite field of characteristic $p$ that contains exactly $q$ elements. Let $F(x)$ be a polynomial over $\mathscr{K}$ of degree $f, f > 0$, and let $r + 1$ denote the number of distinct values $F(\tau)$ as $\tau$ ranges over $\mathscr{K}$. Carlitz, Lewis, Mills, and Straus [1] pointed out that $r \geqq [(q-1)/f]$, and raised the question of determining all polynomials for which $r = [(q-1)/f]$. The cases $r = 0$ and $r = 1$ are special cases that do not fit into the general pattern. These are treated in [1], and do not concern us here. Thus we arrive at the statement of our main problem: For what polynomials $F(x)$ do we have

$$(\mathrm{I}) \qquad r = [(q-1)/f] \geqq 2?$$

Carlitz, Lewis, Mills, and Straus [1] determined all polynomials with $f < 2p + 2$ for which (I) holds. In the present paper this result is extended—all polynomials with $f \leqq \sqrt{q}$ for which (I) holds are determined. These are polynomials of the form

$$F(x) = \alpha L^v + \gamma ,$$

where $L$ is a polynomial that factors into distinct linear factors over $\mathscr{K}$ and that has the form

$$L = \beta + \sum_i \varphi_i x^{p^{ki}} ,$$

and where $v$ and $k$ are integers such that $v \mid (p^k - 1)$ and $q$ is a power of $p^k$. Regardless of the size of $f$ our present methods give a great deal of information about $F(x)$. Furthermore many of the proofs of [1] can be shortened and simplified by using the results of § 1 of the present paper.

The results of [1] provide a complete answer for the case $q = p$. In the present paper the problem is completely solved for the case $q = p^2$.

**1. Preliminaries.** Let $\mathscr{K}$ be a finite field with $q$ elements and characteristic $p$. We use Greek letters for elements of $\mathscr{K}$, and small Latin letters, other than $x$, for nonnegative integers. We use capital letters for polynomials in one variable over $\mathscr{K}$. The polynomials denoted by $A, B, C, D, E$ and the integers denoted by $a, b, c, d, e$

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vary from proof to proof. The polynomials and integers denoted by other letters, except $i$ and $j$, remain the same throughout the paper.

Let $F = F(x)$ be a polynomial over $\mathscr{K}$ of degree $f, f > 0$. Let $\gamma_0, \gamma_1, \cdots, \gamma_r$ denote the distinct values assumed by $F(\tau)$ as $\tau$ ranges over $\mathscr{K}$. It follows easily from the fact that a polynomial of degree $f$ has at most $f$ roots, that $r + 1 \geqq q/f$. This is equivalent to $r \geqq [(q-1)/f]$. We intend to study the question raised in [1] of characterizing those polynomials for which $r = [(q-1)/f]$. The cases $r = 0$ and $r = 1$ were fully treated in [1]. Hence we make the assumption that

$$r = [(q-1)/f] \geqq 2 . \tag{1}$$

Subtracting the constant $\gamma_0$ from $F$ does not change the value of $r$. Thus it is sufficient to consider the case $\gamma_0 = 0$. In the first two sections of this paper, we assume that

$$\gamma_0 = 0 .$$

Then $\gamma_i \neq 0$ for $i > 0$. We now set

$$F_i = F - \gamma_i , \qquad 0 \leqq i \leqq r .$$

The polynomials $F_i$ are relatively prime in pairs, and each of them has at least one root in $\mathscr{K}$. Let $\pi_{i1}, \pi_{i2}, \cdots, \pi_{il_i}$ be the distinct roots of $F_i$ that lie in $\mathscr{K}$ and set

$$L_i = \prod_{j=1}^{l_i} (x - \pi_{ij}) , \qquad 0 \leqq i \leqq r .$$

Then $l_i = \deg L_i \geqq 1$, $0 \leqq i \leqq r$, and[1]

$$x^q - x = \prod_{i=0}^{r} L_i . \tag{2}$$

Now set $F_i = L_i U_i$, $0 \leqq i \leqq r$, and

$$G = \prod_{i=0}^{r} U_i . \tag{3}$$

Then the $L_i$, the $U_i$ and $G$ are polynomials over $\mathscr{K}$, and

$$(x^q - x)G = \prod_{i=0}^{r} F_i . \tag{4}$$

Now (4) and (1) give us an upper bound on the degree of $G$, namely

$$\deg G = (r+1)f - q \leqq q - 1 + f - q = f - 1 .$$

[1] The relations (2), (3), (4), (5), (6), and (7) can all be found in [1] under the assumption that the leading coefficient of $F$ is 1.

Thus we have

$$(5) \qquad \deg G < f\,.$$

Set $u_i = \deg U_i$, $0 \leqq i \leqq r$. We already have $F = F_0$ by the assumption $\gamma_0 = 0$. We set $L = L_0$, $U = U_0$, $l = l_0$, and $u = u_0$.

We now differentiate both sides of (2) and obtain $-1 \equiv L'L^*$ $(\bmod L)$, where $L^* = L_1L_2\cdots L_r$. Hence $G \equiv -L'L^*G$ $(\bmod LG)$. Since $F = LU$ and $U \mid G$, it follows that $F \mid LG$ and thus

$$G \equiv -L'L^*G \qquad (\bmod F)\,.$$

Now

$$L^*G = U\prod_{i=1}^{r}(L_iU_i) = U\prod_{i=i}^{r}(F-\gamma_i) \equiv -\zeta U \qquad (\bmod F)\,,$$

where

$$\zeta = -\prod_{i=1}^{r}(-\gamma_i) \neq 0\,.$$

Hence $G \equiv \zeta L'U$ $(\bmod F)$. Since $\deg(\zeta L'U) < \deg(LU) = f$ and $\deg G < f$, we must have

$$(6) \qquad G = \zeta L'U\,.$$

By symmetry it follows that

$$(7) \qquad G = \zeta_i L_i' U_i\,, \qquad 0 \leqq i \leqq r\,,$$

for suitable nonzero elements $\zeta_i$ of $\mathscr{K}$.

We next derive another expression for $G$.

LEMMA 1. *There exists a nonzero element $\theta$ in $\mathscr{K}$ such that $G = \theta F'$.*

*Proof.* Since $F' = F_i' = L_i'U_i + L_iU_i'$, it follows from (7) that

$$L_iU_i' = F' - G/\zeta_i\,, \qquad 0 \leqq i \leqq r\,.$$

Therefore $L_0U_0' = LU'$, $L_1U_1'$, and $L_2U_2'$ are linearly dependent. Thus there exist $\lambda$, $\lambda_1$, and $\lambda_2$ in $\mathscr{K}$, not all zero, such that

$$\lambda LU' + \lambda_1L_1U_1' + \lambda_2L_2U_2' = 0\,.$$

Multiplying this relation by $UU_1U_2$ and noting that $LU = F$, $L_1U_1 = F - \gamma_1$, $L_2U_2 = F - \gamma_2$, we obtain

$$(8) \quad (\lambda U'U_1U_2 + \lambda_1UU_1'U_2 + \lambda_2UU_1U_2')F = \lambda_1\gamma_1UU_1'U_2 + \lambda_2\gamma_2UU_1U_2'\,.$$

Now the degree of the right side of (8) is less than $u + u_1 + u_2$ and

$$u + u_1 + u_2 \leqq \deg G < f = \deg F\ .$$

This is possible only if we have

$$(9) \qquad \lambda U' U_1 U_2 + \lambda_1 U U_1' U_2 + \lambda_2 U U_1 U_2' = 0\ .$$

The constants $\lambda$, $\lambda_1$, and $\lambda_2$ are not all zero. Without loss of generality we suppose $\lambda_2 \neq 0$. Then (9) gives us $U_2 \mid U U_1 U_2'$. Since $U_2 \mid F_2$, $U_2$ must be relatively prime to both $F$ and $F_1$. Hence $U_2$ is relatively prime to $UU_1$, and $U_2 \mid U_2'$. This implies that $U_2' = 0$. Hence

$$F' = F_2' = L_2' U_2 + L_2 U_2' = L_2' U_2 = G/\zeta_2.$$

Thus $G = \zeta_2 F'$, which completes this proof.

Lemma 1 is false for $r \leqq 1$—counter examples can be readily constructed.

LEMMA 2. *For each $j$, $0 \leqq j \leqq r$, $U_j$ is of the form*

$$U_j = L_j^{w_j} H_j^p\ ,$$

*where $w_j$ is a nonnegative integer, $H_j$ is a polynomial over $\mathscr{K}$, and $L_j \nmid H_j$.*

*Proof.* By symmetry it is sufficient to prove the lemma for the case $j = 0$. Combining (6) with Lemma 1 we obtain

$$\zeta L' U = G = \theta F' = \theta L' U + \theta L U'\ .$$

Thus

$$(10) \qquad \theta L U' = (\zeta - \theta) L' U\ .$$

We set $U = L^w A$, where $L \nmid A$ and $w \geqq 0$. Then substitution in (10) gives us

$$\theta w L^w L' A + \theta L^{w+1} A' = (\zeta - \theta) L' L^w A.$$

This reduces to

$$\theta L A' = (\zeta - \theta - w\theta) L' A.$$

Thus $L \mid (\zeta - \theta - w\theta) L' A$. Since $L$ is the product of distinct linear factors, it follows that $L$ and $L'$ are relatively prime. Since $L \nmid A$, this implies that $\zeta - \theta - w\theta = 0$. Therefore $\theta L A' = 0$. It follows that $A' = 0$. Hence $A = H^p$ for some polynomial $H$. Then we have $L \nmid H$ and $U = L^w H^p$, which completes this proof.

We now suppose, without loss of generality, that

$$(11) \qquad l \leqq l_j\ , \qquad 0 \leqq j \leqq r\ .$$

LEMMA 3. *Under the assumption* (11), *the constants* $w_j$ *of Lemma* 2 *satisfy*

$$w_1 = w_2 = \cdots = w_r = 0 .$$

*Proof*. Combining (3) and (6) we obtain

$$\zeta L' U = G = U U_1 U_2 \cdots U_r .$$

Now suppose $1 \leqq j \leqq r$. Then $U_j \mid L'$, and hence

$$u_j \leqq \deg L' < l \leqq l_j .$$

Therefore $L_j \nmid U_j$, so that we have $w_j = 0$. This completes the proof.

Set $H = H_0$ and $v = w_0 + 1$. Then from Lemmas 2 and 3 we obtain

$$F = LU = L^v H^p , \tag{12}$$

and

$$F_i = L_i U_i = L_i H_i^p , \qquad 1 \leqq i \leqq r , \tag{13}$$

where $L \nmid H$, $L_i \nmid H_i$. Moreover

$$\zeta L' = G/U = U_1 U_2 \cdots U_r = (H_1 H_2 \cdots H_r)^p .$$

Thus $L' = S^p$, where $S = \zeta^{-1/p} H_1 H_2 \cdots H_r$. Therefore $L$ is of the form

$$L = x S^p + T^p , \tag{14}$$

where $T$, as well as $S$, is a polynomial over $\mathscr{K}$.

**2. The polynomial $R(x)$.** Set

$$R(x) = \prod_{i=1}^{r} (x - \gamma_i) = \sum_{j=0}^{r} \rho_j x^j ,$$

where $\rho_j \in \mathscr{K}$, $0 \leqq j \leqq r$, $\rho_r = 1$. From (4) and (6) we obtain

$$LUR(F) = FR(F) = \prod_{i=0}^{r} F_i = (x^q - x) G = \zeta (x^q - x) L' U .$$

These identities and (12) give us

$$\sum_{j=0}^{r} \rho_j L^{1+vj} H^{pj} = LR(F) = \zeta (x^q - x) L' . \tag{15}$$

Differentiating both sides of (15) and noting that $L'' = 0$ by (14), we get the congruence

$$\rho_0 L' \equiv -\zeta L' \pmod{L} .$$

Since $L' \neq 0$, we obtain

$$\rho_0 = -\zeta . \tag{16}$$

By Lemma 1 we have $F' = G/\theta \neq 0$. Hence $p \nmid v$.

Let $k$ be the smallest positive integer such that $v \mid (p^k - 1)$. The main objective of this section is to show that $1 + vj$ is a power of $p^k$ for every nonzero coefficient $\rho_j$ of $R(x)$.

In the proof of the following lemma the notation $A \| B$ means that $A \mid B$ and $(A, B/A) = 1$.

LEMMA 4. *Let $d$ be a nonnegative integer such that $L'$ is a $p^d$th power and $1 + vr > p^{d-1}$. If $j$ is an integer such that $\rho_j \neq 0$, then either* (i) *$1 + vj$ is a power of $p^k$, or* (ii) *$p^d \mid (1 + vj)$. Moreover $H$ is a $p^{d-1}$st power.*

*Proof by induction on $d$.* The desired result is trivial for $d = 0$. We suppose that it is true for an integer $d$ and show that this implies that it is true for $d + 1$. Thus we assume that $L'$ is a $p^{d+1}$st power and $1 + vr > p^d$. Then the induction hypothesis applies so that $R(x)$ is of the form

$$R(x) = \sum_{i=0}^{c} \omega_i x^{(p^{ki}-1)/v} + \Sigma' \rho_j x^j , \tag{17}$$

where $\omega_i \in \mathscr{K}$, $0 \leqq i \leqq c$, $c = [d/k]$, and the second summation $\Sigma'$ is over all $j$ such that

$$p^d \mid (1+vj) , \quad p^d < 1 + vj , \quad j \leqq r .$$

Moreover $H$ is a $p^{d-1}$st power. Thus

$$H = A^{p^{d-1}} \quad \text{and} \quad F = L^v A^{p^d}$$

for some polynomial $A$ over $\mathscr{K}$. Substitution in (15) gives us

$$\Sigma' \rho_j L^{1+vj} A^{jp^d} = \zeta x^q L' + B , \tag{18}$$

where

$$B = -\zeta x L' - \sum_{i=0}^{c} \omega_i L^{p^{ki}} A^{p^d (p^{ki}-1)/v} .$$

The left side of (18) is a $p^d$th power. Since

$$q \geqq 1 + fr \geqq 1 + vr > p^d$$

and $q$ is a power of $p$, it follows that $p^{d+1} \mid q$. Hence $\zeta x^q L'$ is a $p^{d+1}$st power. Therefore $B$ is a $p^d$th power. Thus we can set

$$\zeta x^q L' = C^{p^{d+1}} \quad \text{and} \quad B = D^{p^d} .$$

Since $1 + vr > p^d$ and $\rho_r \neq 0$, it follows that the left side of (18) does not vanish identically. Let the term corresponding to $j = a$ be the nonzero term of lowest degree in the left side of (18). Thus $a$ is the least integer such that $\rho_a \neq 0$ and $1 + va > p^d$. Then $p^d \mid (1 + va)$, and hence $1 + va \geqq 2p^d$. Because of the way $a$ was chosen we have

$$L^{1+va} A^{ap^d} \,\|\, (\zeta x^q L' + B) . \tag{19}$$

Extracting the $p^d$th roots of both sides of (19) we get

$$L^{(1+va)p^{-d}} A^a \,\|\, (C^p + D).$$

Since $1 + va \geqq 2p^d$ this gives us $L^2 A^a \mid (C^p + D)$. By differentiation we obtain

$$LA^{a-1} \mid D' . \tag{20}$$

Now

$$\deg D' < p^{-d} \deg B \leqq p^{-d} \deg \{L^{p^{kc}} A^{p^d (p^{kc}-1)/v}\} \leqq \deg \{LA^{(p^{kc}-1)/v}\} .$$

Since

$$a > (p^d - 1)/v \geqq (p^{kc} - 1)/v ,$$

we have $(p^{kc} - 1)/v \leqq a - 1$, and

$$\deg D' < \deg (LA^{a-1}) .$$

Combining this with (20) we get $D' = 0$. Thus $D$ must be a $p$th power, and $B$ a $p^{d+1}$st power. Thus the right side of (19) is a $p^{d+1}$st power. Hence the left side of (19) is also a $p^{d+1}$st power. Now $L \nmid H$. Since $L$ is the product of distinct linear factors we have $L \nmid A$, $p^{d+1} \mid (1 + va)$, and $A^a$ is a $p$th power. Hence $p \nmid a$, and $A$ itself is a $p$th power. It follows that $H$ is a $p^d$th power. Suppose there is a $b$ such that $\rho_b \neq 0$, $1 + vb$ is not a power of $p^k$, and $p^{d+1} \nmid (1 + vb)$. Without loss of generality suppose that $b$ is the smallest integer with these properties. By (17) we have $1 + vb > p^d$, and by (18) we have

$$L^{1+vb} A^{bp^d} \,\|\, \{\zeta x^q L' + B - \Sigma'' \rho_j L^{1+vj} A^{jp^d}\} , \tag{21}$$

where $\Sigma''$ is over those $j$ such that $j < b$, $p^{d+1} \mid (1 + vj)$. The right side of (21) is a $p^{d+1}$st power. Hence the left side of (21) is also a $p^{d+1}$st power. Therefore $p^{d+1} \mid (1 + vb)$, a contradiction. It follows that for every $j$ such that $\rho_j \neq 0$, either $1 + vj$ is a power of $p^k$ or $p^{d+1} \mid (1 + vj)$. This establishes the desired result for $d + 1$, and completes this proof.

LEMMA 5. *Suppose there exists an integer $d$ such that $L'$ is a $p^d$th power but not a $p^{d+1}$st power, and $1 + vr > p^d$. Then $v = 1$ and $p^{d+1} \nmid (1 + r)$.*

*Proof*. Since $L'$ is a $p$th power by (14), we have $d \geqq 1$. By Lemma 4 we have

$$R(x) = \sum_{i=0}^{c} \omega_i x^{(p^{ki}-1)/v} + \Sigma^* \rho_j x^j + x^r ,$$

where the $\omega_i$ are elements of $\mathscr{K}$, $c = [d/k]$, and the summation $\Sigma^*$ is over all $j$ such that $p^d \mid (1 + vj)$, $p^d < 1 + vj$, $j < r$. Moreover since $1 + vr > p^d$ and $\rho_r \neq 0$, we have $p^d \mid (1 + vr)$. Furthermore $H$ is a $p^{d-1}$st power. Since $\zeta \in \mathscr{K}$, it follows that $\zeta L'$ is a $p^d$th power but not a $p^{d+1}$st power. Thus we can set

$$H = A^{p^{d-1}} \quad \text{and} \quad \zeta L' = C^{p^d} ,$$

where $C$ is not a $p$th power. Substitution in (15) gives us

$$L^{1+vr} A^{rp^d} = x^q C^{p^d} + B , \tag{22}$$

where

$$\begin{aligned} B &= -\zeta x L' - LR(F) + LF^r \\ &= -\zeta x L' - \sum_{i=0}^{c} \omega_i L^{p^{ki}} A^{p^d (p^{ki}-1)/v} - \Sigma^* \rho_j L^{1+vj} A^{jp^d} . \end{aligned}$$

Now the left side of (22) is a $p^d$th power. Moreover

$$q \geqq 1 + fr \geqq 1 + vr > p^d ,$$

so that $p^{d+1}\ q$. Therefore $B$ is a $p^d$th power, say $B = D^{p^d}$. Extracting the $p^d$th roots of both sides of (22) we obtain

$$L^{(1+vr)p^{-d}} A^r = x^{qp^{-d}} C + D . \tag{23}$$

Differentiation now yields

$$L^{-1+(1+vr)p^{-d}} A^{r-1} \{(1 + vr)p^{-d} L'A + rLA'\} = x^{qp^{-d}} C' + D' . \tag{24}$$

since $p^{d+1} \mid q$. Multiplying (24) by $C$, (23) by $C'$, and subtracting, we get

$$L^{-1+(1+vr)p^{-d}} A^{r-1} E = CD' - C'D , \tag{25}$$

where

$$E = (1 + vr)p^{-d} L'AC + rLA'C - LAC' .$$

Now $A \mid H$ and therefore $LA \mid F$. Moreover

$$C \mid L' = G/(\zeta U) = \zeta^{-1} U_1 U_2 \cdots U_r \mid F_1 F_2 \cdots F_r .$$

Hence $C$ is relatively prime to $LA$. Since $C$ is not a $p$th power we have $C' \neq 0$. Hence $C \nmid LAC'$. It follows that $E \neq 0$. From (25) we obtain $CD' \neq C'D$ and

$$L^{-e+(1+vr)p^{-d}} A^{r-1} \mid (CD' - C'D) , \tag{26}$$

where

$$e = \begin{cases} 0 & \text{if } p^{d+1} \mid (1 + vr) , \\ 1 & \text{if } p^{d+1} \nmid (1 + vr) . \end{cases}$$

Comparing degrees in (26) we obtain

$$(1 + vr - ep^d)l + p^d(r - 1) \deg A < p^d \deg (CD) = \deg (L'B) . \tag{27}$$

Now the leading term of $R(x)$ is $x^r$ and $R(x) \neq x^r$. Set $b = \deg \{R(x) - x^r\}$. Then we have $0 \leqq b < r$ and

$$\begin{aligned} \deg B &\leqq \deg (LF^b) \\ &= (1 + vb)l + bp^d \deg A \leqq (1 + vb)l + (r - 1)p^d \deg A . \end{aligned}$$

Substitution in (27) gives us, after simplification,

$$v(r - b)l < ep^d l + \deg L' < (ep^d + 1)l .$$

Hence $v(r - b) \leqq ep^d$. Therefore $e \neq 0$. Hence $e = 1$ and

$$v(r - b) \leqq p^d.$$

Since $p^d \mid (1 + vr)$ and $1 + vr > p^d$, we have $1 + vr \geqq 2p^d$ and

$$1 + vb = 1 + vr - v(r - b) \geqq p^d .$$

Since $\rho_b \neq 0$, this gives us $p^d \mid (1 + vb)$. Since $p^d \mid (1 + vr)$, it follows that $p^d \mid v(r - b)$ and $p \nmid v$. Hence $v(r - b) = p^d$ and $v = 1$. Finally since $e = 1$ we have

$$p^{d+1} \nmid (1 + vr) = 1 + r ,$$

which completes this proof.

LEMMA 6. *If $d$ is an integer such that $p^d < 1 + vr$, then $L'$ is a $p^{d+1}$st power.*

*Proof.* Suppose the result is false. Then $L'$ is not a $p^{d+1}$st power and $p^d < 1 + vr$. Without loss of generality we suppose that $L'$ is a $p^d$th power. By Lemma 5 we have $v = 1$ and $p^{d+1} \nmid (1 + r)$.

Therefore $k = 1$ and $p^d < 1 + r$. It follows from Lemma 4 that $R(x)$ is of the form

$$R(x) = \sum_{i=0}^{d-1} \omega_i x^{p^i - 1} + \Sigma^+ \rho_j x^j ,$$

where the summation $\Sigma^+$ is over all $j$ such that $p^d \mid (1 + j)$, $j \leqq r$. Moreover $H$ is a $p^{d-1}$st power and $p^d \mid (1 + r)$. Now

$$FR(F) = \prod_{i=0}^{r} (F - \gamma_i) = \prod_{i=0}^{r} F_i = (x^q - x)G$$

by (4), so that

$$\Sigma^+ \rho_j F^{j+1} = x^q G + B , \tag{28}$$

where $\deg B \leqq p^{d-1} f$. The left side of (28) is a $p^d$th power. Moreover $q \geqq 1 + fr \geqq 1 + r > p^d$, so that $x^q$ is a $p^{d+1}$st power. Since $G = \zeta L' U$ and $U = L^{v-1} H^p = H^p$, it follows that $G$ is a $p^d$th power. Hence $B$ is also a $p^d$th Power. We set

$$G = C^{p^d} \quad \text{and} \quad B = D^{p^d} .$$

Then, extracting the $p^d$th roots of both sides of (28), we get

$$\sum_{j=1}^{a} \xi_j F^j = x^{qp^{-d}} C + D , \tag{29}$$

where $a = (r + 1)p^{-d} \geqq 2$, the $\xi_j$ are in $\mathscr{K}$, $\xi_a = 1$, and $\deg D \leqq f/p$. Now $p \nmid a$ since $p^{d+1} \nmid (r + 1)$. We set $\bar{F} = F + \xi_{a-1}/a$. Then (29) becomes

$$\sum_{j=0}^{a} \eta_j \bar{F}^j = x^{qp^{-d}} C + D , \tag{30}$$

where the $\eta_j$ are in $\mathscr{K}$, $\eta_a = 1$, and $\eta_{a-1} = 0$. Differentiating (30) we obtain

$$\sum_{j=1}^{a} j \eta_j \bar{F}^{j-1} \bar{F}' = x^{qp^{-d}} C' + D' . \tag{31}$$

Eliminating $x^{qp^{-d}}$ from (30) and (31) we get

$$\eta_0 C' + \sum_{j=1}^{a} \eta_j \bar{F}^{j-1} (C' \bar{F} - jC\bar{F}') = C'D - CD' .$$

Since $\eta_{a-1} = 0$, it follows that

$$\bar{F}^{a-1} (C' \bar{F} - aC\bar{F}') = C'D - CD' - E , \tag{32}$$

where

$$\deg E < (a - 2)f + \deg C .$$

Now

$$\deg C = p^{-d} \deg G < p^{-d} f \leqq f/p$$

by (5). Hence $\deg E < (a-1)f$, and

$$\deg (C'D - CD') < \deg (CD) < 2f/p \leqq (a-1)f .$$

Therefore

$$\deg (C'D - CD' - E) < (a-1)f = \deg \bar{F}^{a-1} ,$$

and (32) yields

$$C'\bar{F} = aC\bar{F}' .$$

Now $\bar{F}' = F' = \theta^{-1}G \neq 0$ by Lemma 1. Therefore $aC\bar{F}' \neq 0$. Hence $C' \neq 0$ and thus $C \nmid C'$. It follows that $(\bar{F}, C) \neq 1$. Since

$$C^{p^d} = G = \prod_{i=0}^{r} U_i$$

we have $(\bar{F}, U_b) \neq 1$ for some $b, 0 \leqq b \leqq r$. Hence $(\bar{F}, F_b) \neq 1$. Since $\bar{F} - F_b \in \mathscr{K}$, we must have $\bar{F} = F_b$. Therefore

$$C'F_b = aCF_b' .$$

Since $v = 1$, we have $F_b = L_b H_b^p$, whether or not $b = 0$. Hence

$$C'L_b H_b^p = aCL_b' H_b^p ,$$

and $C'L_b = aCL_b'$. Now $L_b$ is relatively prime to $L_b'$. Therefore $L_b \mid C$. Since $v = 1$ we have

$$C^{p^d} = G = \prod_{i=0}^{r} U_i = \prod_{i=0}^{r} H_i^p .$$

It follows that $L_b \mid H_0 H_1 \cdots H_r$. On the other hand $L_b \nmid H_b$, while for $i \neq b$ we have $(L_b, H_i) = 1$. Therefore $L_b \nmid H_0 H_1 \cdots H_r$, a contradiction. This completes the proof of this lemma.

We are now in a position to prove the most general theorem of this paper. We drop the assumption $\gamma_0 = 0$.

THEOREM 1. *Let $\mathscr{K}$ be a finite field of characteristic $p$ that contains exactly $q$ elements. Let $F(x)$ be a polynomial over $\mathscr{K}$ of degree $f, f > 0$. Let $\gamma_0, \gamma_1, \cdots, \gamma_r$ be the distinct values $F(\tau)$ as $\tau$ ranges over $\mathscr{K}$, and let $l_i$ denote the number of distinct roots in $\mathscr{K}$ of the polynomial $F(x) - \gamma_i$. Let the $\gamma_i$ be arranged in such a way that $l_0 \leqq l_i$, $1 \leqq i \leqq r$. Set $L = \Pi(x - \pi)$, where the product is over the distinct roots $\pi$ of $F(x) - \gamma_0$ that lie in $\mathscr{K}$. Suppose that*

$r = [(q-1)/f] \geqq 2$. *Then there exist positive integers* $v, k, m$; *a polynomial* $N$ *over* $\mathscr{K}$; *and* $\omega_0, \omega_1, \cdots, \omega_m$ *in* $\mathscr{K}$ *such that* $L \nmid N$, $v \mid (p^k - 1)$, $1 + vr = p^{mk}$, $L'$ *is a* $p^{mk}$th *power*, $\omega_0 \neq 0$, $\omega_m = 1$,

$$F(x) = L^v N^{p^{mk}} + \gamma_0 ,$$

$$\prod_{i=1}^{r} (x - \gamma_i + \gamma_0) = \sum_{i=0}^{m} \omega_i x^{(p^{ki}-1)/v} , \tag{33}$$

*and*

$$\sum_{i=0}^{m} \omega_i L^{p^{ki}} N^{p^{km}(p^{ki}-1)/v} = -\omega_0(x^q - x)L' . \tag{34}$$

*Proof.* Without loss of generality we can suppose that $\gamma_0 = 0$, so that our previous discussion applies. Let $d$ be the integer such that

$$p^d \geqq 1 + vr > p^{d-1} .$$

It follows from Lemma 6 that $L'$ is a $p^d$th power. We now apply Lemma 4 to conclude that either $1 + vr$ is a power of $p^k$ or $p^d \mid (1 + vr)$. In either case we must have $p^d = 1 + vr$. Since $k$ is the smallest positive integer such that $v \mid (p^k - 1)$, it follows that $k \mid d$. We put $m = d/k$. Then $L'$ is a $p^{mk}$th power and $1 + vr = p^{mk}$. Applying Lemma 4 again we find that $R(x)$ is of the form

$$R(x) = \sum_{i=0}^{m} \omega_i x^{(p^{ki}-1)/v} ,$$

so that (33) holds. Moreover $H$ is a $p^{d-1}$st power by Lemma 4, and therefore $H^p$ is a $p^{mk}$th power. Thus there is a polynomial $N$ over $\mathscr{K}$ such that

$$F = L^v H^p = L^v N^{p^{mk}} .$$

Furthermore since $L \nmid H$, it follows that $L \nmid N$. Using (16) we obtain $\omega_0 = \rho_0 = -\zeta \neq 0$. It follows at once from (33) that $\omega_m = 1$. Finally we substitute in (15) to obtain (34). This completes the proof of the theorem.

In the next two sections we apply Theorem 1 to a number of special cases.

**3. A special case.** There are two general types of polynomials known for which (1) holds [1, § 5]. For every polynomial of the first type both $L'$ and $N$ are constants. Thus this case is of special interest. Here we have the following result:

LEMMA 7. *Suppose that $L'$ and $N$ are both constants. Then $q$ is a power of $p^k$, and $F$ is of the form*

$$F = \alpha L^v + \gamma,\ L = \beta + \sum_{j=0}^{d} \varphi_j x^{p^{kj}}\ , \tag{35}$$

*where $L$ factors into distinct linear factors over $\mathscr{K}$ and $v \mid (p^k - 1)$.*

*Proof.* Since $N$ is a constant it follows from Theorem 1 that $F = \alpha L^v + \gamma$, where $\alpha \in \mathscr{K}$ and $\gamma = \gamma_0 \in \mathscr{K}$. Suppose that $L$ is not of the form given in (35). Then, since $L'$ is a constant, we can write

$$L = \beta + \sum_{j=0}^{c} \varphi_j x^{p^{kj}} + \sum_{j=a}^{l/p} \delta_j x^{pj} \tag{36}$$

where $a$ and $c$ are integers such that

$$p^{k(c+1)} > pa > p^{kc},\ l \geqq pa\ ,$$

and $\delta_a \neq 0$. Moreover $L' = \varphi_0 \neq 0$. Now (34) becomes

$$\sum_{i=0}^{m} \chi_i L^{p^{ki}} = -\omega_0 \varphi_0 (x^q - x)\ , \tag{37}$$

where the $\chi_i$ are in $\mathscr{K}$, $\chi_0 = \omega_0 \neq 0$, and $\chi_m \neq 0$. Substituting (36) in (37) we get

$$\psi + \sum_{j=0}^{c} \psi_j x^{p^{kj}} + \chi_0 \delta_a x^{pa} + \sum_{j=pa+1}^{lp^{km}} \sigma_j x^j = -\omega_0 \varphi_0 (x^q - x)\ ,$$

for suitable $\psi$, $\psi_j$, $\sigma_j$ in $\mathscr{K}$. Since $\chi_0 \delta_a \neq 0$, this implies that either $pa = 1$ or $pa = q$. Comparing degrees we obtain

$$q = lp^{km} > l \geqq pa.$$

Clearly $pa \neq 1$. This contradiction implies that $L$ is of the desired form, which completes this proof.

The converse of Lemma 7 is already known [1]: *If $q$ is a power of $p^k$, and if $F$ is of the form* (35), *then the polynomial $F$ satisfies the equality $r = [(q-1)/f]$.* This was proved in [1] as follows: Let $\pi$ be a root of $L$. Replacing $x$ by $x + \pi$ we can assume that $\beta = 0$. Let $l = \deg L$ as before, and set $L(x) = L$. Because of the form of $L$ the values assumed by $L(\tau)$ as $\tau$ ranges over $\mathscr{K}$ form a vector space over the subfield $GF(p^k)$. Since we have assumed that $L$ factors into distinct linear factors over $\mathscr{K}$, it follows that $L$ has exactly $l$ distinct roots in $\mathscr{K}$. Therefore this vector space contains exactly $q/l$ distinct elements. Then since $F = \alpha L^v + \gamma$, where $v \mid (p^k - 1)$, it follows that the number of values assumed by $F(\tau)$ as

$\tau$ ranges over $\mathscr{K}$ is exactly

$$1 + (-1 + q/l)/v = 1 + (q - l)/f = 1 + [(q - 1)/f] .$$

Hence $r = [(q - 1)/f]$.

Thus we have a complete characterization of those polynomials for which $r = [(q - 1)/f] \geqq 2$, subject to the condition that $L'$ and $N$ are both constants. One significance of this result can be seen from the following lemma:

LEMMA 8. *If $f \leqq \sqrt{q}$, and $r = [(q - 1)/f] \geqq 2$, then $L'$ and $N$ are both constants.*

*Proof.* Theorem 1 applies so that we have $1 + rv = p^{mk}$, and $f = vl + p^{mk} \deg N$. Moreover $f^2 \leqq q$ and $r = [(q - 1)/f]$ so that

$$f \leqq q/f \leqq r + 1 = 1 + (p^{mk} - 1)/v \leqq p^{mk} .$$

Thus $p^{mk} \deg N < f \leqq p^{mk}$, $\deg N = 0$, and $N$ is a constant. Furthermore $L'$ is a $p^{mk}$th power by Theorem 1 and $\deg L' < l \leqq f \leqq p^{mk}$. Hence $L'$ is also a constant, and the proof of this lemma is complete.

The above results give us a complete characterization of those polynomials $F$ for which $r = [(q - 1)/f] \geqq 2$ and $0 < f \leqq \sqrt{q}$. Now suppose that $r = [(q - 1)/f] < 2$ and $0 < f \leqq \sqrt{q}$. Then

$$2 > (q - 1)/f \geqq (f^2 - 1)/f ,$$

$f^2 - 2f - 1 < 0$, and thus $f = 1$ or $f = 2$. Now $q$ is a prime power and $f^2 \leqq q < 2f + 1$. Hence we have either (i) $f = 1$ and $q = 2$, or (ii) $f = 2$ and $q = 4$. If $f = 1$, then $F$ is clearly of the form (35) with $v = k = 1$ and $d = 0$. If $f = 2$ and $q = 4$, then $r = 1$, and since $F_0$ and $F_1$ together have 4 distinct roots in $\mathscr{K}$, it follows that $F_0$ has two distinct roots in $\mathscr{K}$, so that $F$ is still of the form (35), this time with $p = 2$ and $v = k = d = 1$. Thus we see that the condition $r \geqq 2$ can be dropped here. Combining all these results we obtain one of our major results:

THEOREM 2. *Let $F(x)$ be a polynomial over the finite field $\mathscr{K}$ of characteristic $p$ and let $q$ denote the number of elements of $\mathscr{K}$. Let $r + 1$ denote the number of distinct values assumed by $F(\tau)$ as $\tau$ ranges over $\mathscr{K}$, and let $f$ be the degree of $F(x)$. Suppose that $0 < f \leqq \sqrt{q}$. Then*

$$r = [(q - 1)/f]$$

*if and only if $F$ is of the form*

$$F = \alpha L^v + \gamma ,$$

*where $L$ is a polynomial that factors into distinct linear factors over $\mathscr{K}$ and that has the form*

$$L = \beta + \sum_{i=0}^{d} \varphi_i x^{p^{ki}} .$$

*and where $v$ and $k$ are integers such that $v \mid (p^k - 1)$, $q$ is a power of $p^k$, and $\alpha$, $\beta$, $\gamma$, and the $\varphi_i$ are elements of $\mathscr{K}$.*

**4. The cases $q = p$ and $q = p^2$.** The results of § 1 enable us to treat the case $q = p$ quickly.

Suppose $q = p$ and $r = [(q-1)/f] \geqq 2$. If $\gamma_0 = 0$, then the results of § 1 apply, so that

$$F = L^v H^p, \qquad L = xS^p + T^p$$

by (12) and (14). Since

$$\deg F = f \leqq \frac{1}{2}(q-1) = \frac{1}{2}(p-1) < p ,$$

the polynomials $H$, $S$, and $T$ are all constants. Thus $F$ is of the form $\alpha(x+\beta)^v$ and $v = f$. It is easily shown that $v \mid (q-1)$ here. Dropping the assumption $\gamma_0 = 0$, we see that if $q = p$ and $r = [(q-1)/f] \geqq 2$, then $f \mid (q-1)$ and $F$ is of the form

$$F = \alpha(x+\beta)^f + \gamma .$$

We note that in this case $L'$ and $N$ must both be constants, so that we could have obtained this result from Lemma 7.

Let us now consider the case $q = p^2$. Comparing the degrees of the two sides of (34) we obtain

$$p^{mk} l + r p^{mk} \deg N = q + \deg L' \leqq q + l - 1 = p^2 + l - 1 .$$

Therefore

$$pl + p \deg N \leqq p^2 + l - 1 . \tag{38}$$

Thus $pl \leqq p^2 + l - 1$ or $l \leqq p + 1$. Since $L'$ is a $p$th power, it follows that $l \equiv 0$ or $1 \pmod p$. Therefore $l = 1$, $p$, or $p + 1$. If $l = p$ or $p + 1$, the inequality (38) gives us

$$p \deg N \leqq p^2 - l(p-1) - 1 \leqq p - 1 ,$$

$\deg N = 0$ and $N$ is a constant. If $l = 1$, then $L$ is of the form $x + \beta$, $L' = 1$, and (34) gives us

$$N \mid (-\omega_0 x^q + w_0 x - \omega_0 L) = -\omega_0(x^q + \beta) = -\omega_0 L^q .$$

Thus in case $l = 1$, we see that $N$ is a constant times a power of $L$. Since $L \nmid N$, this implies that $N$ is a constant. Thus $N$ is a constant in all three cases.

If $L'$ is also a constant then Lemma 7 applies, and $F$ is of the form (35) with either (i) $l = 1$, $d = 0$, and $v \mid (p^2 - 1)$, or (ii) $l = p$, $k = d = 1$, and $v \mid (p - 1)$.

Now suppose that $L'$ is not a constant. Since $L'$ is a $p^{mk}$th power by Theorem 1, we must have $l = p + 1$ and $m = k = 1$. Since $N$ is a constant we have $F = \alpha L^v + \gamma$, where $\alpha \in \mathscr{K}$ and $\gamma = \gamma_0 \in \mathscr{K}$. Moreover $L$ is of the form $L = xS^p + T^p$ by (14). Since $L$ has leading coefficient 1, $S$ is of the form $S = x + \varphi$. Moreover $T$ is of the form $T = \mu x + \nu$. Now (34) becomes

$$\omega_0 L + \chi L^p = -\omega_0(x^q - x)S^p ,$$

where $\chi \in \mathscr{K}$. Comparing leading coefficients we see that $\chi = -\omega_0$. Therefore

$$L^p = (x^q - x)S^p + L = x^{p^2}S^p + T^p .$$

Extracting $p$th roots we obtain $L = x^p S + T$. Thus

$$xS^p + T^p = x^p S + T ,$$

or

$$x^{p+1} + \mu^p x^p + \varphi^p x + \nu^p = x^{p+1} + \varphi x^p + \mu x + \nu . \tag{39}$$

Comparing the coefficients of $x$ in (39) we obtain $\mu = \varphi^p$. Therefore

$$L = x^p S + T = x^{p+1} + \varphi x^p + \varphi^p x + \nu = (x + \varphi)^{p+1} + \beta ,$$

where $\beta = \nu - \varphi^{p+1}$. Comparing the constant terms of (39) we get $\nu^p = \nu$. Therefore $\nu \in GF(p)$, the prime field of $\mathscr{K}$. Now $\varphi^{p+1} \in GF(p)$. Hence $\beta \in GF(p)$. Since $L$ has distinct roots we have $\beta \neq 0$. Now if $v = 1$, then $F = \alpha L + \gamma$, and $F - \gamma - \alpha\beta$ has exactly one distinct root in $\mathscr{K}$, contradicting (11). Thus $v \geqq 2$. We have shown that if $q = p^2$, $r = [(q - 1)/f] \geqq 2$ and $L'$ is not constant, then $F$ is of the form $\alpha L^v + \gamma$, where $L$ is of the form

$$L = (x + \varphi)^{p+1} + \beta ,$$

where $\beta \in GF(p)$, $\beta \neq 0$, $v \mid (p - 1)$, $v \geqq 2$.

Conversely if $q = p^2$ and $F$ has this form, then $L(\tau) \in GF(p)$ for all $\tau \in \mathscr{K}$, and thus $F$ assumes at most

$$1 + (p - 1)/v = 1 + (q - 1)/f = 1 + [(q - 1)/f]$$

distinct values. Since we always have $r \geqq [(q-1)/f]$, this implies that $r = [(q-1)/f]$.

We have completed the discussion of the case $q = p^2$. We sum up our results for this case in our final theorem:

THEOREM 3. *Let $\mathscr{K}$ be a field of characteristic $p$ that contains exactly $p^2$ elements. Let $F(x)$ be a polynomial over $\mathscr{K}$ of degree $f$, $f > 0$. Let $F(\tau)$ assume exactly $r+1$ distinct values as $\tau$ ranges over $\mathscr{K}$. If $r = [(p^2-1)/f] \geqq 2$, then $F(x)$ has one of the following three forms:*

(i) *$F(x) = \alpha(x+\beta)^v + \gamma$, where $v \mid (p^2-1)$, $\alpha \neq 0$,*

(ii) *$F(x) = \alpha(x^p + \varphi x + \beta)^v + \gamma$, where $x^p + \varphi x + \beta$ has $p$ distinct roots in $\mathscr{K}$, $v \mid (p-1)$, $\alpha \neq 0$,*

(iii) *$F(x) = \alpha\{(x+\varphi)^{p+1} + \beta\}^v + \gamma$, where $\beta \in GF(p)$, $\beta \neq 0$, $v \geqq 2$, $v \mid (p-1)$, and $\alpha \neq 0$.*

*Conversely if $F(x)$ has one of these three forms, then $r = [(q- )/f]$.*

For $q > p^2$, the question of the characterization of all polynomials $F$ for which (1) holds, remains open. The most general types of polynomials known for which (1) holds are described in [1, § 5]. At present it seems unlikely that there are any more.

## REFERENCE

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