ON THE CONVERGENCE OF SEMI-DISCRETE ANALYTIC FUNCTIONS

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1. Introduction. In a previous paper [3], the author has presented the basic concepts and definitions for semi-discrete analytic functions. These functions are defined on two types of semi-lattices (sets of lines in the xy-plane, parallel to the x-axis)—one of which leads to a symmetric theory. We will concern ourselves here only with the symmetric case. These functions satisfy the following defining equation [3] on a region of the semi-lattice

$$(1.1) \qquad \frac{\partial f(z)}{\partial x} = [f(z + ih/2) - f(z - ih/2)]/ih ,$$

where $h > 0$ is the spacing of the semi-lattice. For convenience, we will repeat the definition of the symmetric semi-lattice and its associated odd and even semi-lattices. A grid-line, a_m, is the set of points in the xy-plane such that $y = mh$ where $h > 0$. The union $G(2k, h)$ of the a_m for $m = k$ $(k = 0, \pm 1, \pm 2, \cdots)$ is called the *even* semi-lattice; the union $G(2k + 1, h)$ of the a_m for $m = (2k + 1)/2$ is called the *odd* semi-lattice. The semi-discrete z-plane is the union of $G(2k, h)$ and $G(2k + 1, h)$. It will be denoted by $L(h)$. Additional concepts such as domains, paths, path-integrals, etc., are defined in [3]. The following notational conventions will be employed:

$$(1.2) \qquad f_k = f(x + ihk) = f_k(x) ,$$

and the abbreviation SD will be used to stand for semi-discrete.

2. Sub and super harmonic semi-discrete functions. In the continuous case, it is well-known that if a function $u(x, y)$ is defined over a region R of the plane and if, further, $\Delta^2(u) \geqq 0$ for all $(x, y) \in R$, where Δ^2 denotes the two dimensional Laplacian; then $u(x, y)$ cannot have a maximum on the interior of R. Such a function is said to be *sub-harmonic* in R [2]. Similarly, if the function $u(x, y)$ defined on R satisfies the equation $\Delta^2(u) \leqq 0$ for all $(x, y) \in R$; then $u(x, y)$ cannot have a minimum on the interior of R. Such a function is said to be *super-harmonic* in R [2]. An analogous result holds for semi-discrete functions which are defined on domains of either the even or odd semi-lattice. To be specific, we will consider functions $u(x, y)$ defined on

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domains of $G(2k, h)$ and introduce the notation

$$(2.1)\qquad \begin{aligned} &\text{(a)}\quad hEu(x, y) = u(x, y+h) - u(x, y)\ , \\ &\text{(b)}\quad h\bar{E}u(x, y) = u(x, y) - u(x, y-h)\ . \end{aligned}$$

The semi-discrete Laplacian operators for $G(2k)$ is then

$$(2.2)\qquad \nabla u(x, y) = \frac{\partial^2 u(x, y)}{\partial x^2} + E\bar{E}u(x, y)\ .$$

THEOREM 2.1. *Let $u(x, y)$ be a* SD-*function defined on a semi-discrete domain R of $G(2k, h)$. If $\nabla u \geqq 0$ for all $(x, y) \in R$, then on R*

$$(2.3)\qquad u(x, y) \leqq M\ ,$$

where M is the supremum of $u(x, y)$ on C, the boundary of R.

Proof. The proof of this statement is obtained by a suitable modification of the proof for the "weak maximum theorem" established by Helmbold [1] for semi-discrete harmonic functions. Let C denote the boundary of the SD-domain R of $G(2k, h)$, let $u(x, y)$ be a SD-function on R such that $\nabla u \geqq 0$ for all $(x, y) \in R$, and let M' denote the supremum of $u(x, y)$ on R. Assume that u takes the value M' at a point (t, nh) of the interior $R^0 = R \sim C$ of R. If the adjacent points $(t, (n \pm 1)h)$ are points of R^0, $\partial^2 u/\partial x^2 = u''$ will be continuous at (t, nh) and further $u''_n(t) \leqq 0$. By assumption $\nabla u_n(t) \geqq 0$ which, together with the previous remarks, implies that

$$\text{(a)}\qquad u_n(t) = u_{n+1}(t) = u_{n-1}(t) = M'\ .$$

This argument may be repeated for the sequence of points $(t, (n \pm 1)h)$, $(t, (n \pm 2)h)$, $\cdots$ until a point (t, ph) is reached such that one of its adjacent points is a point of C. If u''_p is continuous, the proof is complete. Otherwise, since u''_p is then at least piecewise continuous, integration of $\nabla u_p \geqq 0$ shows that for some range of values of $\varepsilon > 0$

$$\text{(b)}\qquad u'_p(t+\varepsilon) - u'_p(t) \geqq \varepsilon h^{-2}\{2u_p(\theta) - u_{p+1}(\theta) - u_{p-1}(\theta)\}\ ,$$

where $t \leqq \theta \leqq t + \varepsilon$. Since $u_p = M'$ is a maximum, the left side of (b) is negative. Hence, the bracketed term is negative. Taking the limit of this term as $\varepsilon \to 0$, $\varepsilon > 0$ shows that

$$\text{(c)}\qquad 2M' \leqq u_{p+1}(t^+) + u_{p-1}(t^+)\ .$$

Similarly, we obtain

$$\text{(d)}\qquad 2M' \leqq u_{p+1}(t^-) + u_{p-1}(t^-)\ .$$

Addition of (c) and (d) shows that $M' \leqq M$ where M is the maximum value of $u(x, y)$ on C.

In an identical manner, we establish the following result for super SD-harmonic functions.

THEOREM 2.2. *Let $u(x, y)$ be a SD-function defined on a semi-discrete domain R of $G(2k, h)$. If $\nabla u \leqq 0$ for all $(x, y) \in R$, then on R*

$$u(x, y) \geqq m \ , \tag{2.4}$$

where m is the infimum of $u(x, y)$ on C, the boundary of R.

3. Limit theorem for semi-discrete analytic functions. A SD-function $f(z)$ of the complex variable $z = x + inh$ which is continuous and single-valued on a SD-domain R of $L(h)$ is said to be SD-*analytic* if it satisfies (1.1) for all points $z \in R$ [3]. In addition, if we write $f = u + iv$, then $\nabla u = \nabla v = 0$ on R; that is, u and v are SD-harmonic.

Let us suppose that $L(h)$ is superimposed upon the continuous z-plane, denoted by L_c, with their x and y axes coinciding. Let R_c be a simply-connected finite domain of L_c whose boundary is a Jordan curve. A covering set of rectangles, Q_k, is defined as follows,

$$Q_k = \{(x, y) : \alpha_k \leqq x \leqq \beta_k; (kh - h) \leqq 2y \leqq (kh + h)\} \ ,$$

where α_k is the least value of x in R taken on the strip $kh - h \leqq 2y \leqq kh + h$, and β_k is the greatest value of x in R on this strip. By construction, each point of R_c is also a point of $Q = \bigcup_k Q_k$. The intersection of Q with $L(h)$ forms a SD-domain, $R(h)$, which approximates R_c. We consider the sequence of SD-domains $\{R(h_j); h_1 > h_2 > \cdots\}$ obtained by the above procedure upon successive refinements of the semi-lattice retaining at each step the lines of the previous semi-lattice. In the limit, $R(h_j) \to R_c$. It is shown in [3] that a SD-analytic function is completely determined in $R(h)$ by its values on $C(h)$, the total-boundary of $R(h)$. Therefore, let us assume that an interpolation scheme is established to provide such boundary values for a SD-analytic function $f^{(h)}(z)$ on $R(h)$ from the boundary values of an analytic function $\zeta(z)$ on R_c such that these approximate boundary values tend uniformly to the true boundary values. We consider the sequence of SD-analytic functions $\{f^{(h_j)}(z)\}$ so determined on $\{R(h_j)\}$ and will prove that as $h_j \to 0$, $f^{(h_j)}(z) \to \zeta(z)$.

THEOREM 3.1. *Let R be a domain whose boundary C is a Jordan curve and let R' be a subdomain of R which is bounded by a Jordan curve $C' \subset R$. Consider the set of all possible semi-lattices $G(2k, h)$ parallel to the real axis of the z-plane. Consider also the set of all SD-functions $u^{(h)}(x, y)$ which are uniformly bounded, $|u| \leqq M$ in R,* *and which satisfy in R the equation $\nabla u = 0$. Then, for h sufficiently small, there exists a constant M′ such that*

$$\left|\frac{\partial u^{(h)}}{\partial x}\right| \leqq M' \quad \text{and} \quad |\nabla u^{(h)}| \leqq M'$$

for all $(x, y) \in R$.

Proof. The proof of this statement follows the proof given by Fellow [4] for the discrete case. The sub-domain R' can be covered by a finite number of rectangles contained in R and each of these rectangles can be inclosed in a larger rectangle also contained in R. Following the argument of Feller [4], it will be sufficient to consider, for an arbitrary $\delta > 0$, the two concentric rectangles

$$R = \{(x, y) : |x| < a - \delta, |y| < b\}$$
$$R' = \{(x, y) : |x| < a - \delta, |y| < b - \delta/3\}\,,$$

where b is a multiple of the gap h, and $h < \delta/3$.

To prove the assertion, we shall show that the function

$$\psi(x, y) = \left(\frac{\partial u}{\partial x}\right)^2 \Phi(x, y) + C\{u^2(x, y) + u^2(x, y + h) + u^2(x, y - h)\}$$

where $\Phi(x, y) = (x^2 - a^2)^2(y^2 - b^2)^2$ and C is a large positive constant, to be determined later, satisfies the inequality $\nabla(\psi) \geqq 0$.

Assume for the moment that this has been established. Then, by Theorem 2.1, it follows that ψ attains its maximum value on the boundary. However, by definition, $\Phi = 0$ on the boundary and thus in the entire rectangle

$$0 \leqq \psi(P) \leqq 3CM^2$$

where M is the uniform bound on u. Since the second term of ψ is nonnegative, we may conclude that for all $P \in R'$

$$\left(\frac{\partial u}{\partial x}\right)^2 \leqq 3CM^2/\Phi \leqq 3CM^2/(\delta/3)^8$$

[since for small δ, $\Phi \geqq (\delta/2)^4(\delta/3)^4 \geqq (\delta/3)^8$].

Since $(\delta/3)^8 > 0$, taking the last expression for M' establishes the theorem, subject to showing that $\nabla(\psi) \geqq 0$. Only the outline of this calculation will be presented. The complete sequence of steps follows the argument given by Feller [4] using the differential rather than the difference operator on x.

Calculation of $\nabla\psi$ using the fact that u is SD-harmonic [as is u'] gives

$$\begin{aligned}\nabla(\psi) = {} & (u')^2\nabla(\Phi) + \Phi[2(u'')^2 + (Eu')^2 + (\bar{E}u')^2] \\ & + \Phi'(4u'u'') + E\Phi[u_1'Eu' + u'Eu'] \\ & + \bar{E}\Phi[u_{-1}'\bar{E}u' + u'\bar{E}u'] + C[2(u')^2 + (Eu)^2 + (\bar{E}u)^2] \\ & + C[2(u_1')^2 + (Eu_1)^2 + (\bar{E}u_1)^2 + 2(u_{-1}')^2 + (Eu_{-1})^2 + (\bar{E}u_{-1})^2]\end{aligned} \tag{a}$$

where $u_{\pm 1} = u(x, y \pm h)$. Since $|\partial\Phi/\partial x| = 4|x(y^2 - b^2)|\sqrt{\Phi}$, a constant λ exists such that for all points of $R\,|\Phi'| < \lambda\sqrt{\Phi}$. Similar bounds exist for $E\Phi$ and $\bar{E}\Phi$. Further, in R, $\nabla(\Phi)$ is bounded. Accordingly we assume that λ is so chosen that on R

$$|\nabla(\Phi)| < \lambda\,, \quad |\Phi'| < \lambda\sqrt{\Phi}\,, \quad |E\Phi| < \lambda\sqrt{\Phi}\,, \quad |\bar{E}\Phi| < \lambda\sqrt{\Phi}\,.$$

For an arbitrary $\varepsilon > 0$, we see that

$$|u'u''\Phi'| \leqq \left(\frac{u'}{\varepsilon}\right)^2 + \varepsilon^2\lambda^2\Phi(u'')^2\,.$$

With such bounds established for the various terms which appear in (a), the following inequality is obtained.

$$\begin{aligned}\nabla(\psi) \geqq {} & [(Eu')^2 + (\bar{E}u')^2 + 2(u'')^2]\Phi(1 - 2\varepsilon^2\lambda^2) \\ & + 2(u')^2[C - 3/\varepsilon^2] + C[(Eu)^2 + (\bar{E}u)^2 + (Eu_1)^2] \\ & + C[(\bar{E}u_1)^2 + (Eu_{-1})^2 + (\bar{E}u_{-1})^2] + (u_1')^2[2C - 1/\varepsilon^2] \\ & + (u_{-1}')^2[2C - 1/\varepsilon^2]\,.\end{aligned} \tag{b}$$

Selecting ε so that $\varepsilon^2\lambda^2 = 1/2$, the first term on the right in (b) vanishes. Finally, choosing $C \geqq 3/\varepsilon^2$, the remaining terms on the right in (b) will be positive. That is, $\nabla(\psi) \geqq 0$.

THEOREM 3.2. *Let $\{u^{(h)}(x, y)\}$ be the set of uniformly bounded* SD-*functions considered in Theorem* 3.1. *This set is a family of equi-continuous functions on R.*

Proof. In Theorem 3.1 we established the existence of a uniform bound for the set $\{\partial u^{(h)}/\partial x\}$ and also $\{Eu^{(h)}\}$. Let M denote this bound. (1) Given $\varepsilon > 0$, let P, Q be two points on a line of the semi-lattice such that $\overline{PQ} < \varepsilon/M$; that is, $|x_P - x_Q| < \varepsilon/M$, where x_P denotes the x-coordinate of P and x_Q denotes the x-coordinate of Q. Then

$$|u^{(h)}(P) - u^{(h)}(Q)| = \left|\int_{x_Q}^{x_P} \frac{\partial u^{(h)}}{\partial t}\,dt\right| \leqq [M^2(x_P - x_Q)^2]^{1/2} \leqq \varepsilon\,.$$

(2) Given $\varepsilon > 0$, let P, Q be two points of R which lie on a vertical line in R such that $|y_P - y_Q| < \varepsilon/Mh$.

$$|u^{(h)}(P) - u^{(h)}(Q)| = h\left|\sum_{y=y_Q}^{y=y_P} Eu^{(h)}\right|\,.$$

Thus,

$$|u^{(h)}(P) - u^{(h)}(Q)| \leqq |y_P - y_Q| Mh \leqq \varepsilon .$$

(3) Given $\varepsilon > 0$, let P, Q be two arbitrary points of R such that $\overline{PQ} < \delta(\varepsilon)$. Let P' lie on the same vertical line as P and have the same y-coordinate as Q; i.e., $P' = (x_P, y_Q)$. Then

$$|u^{(h)}(P) - u^{(h)}(Q)| \leqq |u^{(h)}(P) - u^{(h)}(P')| + |u^{(h)}(P') - u^{(h)}(Q)| .$$

Application of the two previous cases completes the proof.

By Theorem 3.2, if $\{f^{(h)} = u^{(h)} + iv^{(h)}\}$ is a set of uniformly bounded SDA functions, this set is a family of equicontinuous functions which, by Kellogg [2], contains a subsequence converging uniformly in R' to a continuous limit. Since R' was an arbitrary closed sub-domain of R, we can choose a sequence of such regions $R' \subset R'' \subset \cdots \subset R$ whose sum is R and find successive subsequences of $f^{(h_1)}, f^{(h_2)}, \cdots$ which converge in each of these regions to a continuous function. The resultant diagonal subsequence will converge uniformly to a continuous function in all of R. Since successive differences and derivatives of SD-harmonic functions are again SD-harmonic, the arguments in Theorems 3.1 and 3.2 can be repeated to show that there is a subsequence of the final subsequence whose first derivative and first difference ratio also converge in R. Thus, we can find a final subsequence which will have an arbitrary number of successive derivatives or differences which converge in R. Denote this final convergent subsequence by $\{f_*^{(h)}\}$ and let $\zeta(z)$ be the continuous function in R to which it converges.

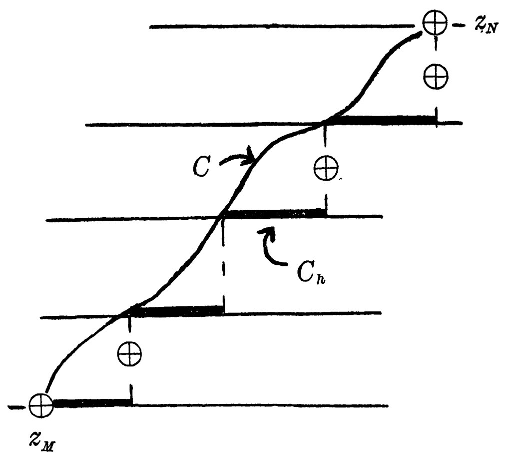
Let C be a rectifiable curve in L_c. By the construction of Q, each point of C is a point of Q. Consider a rectangle Q_k of Q which contains a segment C_k of C. To be explicit, we will assume that C_k intersects $Q_k \cap L(h)$ at the three points $p_1 = (x_1, h(k-1)/2)$, $p_2 = (x_2, hk/2)$, and $p_3 = (x_3, h(k+1)/2)$, and that the positive direction is from p_1 to p_3. The remaining possibilities can be treated by suitable modifications of the following discussion. On $Q_k \cap L(h)$, three SD-paths may be defined. The *upper* SD-path consists of the points p_1, $(x_1, hk/2)$, and the line segment from x_1 to x_3 with $y = h(k+1)/2$. The *lower* SD-path is the line segment from x_1 to x_3 with $y = h(k-1)/2$, the points $(x_3, hk/2)$, and p_3. The *mixed* SD-path consists of the line segment from x_1 to x_2 with $y = h(k-1)/2$, the point p_2, and the line segment from x_2 to x_3 with $y = h(k+1)/2$. At least one of these SD-paths must lie within $R(h)$ and will be chosen to be the SD approximation of the segment C_k. The SD-Cauchy theorem [3] shows that it is immaterial which SD-path is chosen if more than one of these approximating SD-paths lies within $R(h)$. The SD-path on $R(h)$ which approximates C is the union of the SD-paths chosen to approximate its segments, C_k.

THEOREM 3.3. *Let $\zeta(z)$ be a continuous function on a domain R and let C be a rectifiable [or Jordan] curve which is contained in R. If C_h is a* SD-*path contained in R_h which approximates C, then*

$$\lim_{h\to 0}\int_{C_h}\zeta(z)\delta z=\int_C\zeta(z)dz\,. \tag{3.1}$$

Proof. By the definition for SD-path integration [3],

$$\int_{C_h}\zeta\delta z=\sum_{p=M}^{N-1}\int_{x_p}^{x_{p+1}}\zeta_p(t)dt+ih\sum_{p=M}^{N-2}\zeta_{p+(1/2)}(x_{p+1})\,,$$



where C_h is a SD-path joining $z_M=x_M+iM$ and $z_N=x_N+iN$. We note that as $h\to 0$, so must $|x_p-x_{p+1}|\to 0$. Since ζ is continuous, there exists a value λ_p where $x_p\leqq\lambda_p\leqq x_{p+1}$ such that

$$\int_{C_h}\zeta\delta z=\sum_{p=M}^{N-1}[x_{p+1}-x_p]\zeta(\lambda_p)+ih\sum_{p=M}^{N-2}\zeta_{p+(1/2)}(x_{p+1})\,.$$

As $h\to 0$ the right side of the above converges to the value of the path-integral of the continuous function ζ along the path C.

THEOREM 3.4. *Let $R(h_k)$ denote a sequence of semi-lattices on a domain R such that $h_k\to 0$, and let $f^{(h_k)}$ be semi-discrete analytic on $R(h_k)$. If the collection of these $f^{(h_k)}$ is uniformly bounded in R, then it contains a subsequence that converges everywhere in R to a function $\zeta(z)$ that is analytic in R.*

Proof. This subsequence is the final subsequence obtained in the previous discussion. Let C denote an arbitrary closed rectifiable path in R and let C_h be a closed SD-path on $R(h_k)$ which approximates C. Then

(a) $$\lim_{h\to 0} \oint_{C_h} f_*^{(h_k)} \delta z = \oint_C \zeta(z) dz ,$$

where $\{f_*^{(h_k)}\}$ is the subsequence which converges to ζ. To establish (a) we consider

(b) $$\left| \oint_{C_h} f_*^{(h_k)} \delta z - \oint_C \zeta(z) dz \right| \leqq \left| \oint_{C_h} (f_*^{(h_k)} - \zeta) \delta z \right| + \left| \oint_{C_h} \zeta \delta z - \oint_C \zeta dz \right| .$$

Since $f_*^{(h_k)} \to \zeta$, given $\varepsilon > 0$ there exists $\delta_1(\varepsilon) > 0$ such that the first term on the right side of (b) is smaller than $\varepsilon/2$ provided $h_k < \delta_1$. Similarly by Theorem 3.3, there exists $\delta_2(\varepsilon) > 0$ such that the second term on the right side of (b) is smaller than $\varepsilon/2$ provided $h_k < \delta_2$. Thus, on letting $\delta = \max(\delta_1, \delta_2)$

(c) $$\left| \oint_{C_h} f_*^{(h_k)} \delta z - \oint_C \zeta dz \right| < \varepsilon ,$$

provided $h_k < \delta$. This establishes (a). However, since $f_*^{(h_k)}$ is SDA for each h_k, the left side of (a) is always zero. Thus

(d) $$\oint_C \zeta(z) dz = 0 .$$

Since C is an arbitrary closed rectifiable curve of R and ζ is continuous, by Morera's theorem $\zeta(z)$ is analytic in R.

To complete the discussion we must show that the limit function $\zeta(z) = U(z) + iV(z)$ of the chosen subsequence $\{f_*^{(h_k)}\}$ satisfies the given boundary condition $\zeta = \psi(s)$ on C, the boundary of R. It is sufficient for this purpose to consider the real-valued function $U = Re\{\zeta\}$ and show that $U = Re\{\psi(s)\}$ on C. Let Q be a fixed point of C. By hypothesis we can construct a circle lying outside C and intersecting C only at the point Q, see Feller [4]. We denote the center of this circle by A, its radius by ρ, and let P denote an arbitrary point of R whose distance from A is r.

For an arbitrary $\varepsilon > 0$, we define the functions [4]

(3.2) $$U_1(P) = F(Q) + \varepsilon + K\left(\frac{1}{\rho} - \frac{1}{r}\right) ,$$

and

$$U_2(P) = F(Q) - \varepsilon - K\left(\frac{1}{\rho} - \frac{1}{r}\right) ,$$

where $F = Re\{\psi\}$ and K is a positive constant to be determined later. On any semi-lattice

(3.3) $$\nabla U_1(P) = -K[r^{-3} + 0(h)] < 0 ,$$

and

$$\nabla U_2(P) > 0$$

in R provided that h is sufficiently small. Now if $u(P)$ is a solution of the differential-difference equation $\nabla u = 0$ for the semi-lattice, by (3.3) the function $U_1(P) - u(P)$ is SD super-harmonic for $P \in R$. Accordingly, by Theorem 2.2, it assumes its minimum on C. Similarly, the function $U_2(P) - u(P)$ is SD sub-harmonic and by Theorem 2.1 assumes its maximum on C.

The argument given by Feller [4] now applies directly. We consequently establish that

$$\overline{\lim_{P \to Q}}\, U(P) \leqq F(Q) ,$$

and

$$\underline{\lim_{p \to Q}}\, U(P) \geqq F(Q)$$

which completes the proof.

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