SOME APPLICATIONS OF MEANS OF CONVEX BODIES

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Let A be a real, positive definite, $n \times n$ matrix; with A we associate, in the Euclidean n-space R_n, the ellipsoid $E(A)$ of points x for which

$$(x, Ax) \leqq 1$$

where (x, y) denotes the usual inner product. In references [5], [6], [7] certain means of convex bodies were studied. It will be shown here that two particular means of ellipsoids of the type $E(A)$ correspond to two simple combinations of the corresponding matrices A. The applications mentioned in the title rest upon this correspondence. The first two give results about positive definite matrices, including a refinement of a determinant inequality of Minkowski; the third application shows the existence of a set of unique ellipsoids related to a convex body by a set of similar extremal problems, the classical Loewner ellipsoid being a particular instance.

Throughout this paper the letters A and B, sometimes with distinguishing marks, denote real, positive definite, $n \times n$ matrices. The distance from x to the origin is written $\|x\|$.

1. The distance and support functions of $E(A)$ are:

$$F(x) = \sqrt{(x, Ax)}\ , \qquad H(x) = \sqrt{(x, A^{-1}x)}\ .$$

In the first case, if $x \neq 0$, we have $F(x) = \|x\|/\|z\|$ where $x/\|x\| = z/\|z\|$ and $(z, Az) = 1$, and so

$$\begin{aligned}\|x\|/\|z\| &= \|x\|\sqrt{(z/\|z\|, Az/\|z\|)}\\ &= \|x\|\sqrt{(x/\|x\|, Ax/\|x\|)} = \sqrt{(x, Ax)}\ .\end{aligned}$$

In the second case

$$H(x) = \max_y\,(x, y) \qquad \text{where} \qquad (y, Ay) = 1\ .$$

We represent y in the form $\lambda A^{-1}x + v$ where $(x, v) = 0$. Then

$$(y, Ay) = \lambda^2(x, A^{-1}x) + (v, Av)\ ,$$

whence

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$$(x, y) = \lambda(x, A^{-1}x) = \sqrt{(x, A^{-1}x)}\sqrt{[1 - (v, Av)]} ,$$

and the maximum is attained for $v = 0$.

The polar reciprocal $\hat{E}(A)$ of $E(A)$ with respect to the unit sphere $E(I)$ has $H(x)$ as its distance function, $F(x)$ as its support function. Consequently

$$\hat{E}(A) = E(A^{-1}) .$$

In [5] the p-dot mean of two convex bodies K_0, K_1 in R_n, which have the origin as a common interior point, was defined for $p \geqq 1$ to be to convex body $\dot{M}_p(K_0, K_1; \vartheta)$ whose distance function is

$$[(1 - \vartheta)F_0^p(x) + \vartheta F_1^p(x)]^{1/p}$$

where F_i is the distance function of K_i and $0 \leqq \vartheta \leqq 1$. From this it follows that $\dot{M}_2(E(A_0), E(A_1); \vartheta)$ has the distance function

$$\sqrt{[(1 - \vartheta)(x, A_0x) + \vartheta(x, A_1x)]} = \sqrt{(x, [(1 - \vartheta)A_0 + \vartheta A_1]x)} .$$

Thus

$$(2) \qquad \dot{M}_2(E(A_0), E(A_1); \vartheta) = E((1 - \vartheta)A_0 + \vartheta A_1) .$$

In [7] the p-mean $M_p(K_0, K_1; \vartheta)$ was defined for $p \geqq 1$ to be the convex body whose support function is

$$[(1 - \vartheta)H_0^p(x) + \vartheta H_1^p(x)]^{1/p}$$

where H_i is the support function of K_i. Therefore, by reasoning similar to the preceding, we have

$$(3) \qquad M_2(E(A_0), E(A_1); \vartheta) = E([(1 - \vartheta)A_0^{-1} + \vartheta A_1^{-1}]^{-1}) .$$

2. Our first application is based on the inclusion

$$(4) \qquad \dot{M}_2(K_0, K_1; \vartheta) \subseteqq M_2(K_0, K_1; \vartheta) ,$$

established in [5] and [7][1] with equality if and only if $K_0 = K_1$, and the observation that

$$E(A) \subseteqq E(B)$$

if and only if $A - B$ is positive semi-definite. For the latter we write $A \geqq B$; we call such an inequality strict if $A - B$ is not a zero matrix. From (2), (3) and (4) we have

$$(5) \qquad E((1 - \vartheta)A_0 + \vartheta A_1) \subseteqq E([(1 - \vartheta)A_0^{-1} + \vartheta A_1^{-1}]^{-1}) .$$

[1] The inclusion is not specifically mentioned, but in [7] it is proved that $M_1 \subseteqq M_p$ for $p > 1$ and in [5] that $\dot{M}_p \subseteqq \dot{M}_1$ and $\dot{M}_1 \subseteqq M_1$.

Hence, from (5) we obtain an "inequality of arithmetic and harmonic means" for positive definite matrices.

THEOREM 1. *If* A_0, A_1 *are any two real, positive definite,* $n \times n$ *matrices, then*

$$(1-\vartheta)A_0 + \vartheta A_1 \geqq [(1-\vartheta)A_0^{-1} + \vartheta A_1^{-1}]^{-1}$$

for $0 \leqq \vartheta \leqq 1$. *The inequality is strict except in the trivial cases* $A_0 = A_1$ *or* $\vartheta = 0, 1$.

3. The next application is a refinement of the following determinant inequality of Minkowski, cf [1], p. 70.

$$\det^{1/n}(A_0 + A_1) \geqq \det^{1/n} A_0 + \det^{1/n} A_1 .$$

Let V be the volume functional. In [5] it was shown that

$$(6) \qquad V(\dot{M}_p(K_0, K_1; \vartheta)) \leqq [(1-\vartheta)V^{-p/n}(K_0) + \vartheta V^{-p/n}(K_1)]^{-n/p}$$

with equality if and only if $K_0 = \lambda K_1$ for some $\lambda > 0$. Since

$$V(E(A)) = \pi^{n/2}/\Gamma(1 + n/2)\sqrt{\det A} ,$$

we have, with $p = 2$ in (6),

$$(7) \qquad \det[(1-\vartheta)A_0 + \vartheta A_1] \geqq [(1-\vartheta)\det^{1/n} A_0 + \vartheta \det^{1/n} A_1]^n$$

with equality if and only if $A_0 = \lambda A_1$ for some $\lambda > 0$. With a slight change in notation, this is Minkowski's determinant inequality.

If L is any k-dimensional linear subspace of R_n, then

$$\dot{M}_2(E(A_0) \cap L, E(A_1) \cap L; \vartheta) = \dot{M}_2(E(A_0), E(A_1); \vartheta) \cap L .$$

Consequently, by letting A' be the $k \times k$, positive definite matrix associated with $E(A) \cap L$, we obtain

$$E((1-\vartheta)A_0' + \vartheta A_1') = E([(1-\vartheta)A_0 + \vartheta A_1]') .$$

To this we apply (7), with $n = k$, to get

$$(8) \qquad \det[(1-\vartheta)A_0 + \vartheta A_1]' \geqq [(1-\vartheta)\det^{1/k} A_0' + \vartheta \det^{1/k} A_1']^k .$$

Let us define $|A|_k$ to be the product of the k least eigenvalues of A, repeated eigenvalues being counted according to their multiplicity. The inequality

$$\det A' \geqq |A|_k$$

with equality if and only if L is the k-dimensional space spanned by the eigenvectors corresponding to the k least eigenvalues of A, is essentially Theorem 20, p. 74 of [1].

In (8) choose L to be the linear subspace spanned by those eigenvectors of $(1-\vartheta)A_0+\vartheta A$, which correspond to the k smallest eigenvalues of $(1-\vartheta)A_0+\vartheta A_1$. By (9):

$$\det A_0' \geqq |A_0|_k\,, \qquad \det A_1' \geqq |A_1|_k\,, \tag{10}$$

and so (8) becomes

$$|(1-\vartheta)A_0+\vartheta A_1|_k \geqq [(1-\vartheta)\,|A_0|_k^{1/k}+\vartheta\,|A_1|_k^{1/k}]^k\,. \tag{11}$$

There is equality in (8) if and only if, for some $\lambda>0$,

$$A_0'=\lambda A_1'$$

and equality in (10) if and only if the subspaces L appropriate to $|A_0|_k$, $|A_1|_k$ are the same. Hence, in (11), there is equality if and only if the following conditions are met. Let $x_1, \cdots, x_k$ be eigenvectors of A_0 corresponding to the k smallest eigenvalues $\lambda_1 \leqq \cdots \leqq \lambda_k$. These are eigenvectors of A_1 corresponding to the k smallest eigenvalues of A_1 which are of the form $\lambda\lambda_1 \leqq \cdots \leqq \lambda\lambda_k$ for some $\lambda>0$.

Inequality (11), which includes (7) when $k=n$, is an improvement of a result of Ky Fan, cf. [1], Theorem 21, p. 74, in which the right side of (11) is replaced by the geometric mean $|A_0|_k^{1-\vartheta}\,|A_1|_k^{\vartheta}$ since the power mean of order $1/k$ appearing on the right side of (11) exceeds this geometric mean.

If we define ${}_k|A|$ to be the product of the k greatest eigenvalues of A, then

$$|A^{-1}|_k = 1/{}_k|A|\,. \tag{12}$$

We apply (11) to $(1-\vartheta)A_0^{-1}+\vartheta A_1^{-1}$ and obtain, after taking reciprocals,

$$1/|(1-\vartheta)A_0^{-1}+\vartheta A_1^{-1}|_k \leqq [(1-\vartheta)\,{}_k|A_0|^{-1/k}+\vartheta\,{}_k|A_1|^{-1/k}]^{-k}\,.$$

With the use of (12) on the left side, we have finally

$${}_k|[(1-\vartheta)A_0^{-1}+\vartheta A_1^{-1}]^{-1}| \leqq [(1-\vartheta)\,{}_k|A_0|^{-1/k}+\vartheta\,{}_k|A_1|^{-1/k}]^{-k}$$

as a "dual" result to (11). The cases of equality are given by the conditions for equality in (11) with the word "smallest" replaced by "greatest" throughout.

The last application concerns a generalization of the Loewner ellipsoid of a convex body K. Let x be an interior point of K. The classical Loewner ellipsoid is that *unique* ellipsoid, centred at x and containing K, which has minimum volume, cf. [3]. Let us take the point x to be the origin and denote the mean cross-sectional measures W_ν, $\nu=0,1,\cdots,n-1$, of $E(A)$ by $W_\nu(A)$; for their definition see

[2]. In particular $W_0(A) = V(E(A))$. We will show that, for each ν there is a unique ellipsoid $E(A)$ containing K for which $W_\nu(A)$ is a minimum.

It is clear that $W_\nu(A)$ depends continuously on the entries a_{ij} of A. Moreover, when we restrict the ellipsoids $E(A)$ not only to contain K, but also to be contained in the sphere $E(I/\rho^2)$, the domain of definition of the functions $W_\nu(A)$ is closed and bounded. Consequently each of the functions $W_\nu(A)$ attains a minimum. Furthermore, if the radius of the bounding sphere $E(I/\rho^2)$ is chosen to be sufficiently large, the minimum of $W_\nu(A)$ and the matrix or matrices for which it is attained will be independent of ρ. Thus the uniqueness is the only point in question.

In [6] inequality (6) was extended to read

$$(13) \quad W_\nu^{1/(n-\nu)}[\dot{M}_p(K_0, K_1; \vartheta)] \leqq [(1-\vartheta)W_\nu^{-p/(n-\nu)}(K_0) + \vartheta W_\nu^{-p/(n-\nu)}(K_1)]^{-1/p}$$

for $p = 1$, with equality if and only if $K_0 = \lambda K_1$ for some $\lambda > 0$. Inequality (13) is true for all $p \geqq 1$ however. This can be shown from the special case $p = 1$ in the following fashion. We make the usual type of reduction to the special case in which $W_\nu(K_i) = 1$, $i = 0, 1$, by setting:

$$\lambda_i = W_\nu^{1/(n-\nu)}(K_i) , \qquad K_i = \lambda_i K_i' ,$$
$$\vartheta' = \vartheta\lambda_1^{-p}/[(1-\vartheta)\lambda_0^{-p} + \vartheta\lambda_1^{-p}] .$$

Then

$$\dot{M}_p(K_0', K_1'; \vartheta') = \dot{M}_p(K_0, K_1; \vartheta)/\mu$$

where

$$\mu = [(1-\vartheta)\lambda_0^{-p} + \vartheta\lambda_1^{-p}]^{-1/p} .$$

Since $W_\nu(K_i') = 1$, in order to prove (13) it is enough to prove

$$W_\nu(\dot{M}_p(K_0', K_1'; \vartheta') \leqq 1 .$$

This has been shown to be true for $p = 1$. By Theorem 2 of [5]

$$\dot{M}_p(K_0', K_1'; \vartheta') \subseteqq \dot{M}_1(K_0', K_1'; \vartheta')$$

with equality if and only if $K_0' = K_1'$. These assertions, together with the monotonic character of W_ν cf. [2], p. 50, prove (13) and establish the cases of equality. Naturally we will use (13) for $p = 2$.

Let A_ν be a matrix which is a solution of the minimum problem:

$$K \subseteqq E(A) , \qquad W_\nu(A) = \text{minimum}.$$

Suppose A_ν' is a second solution. From

$$K \subseteqq E(A_\nu)\ , \qquad K \subseteqq E(A'_\nu)$$

we have

$$K \subseteqq E((1-\vartheta)A_\nu + \vartheta A'_\nu)\ ;$$

from (13) we have

$$W_\nu((1-\vartheta)A_\nu + \vartheta A'_\nu) \leqq W_\nu(A_\nu) = W_\nu(A'_\nu)$$

with equality in the inequality if and only if $A_\nu = \lambda A'_\nu$. The last equality shows that we must have $\lambda = 1$ and so A_ν is unique.

In a similar way we can establish that, given K and an interior point of K which we take as the origin, there is a unique ellipsoid $E(B_\nu)$ which is contained in K for which is a maximum. The only difference is the use of Theorem 2 of [7] in lieu of inequality (13).

We summarize:

Theorem 2. *Given a convex body K in Euclidean n-space and an interior point of K which we take as the origin, there are positive definite $n \times n$ matrices $A_\nu, B_\nu, \nu = 0, 1, \cdots, n-1$ such that, among the ellipsoids $E(A)$ which contain K, $E(A_\nu)$ is the unique, outer, Loewner ellipsoid minimizing W_ν and among the ellipsoids $E(B)$ which are contained in K, $E(B_\nu)$ is the unique inner, Loewner ellipsoid maximizing W_ν.*

We close with several observations. Suppose $\hat{K}$ is the polar reciprocal of K with respect to $E(I)$, then, in the notation of Theorem 2, $E(B_\nu^{-1})$ is the νth outer Loewner ellipsoid of $\hat{K}$ while $E(A_\nu^{-1})$ is the νth inner Loewner ellipsoid. To prove this, we denote the outer and inner Loewner ellipsoids of $\hat{K}$ with respect to the origin by $E(\hat{A}_\nu)$, $E(\hat{B}_\nu)$ respectively. If $K_0 \subseteqq K_1$, then $\hat{K}_0 \supseteqq \hat{K}_1$. Consequently, by (1),

$$\hat{E}(A_\nu) = E(A_\nu^{-1}) \subseteqq \hat{K}\ , \qquad \hat{E}(B_\nu) = E(B_\nu^{-1}) \supseteqq \hat{K}\ .$$

Therefore

$$E(A_\nu^{-1}) \subseteqq E(\hat{B}_\nu)\ , \qquad E(B_\nu^{-1}) \supseteqq E(\hat{A}_\nu)\ .$$

Applying the same argument to $\hat{A}_\nu$ and $\hat{B}_\nu$ we get

$$E(\hat{A}_\nu^{-1}) \subseteqq E(B_\nu)\ , \qquad E(\hat{B}_\nu^{-1}) \supseteqq E(A_\nu)\ .$$

In terms of the ordering of positive definite matrices, these inclusions become

$$A_\nu^{-1} \geqq \hat{B}_\nu\ , \quad \hat{A}_\nu \geqq B_\nu^{-1}\ , \quad \hat{A}_\nu^{-1} \geqq B_\nu\ , \quad A_\nu \geqq \hat{B}_\nu^{-1}\ . \tag{14}$$

Now when $B \geqq A$, then $A^{-1} \geqq B^{-1}$ since, from the first condition we have

$$E(A) \supseteqq E(B)$$

and, by taking polar reciprocals, we obtain

$$E(A^{-1}) \subseteqq E(B^{-1}) .$$

Apply this to the last inequality of (14). Taken together with the first inequality of (14), this yields

$$A_\nu^{-1} \geqq \hat{B}_\nu \geqq A_\nu^{-1} .$$

Thus $\hat{B}_\nu - A_\nu^{-1}$ is both positive and negative semi-definite. Hence

$$A_\nu^{-1} = \hat{B}_\nu .$$

By a similar argument it is shown that

$$B_\nu^{-1} = \hat{A}_\nu .$$

Part of Theorem 2 remains true even if the centre of the ellipsoids to be considered does not lie within K. We give this as a corollary.

COROLLARY TO THEOREM 2. *Given a convex body K, not necessarily containing the origin, there are positive definite matrices A_ν, $\nu = 0, 1, \cdots, n-1$, such that, among the ellipsoids $E(A)$ which contain K, $E(A_\nu)$ is the unique outer Loewner ellipsoid minimizing W_ν.*

Suppose $E(A)$ contains K; since $E(A)$ is centred at the origin it also contains a sufficiently small sphere $E(\rho I)$ and so, by the convexity of $E(A)$, $E(A)$ contains

$$K' = \overline{K \cup E(\rho I)}$$

where the bar denotes the convex closure. Conversely, if $E(A)$ contains K' it contains the subset K. We claim as proof of the corollary that the outer Loewner ellipsoid $E(A_\nu)$ of K' is also that of K. Indeed $E(A_\nu)$ contains K and if an ellipsoid $E(A'_\nu)$ contains K and is such that

$$W_\nu(A'_\nu) \leqq W_\nu(A_\nu)$$

then $E(A'_\nu)$ must contain K' and so, by Theorem 2, $A'_\nu = A_\nu$.

Let x be the interior point mentioned in Theorem 2 and let $E(A_\nu(x))$, $E(B_\nu(x))$ be the νth outer and inner Loewner ellipsoids of K which are centred at x. We allow x to vary and so generate two collections of ellipsoids $\{E(A_\nu(x))\}$ and $\{E(B_\nu(x))\}$. For $\nu = 0$ Danzer, Laugwitz and Lenz in [4] have shown that in the first collection there is a unique ellipsoid for which the volume W_0 is a minimum and in the second collection there is a unique ellipsoid for which the volume is a maximum. We have not been able to decide if this is also true for $\nu = 1, 2, \cdots, n-1$ with W_ν in place of the volume.

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