INFINITE PRODUCTS OF ISOLS

ERIK ELLENTUCK

Introduction. In [1] the notion of the sum of an infinite number of isols is introduced. In this paper we shall similarly attack the problem of the product of an infinite number of isols. Before proceeding to this it is necessary to review the concept of exponentiation. Let $\varepsilon = \{0, 1, \cdots\}$ be the set of nonnegative integers. f *is a finite function* if $\delta f = \varepsilon$ (δf, ρf are domain, range of f respectively) and $\{x : f(x) \neq 0\}$ is finite. The set $\{x : f(x) \neq 0\}$ is called the *essential domain* of f (denoted $\delta_e f$) and the set $\{f(x) : f(x) \neq 0\}$ the *essential range* of f (denoted $\rho_e f$). f is a *finite function from the set* β *into the set* α if $\delta_e f \subseteqq \beta$ and $\rho_e f \subseteqq \alpha$. It can be shown (cf. [3], 181) that there exists a recursive function $r_n(x)$ to two variables such that

(1) All finite functions are generated without repetitions in the sequence $\{r_n(x)\}$.

(2) From n, one can effectively find $r_n(x)$.

(3) From $r_n(x)$, one can effectively find n. Then for any subsets α and β of ε we define

$$\alpha^\beta = \{n : r_n(x) \text{ is a finite function from } \beta \text{ into } \alpha\} .$$

In case α and β are finite it is necessary that $0 \in \alpha$ in order to make α^β have m^n elements where α has m elements and β has n elements. If $A \neq 0$ we let $A^B = \mathrm{Req}(\alpha^\beta)$ where $0 \in \alpha \in A, \beta \in B$. Otherwise $0^B = 1$ if $B = 0, 0^B = 0$ if $B > 0$.

Let $R = \mathrm{Req}(\varepsilon)$. It is known (cf. [3], 189) that $2^R = R$. Since we would like an infinite product of identical factors to reduce to an exponentiation, we see that an infinite product of isols may not be an isol. On the other hand, if X is an infinite isol, then so is 2^X (cf. [3], 182). Thus depending on which exponent we use to formalize the concept of an infinite product of repeated factors we may or may not obtain an isol.

A one-to-one function t_n from ε into ε is *regressive* (cf. [1]) if there is a partial recursive function $p(x)$ such that $\rho t \subseteqq \delta p$ and $p(t_0) = t_0, (\forall n) (p(t_{n+1}) = t_n)$. A set is *regressive* if it is finite or the range of a regressive function. A set is *retraceable* if it is finite or the range of a strictly increasing regressive function. There is no loss of generality by also supposing that p has the following additional properties: $\rho p \subseteqq \delta p$ and $(\forall x) (x \in \delta p \rightarrow (\exists n) (p^{n+1}(x) = p^n(x)))$ (superscript denotes iterate). Define p^* by $\delta p^* = \delta p$ and $p^*(x) = (\mu n) (p^{n+1}(x) = p^n(x))$. Define $\bar{p}$ by $\delta \bar{p} = \delta p$ and $\rho_{\bar{p}(x)} = \{p(x), \cdots, p^n(x)\}$ where $n = p^*(x)$. Two one-to-one

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functions t_n and t'_n from ε into ε are *recursively equivalent* (denoted $t_n \simeq t'_n$) if there is a partial isomorphism f such that $\rho t \subseteqq \delta f$ and $(\forall n)\,(f(t_n) = t'_n)$. The following propositions are proven in [1]. Let $\tau = \rho t$ and $\tau' = \rho t'$ where t_n and t'_n are regressive functions. Then $\tau \simeq \tau'$ if and only if $t_n \simeq t'_n$. Let $t_n \simeq t'_n$. Then t_n is a regressive function if and only if t'_n is a regressive function. Let $\tau \simeq \tau'$. Then τ is a regressive set if and only if τ' is a regressive set. Every regressive function is recursively equivalent to a strictly increasing regressive function. Every regressive set is recursively equivalent to a retraceable set. An RET is *regressive* if it consists of regressive sets. Let Λ_R be the collection of all regressive isols. It is not difficult to show that there are at least $\mathfrak{c}$ of them, and that each contains a retraceable set.

Let V be the class of all subsets of ε, let Q be the class of all finite subsets of ε. A mapping $\Phi\colon V \rightarrow V$ is called a *combinatorial operator* if (1) $\alpha \in Q$ implies $\Phi(\alpha) \in Q$, (2) the cardinality of $\Phi(\alpha)$ is determined by that of α. (3) Φ possesses a quasi inverse Φ^{-1} such that for any $x \in \bigcup\{\Phi(\alpha);\, \alpha \in V\}$, $\Phi^{-1}(x) \in Q$ and $x \in \Phi(\beta)$ if and only if $\Phi^{-1}(x) \subseteqq \beta$. Let $\{\rho_n\}$ be a one-to-one effective enumeration of Q with $\rho_0 = \phi$. Φ is called a *recursive combinatorial operator* if there is a recursive function $g(x)$ such that $\Phi(\rho_n) = \rho_{g(n)}$. It is well known that a function F is recursive combinatorial if and only if it is induced by a recursive combinatorial operator Φ in the sense that $F(\operatorname{Req}\alpha) = \operatorname{Req}\Phi(\alpha)$ for every $\alpha \in V$.

Infinite products. In this paper we only consider products of an infinite sequence of finite positive isols (positive integers). Let $\{a_n\}$ be an infinite sequence of positive integers. For any regressive function t_n with immune range let $\pi(t_n) = \{n\colon \delta_e r_n \subseteqq \rho t \wedge (\forall x)\,(r_n(t_x) < a_x)\}$.

THEOREM 1. *$\pi(t_n)$ is an isolated set.*

Proof. Suppose that γ is a recursively enumerable set, $\gamma \subseteqq \pi(t_n)$. Let $\delta = \bigcup\{\delta_e r_n\colon n \in \gamma\}$. Then δ is recursively enumerable and $\delta \subseteqq \rho t$. Since ρt is immune, δ is finite. But this implies that γ is finite as well. Hence $\pi(t_n)$ is isolated.

THEOREM 2. *Let t_n, t'_n be regressive functions with immune ranges. If $t_n \simeq t'_n$, then $\pi(t_n) \simeq \pi(t'_n)$.*

Proof. There is a partial isomorphism f such that $\rho t \subseteqq \delta f$ and $f(t_n) = t'_n$ for all n. Let $\delta = \{n : \delta_e r_n \subseteqq \delta f\}$. δ is recursively enumerable. We may define a function g as follows: $\delta g = \delta$ and for every $n \in \delta$, $r_{g(n)}$ is a finite function with $\delta_e r_{g(n)} = f(\delta_e r_n)$ and such that for each $x \in \delta_e r_n$,

$r_{g(n)}(f(x)) = r_n(x)$. g is clearly a partial recursive function and it is not difficult to show that g is one-to-one, $\pi(t_n) \subseteqq \delta g$, and $g(\pi(t_n)) = \pi(t'_n)$.

Thus the recursive equivalence type of $\pi(t_n)$ depends only on the recursive equivalence type of ρt since for regressive functions t_n, t'_n we have $\rho t \simeq \rho t'$ if and only if $t_n \simeq t'_n$. This justifies the

DEFINITION. $\prod_T a_n = \text{Req}\, \pi(t_n)$ where $T \in \Lambda_R$ and t_n is any regressive function with $\rho t \in T$.

By Theorem 1 we know that $\prod_T a_n$ is an isol and that therefore the product operation (for fixed $\{a_n\}$) maps Λ_R into Λ. Let f be a recursive combinatorial function. In [5] it is shown that the partial product $g(m) = \prod_{n=0}^{n=m-1} (1 + f(n))$ is also a recursive combinatorial function. It is possible to evaluate $\prod_T (1 + f(n))$ by using the

THEOREM 3. *If f is a recursive combinatorial function and $g(m) = \prod_{n=0}^{n=m-1} (1 + f(n))$ is its partial product, then for every $T \in \Lambda_R$, $\prod_T (1 + f(n)) = G(T)$ (where G canonically extends g).*

Proof. For every set $\alpha \subseteqq \varepsilon$ and integer, $n \in \varepsilon$ let $1 \oplus \alpha = \{x + 1 : x \in \alpha\}$ and $\alpha_{<n} = \{x : x \in \alpha \wedge x < n\}$. Since $T \in \Lambda_R$ there is a retraceable set $\tau \in T$ which is enumerated by a strictly increasing regressive function t_n.

$$\prod_T (1 + f(n)) = \text{Req}\, \pi(t_n),$$

$$\pi(t_n) = \{n : \delta_e r_n \subseteqq \tau \wedge (\forall x)(r_n(t_x) < 1 + f(x))\} .$$

Let Φ be a recursive combinatorial operator inducing f, and let

$$\bar{\pi}(\tau) = \{n : \delta_e r_n \subseteqq \tau \wedge (\forall x)(x \in \delta_e r_n \rightarrow r_n(x) \in 1 \oplus \Phi(\tau_{<x}))\}.$$

In order to complete our proof we shall show that $\bar{\pi}$ is a recursive combinatorial operator inducing g and that $\bar{\pi}(\tau) \simeq \pi(t_n)$.

Let s be an integer and $\alpha = \{a_1, \cdots, a_s\}$, $a_1 < \cdots < a_s$ a set having exactly s elements. If $n \in \bar{\pi}(\alpha)$, then $\delta_e r_n \subseteqq \alpha$ and for each i, $1 \leqq i \leqq s$, $r_n(a_i) = 0$ (if $a_i \notin \delta_e r_n$) or $r_n(a_i) \in 1 \oplus \Phi(\alpha_{<a_i})$. Since $\alpha_{<a_i}$ contains $i - 1$ elements, $r_n(a_i)$ may assume any one of $1 + f(i-1)$ values. Thus $\bar{\pi}(\alpha)$ contains $g(s)$ elements, i.e. $\bar{\pi}$ induces g. Now let α be any set of integers and let $n \in \bar{\pi}(\alpha)$. We define

$$\bar{\pi}^{-1}(n) = \delta_e r_n + \bigcup \{\Phi^{-1}(r_n(x) - 1) : r_n(x) \neq 0\} .$$

It is clear that $\bar{\pi}^{-1}(n) \subseteqq \alpha$ and that if $r_n(x) \neq 0$, then $\Phi^{-1}(r_n(x) - 1) \subseteqq \varepsilon_{<x}$. Conversely, suppose that $\bar{\pi}^{-1}(n) \subseteqq \beta$. Then $\delta_e r_n \subseteqq \beta$ and $\Phi^{-1}(r_n(x) - 1) \subseteqq \beta$ for $r_n(x) \neq 0$. But $\Phi^{-1}(r_n(x) - 1) \subseteqq \varepsilon_{<x}$ and therefore $\Phi^{-1}(r_n(x) - 1) \subseteqq \beta_{<x}$ for $r_n(x) \neq 0$. Hence $n \in \bar{\pi}(\beta)$. Thus $\bar{\pi}^{-1}$ satisfies the condition of being a quasi inverse function of $\bar{\pi}$. Since $\bar{\pi}$ is clearly effective we see that it is a recursive combinatorial operator inducing g. Thus in

particular $G(T) = \text{Req}\,\bar{\pi}(\tau)$. Since t_n is regressive there is a partial recursive function p such that $\tau \subseteqq \delta p$ and $p(t_{n+1}) = t_n$, $p(t_0) = t_0$ for all n (satisfying all those conditions given in the introduction). Let $\delta = \{n : \delta_e r_n \subseteqq \delta p \wedge (\forall x)\,(x \in \delta_e r_n \rightarrow r_n(x) \in 1 \oplus \Phi(\rho_{\bar{p}(x)}))\}$. Since $\rho_{\bar{p}(x)} = \tau_{<x}$ for $x \in \tau$, $\bar{\pi}(\tau) \subseteqq \delta$. Then for any integer $n \in \delta$ we may effectively calculate $\delta_e r_n$, and for each $x \in \delta_e r_n$ we may effectively calculate $\bar{p}(x)$, $\rho_{\bar{p}(x)}$, $1 \oplus \Phi(\rho_{\bar{p}(x)})$ and finally $s_{n(x)} = \text{card}\,[(\{0\} \cup 1 \oplus \Phi(\rho_{\bar{p}(x)}))_{<r_n(x)}]$. If we let $s_n(x) = 0$ for $x \notin \delta_e r_n$, then there is a partial recursive function $h(n)$ such that $\delta h = \delta$ and $r_{h(n)} = s_n$. It is clear that h is defined on δ and that it is a partial recursive function. Let $m, n \in \delta$ and suppose that $h(m) = h(n)$. Since $\delta_e r_{h(n)} = \delta_e r_n$ it follows that $\delta_e r_m = \delta_e r_n$. For $x \in \delta_e r_m$

$$r_{h(m)}(x) = \text{card}\,[(\{0\} \cup 1 \oplus \Phi(\rho_{\bar{p}(x)}))_{<r_m(x)}]$$

and

$$r_{h(n)}(x) = \text{card}\,[(\{0\} \cup 1 \oplus \Phi(\rho_{\bar{p}(x)}))_{<r_n(x)}]\,.$$

Since $r_{h(m)}(x) = r_{h(n)}(x)$ it is clear that $r_m(x) = r_n(x)$. Thus r_m and r_n are the same function and therefore $m = n$. h is a one-to-one function. If $n \in \bar{\pi}(\tau)$, then $\delta_e r_n \subseteqq \tau$ and $x \in \delta_e r_n$ implies that $r_n(x) \in 1 \oplus \Phi(\rho_{\bar{p}(x)}) = 1 \oplus \Phi(\tau_{<x})$. If $x = t_m$, then $r_n(t_m) \in 1 \oplus \Phi(\{t_0, \cdots, t_{m-1}\})$, $r_{h(n)}(t_m) < 1 + f(m)$ and therefore $h(n) \in \pi(t_n)$. Thus h maps $\bar{\pi}(\tau)$ into $\pi(t_n)$. Finally if $s \in \pi(t_n)$, and $t_m \in \delta_e r_s$ choose n such that $\delta_e r_n = \delta_e r_s$ and such that $r_n(t_m)$ is $r_s(t_m)$th element of $1 \oplus \Phi(\{t_0, \cdots, t_{m-1}\})$. Since $1 \leqq r_s(t_m) \leqq f(m)$ the function r_n is defined. Thus h maps $\bar{\pi}(\tau)$ onto $\pi(t_n)$. $\bar{\pi}(\tau) \simeq \pi(t_n)$.

If we denote the product $\prod_T a_n$ by $T : a_0 \cdot a_1 \cdot a_2 \cdots$, then the following two formulas hold as a consequence of Theorem 3: $T : 1 \cdot 2 \cdot 3 \cdots = T!$ and $T : a \cdot a \cdot a \cdots = a^T$ where $a > 1$. Thus the infinite product operation as defined is consistent with the previously defined exponentiation and factorial operations.

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Shell Development Company
Emeryville, California