A NOTE ON PSEUDO-CREATIVE SETS AND CYLINDERS

PAUL R. YOUNG

1. Notation and Definitions. We will use N to denote the set of all nonnegative integers. Unless specifically mentioned otherwise, all sets are considered subsets of N. If A is a set, $A' = N - A$. Since we consider only sets of nonnegative integers, we will not use Cartesian products of sets but will instead work with images of Cartesian products under some effective mapping. More specifically, if A and B are sets, let $A \otimes B = \{(a, b) \mid a \in A \text{ and } b \in B\}$. Let τ be any one-to-one effective mapping of $N \otimes N$ onto N. Then we define $A \times B$ to be $\tau(A \otimes B)$, and we abbreviate $\tau((a, b))$ to $\langle a, b\rangle$. (This is the notation introduced by Rogers in [4].) Given integers a and b we can always effectively find the integer $\langle a, b\rangle$, and given the integer $\langle a, b\rangle$ we can always effectively find a and b.

In [2], Myhill has called a set a cylinder if it is recursively isomorphic to $B \times N$ for some r.e. set B; however we will follow Rogers in calling a set, A, a cylinder if it is recursively isomorphic to $B \times N$ for any set B. Such a set A is called a cylinder of B.

For definitions of recursive, simple, and creative sets, see [3]. A noncreative, recursively enumerable (r.e.), set A has been called pseudo-creative if for every r.e. set $B \subset A'$ there is an infinite r.e. set $C \subset A'$ such that $B \cap C = \varnothing$. A nonrecursive r.e. set A has been called pseudo-simple if there is an infinite r.e. set $B \subset A'$ such that $A \cup B$ is simple. We will denote the class of all recursive sets by $\mathscr{C}_0$, the class of all simple sets by $\mathscr{C}_1$, the class of all pseudo-simple sets by $\mathscr{C}_2$, the class of all pseudo-creative sets by $\mathscr{C}_3$, and the class of all creative sets by $\mathscr{C}_4$. These classes are pairwise disjoint and every r.e. set falls into one of the classes ([2]).

Let A and B be sets. We write $A \leqq_1 B$ if there is a one-to-one recursive function such that $x \in A$ if and only if $f(x) \in B$, $A \leqq_m B$ if there is some recursive function g such that $x \in A$ if and only if $g(x) \in B$, and $A \leqq_{btt} B$ if A is reducible to B via bounded truth-tables. If there is no recursive function g such that $x \in A$ if and only if $g(x) \in B$, we write $A \nleqq_m B$. If both $A \leqq_m B$ and $B \leqq_m A$, we write $A \equiv_m B$.

2. Introduction and preliminaries. In [2] it is shown that the class of pseudo-creative sets is nonempty by proving that the cylinder of any nonrecursive, noncreative, r.e. set is pseudo-creative. In this

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note we shall show that there is a pseudo-creative set which is not a cylinder, and we shall develop some related facts concerning the relation between pseudo-creative sets and cylinders.

LEMMA 1 (Myhill). *Every creative set is a cylinder. Every recursive set which is infinite and has an infinite complement is a cylinder. The empty set and N are cylinders. If A is pseudo-creative, pseudo-simple, or simple, then any cylinder of A is pseudo-creative. No simple set or pseudo-simple set is a cylinder. If A is r.e., then $A \leqq_1 A \times N$ and $A \times N \leqq_m A$.*

Proof. The proofs are straightforward and may be found in [2]. The requirement in the last assertion that A be r.e. may be omitted.

LEMMA 2. *Let A be a cylinder. Then there exists a one-to-one recursive function f such that $x \in A$ implies that $\{x, f(x), f^2(x), f^3(x), \cdots\}$ is an infinite r.e. subset of A, and $x \in A'$ implies that $\{x, f(x), f^2(x), f^3(x), \cdots\}$ is an infinite r.e. subset of A'.*

Proof. We may assume $A = B \times N$ for some set B. Define $f(\langle x, n\rangle) = \langle x, n+1\rangle$.

LEMMA 3 (Post-Shoenfield). *If B is a r.e. set and if $A \leqq_{btt} B$ where A is creative, then B is either creative or pseudo-creative.*

Proof. In [3] it is shown that B cannot be recursive or simple. In [5] it is shown that B cannot be pseudo-simple.

LEMMA 4. *Let $A \in \mathscr{C}_i$, $B \in \mathscr{C}_j$, and $A \leqq_1 B$. Then $i \leqq j$.*

Proof. The proof follows easily from the definitions and will be omitted.

LEMMA 5 (Fischer). *There is a simple set S such that $S \times S \not\leqq_m S$.*

Proof. See [1].

3. **Results.** An infinite set which contains no infinite r.e. subset is called immune ([3]).

LEMMA 6. *If A is a nonimmune infinite set, then $A \times N \leqq_1 A \times A$.*

Proof. Let B be an infinite r.e. subset of A and let g be a one-to-one recursive function whose range is B. Define $h(\langle a, b\rangle) = \langle a, g(b)\rangle$.

Then h is a one-to-one recursive function and $x \in A \times N$ if and only if $h(x) \in A \times A$.

COROLLARY 1. *Suppose S is simple or pseudo-simple. Then $S \times S$ is pseudo-creative.*

Proof. By Lemma 6, $S \times N \leqq_1 S \times S$. By Lemma 1 $S \times N$ is pseudo-creative and therefore by Lemma 4 $S \times S$ is either pseudo-creative or creative. Since $S \times S \leqq_{btt} S$, by Lemma 3 $S \times S$ cannot be creative.

THEOREM 1. *Let A be an infinite nonimmune set. Then $A \times A \leqq_m A$ implies that $A \times A$ is a cylinder.*

Proof. Suppose $A \times A \leqq_m A$ via the recursive function g. Define $h(\langle a, b\rangle) = \langle g(\langle a, b\rangle), \langle a, b\rangle\rangle$. Then $A \times A \leqq_1 A \times N$ via h. By Lemma 6, $A \times N \leqq_1 A \times A$. Thus $A \times A$ is recursively isomorphic to $A \times N$.

THEOREM 2. *Let A be any infinite r.e. set which is not pseudo-creative. Then $A \times A \leqq_m A$ if and only if $A \times A$ is a cylinder.*

Proof. In view of the preceding theorem, we need only prove that if $A \times A$ is a cylinder then $A \times A \leqq_m A$.

If A is creative or recursive so is $A \times A$, and in this case $A \times A \equiv_m A$ and $A \times A$ is a cylinder. Therefore we may assume that A is simple or pseudo-simple. Let $B \subset A'$ be a r.e. set such that $A \cup B$ is simple. (If A is simple, B is finite.) Let $B_0 = A \times N \cup N \times B$, and let $B_1 = N \times A \cup B \times N$. $B_0 \cup B_1$ is simple, for otherwise there is an infinite r.e. set $C \subset B_0' \cap B_1'$, and this implies that either $\{x \mid (\exists y)[\langle x, y\rangle \in C]\}$ is an infinite r.e. subset of $A' \cap B'$ or $\{y \mid (\exists x)[\langle x, y\rangle \in C]\}$ is an infinite r.e. subset of $A' \cap B'$.

Assume $A \times A$ is a cylinder and let f be the recursive function described in Lemma 2. (So $x \in A \times A$ implies that $\{x, f(x), f^2(x), \cdots\}$ is an infinite r.e. subset of $A \times A$ and $x \in (A \times A)'$ implies that $\{x, f(x), f^2(x), \cdots\}$ is an infinite subset of $(A \times A)'$.)

To obtain a many-one reduction of $A \times A$ to A: Given x, enumerate $\{x, f(x), f^2(x), f^3(x), \cdots\}$, B_0, and B_1. Since $B_0 \cup B_1$ is simple, we must eventually find an integer $\langle c, d\rangle$ either in $\{x, f(x), f^2(x), \cdots\} \cap B_0$ or in $\{x, f(x), f^2(x), \cdots\} \cap B_1$. In the former case define $g(x) = d$; in the latter case define $g(x) = c$. Then $x \in A \times A$ if and only if $g(x) \in A$.

We next modify Theorem 2 to characterize a class of pseudo-creative noncylinders.

COROLLARY 2. *Let A be a r.e. set which is not pseudo-creative. Then $A \times A$ is a pseudo-creative noncylinder if and only if $A \times A \not\leqq_m A$.*

Proof. If A is recursive, $A \times A$ is also recursive and $A \times A$ is many-one equivalent to A. If A is creative, since $A \leqq_1 A \times N \leqq_1 A \times A$, $A \times A$ is also creative and hence many-one equivalent to A. The corollary now follows from Theorem 2 and Corollary 1.

COROLLARY 3. *There exists a pseudo-creative set which is not a cylinder and which is bounded-truth-table reducible to a simple set.*

Proof. By Lemma 5 there is a simple set S such that $S \times S \not\leqq_m S$. Since $S \times S \leqq_{btt} S$, $S \times S$ is the desired set.

Our next theorem shows that Theorem 2 cannot be strengthened to include the pseudo-creative sets.

THEOREM 3. *There is a pseudo-creative set A such that $A \times A$ is a cylinder but $A \times A \not\leqq_m A$.*

Proof. Let S be a simple set such that $S \times S \not\leqq_m S$. Then

$$S \equiv_m S \times N \leqq_1 S \times S \leqq_1 (S \times S) \times N \leqq_1 (S \times N) \times (S \times N) .$$

Let $A = S \times N$. Then $A \times A$ is clearly a cylinder, but $A \times A \leqq_m A$ implies that $S \times S \leqq_m S$, a contradiction. Thus $A \times A \not\leqq_m A$, and by either Lemma 1 or Theorem 2, A is pseudo-creative.

Since any set is many-one equivalent to its cylinder and all creative sets are many-one equivalent, the cylinder of any pseudo-creative set is still pseudo-creative. Thus, since any set is one-to-one reducible to its cylinder, we might hope to subclassify the pseudo-creative sets into cylinders and noncylinders and obtain for the subclassification a result analogous to Lemma 4. In view of the following theorem, such an analogue fails.

THEOREM 4. *There exist pseudo-creative sets A and B such that A is a cylinder and $A \leqq_1 B$, but B is not a cylinder.*

Proof. Let $A = S \times N$ and $B = S \times S$ where S is a simple set such that $S \times S \not\leqq_m S$. By Theorem 2, $S \times S$ is not a cylinder, and by Lemma 6, $A \leqq_1 B$. By Lemma 1 A is pseudo-creative, and by Corollary 1, B is pseudo-creative.

REMARKS. 1. In another paper we shall show that there is a pseudo-creative set which is not a cylinder and which, in contrast to those pseudo-creative noncylinders constructed by using Theorem 2, is not bounded-truth-table reducible either to a simple set or to a pseudo-simple set.

2. The author does not know if there is a simple, pseudo-simple,

or pseudo-creative set A such that $A \times A \leqq_m A$. The question of whether such a set exists is equivalent to the following question: Is it true that if A is a r.e. set, then $A \times A \nleqq_m A$ if and only if A is either recursive or creative?

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